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Recurrences for Callan's Generalization of Narayana Polynomials*

CHEN Xi · YANG Arthur Li Bo · ZHAO James Jing Yu

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Abstract By using Chen, Hou and Mu's extended Zeilberger algorithm, the authors obtain two recurrence relations for Callan's generalization of Narayana polynomials. Based on these recurrence relations, the authors further prove the real-rootedness and asymptotic normality of Callan's Narayana polynomials.

Keywords Asymptotic normality, Callan's Narayana polynomials, central limit theorem, local limit theorem, real zeros.

1 Introduction

For any integers $n > k \geq 0$, the classical Narayana number $N(n,k)$ is given by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1},$$

which appears in OEIS as A001263 in [1]. It is well known that the Narayana numbers refine the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ since

$$\sum_{k=0}^{n-1} N(n,k) = C_n.$$

CHEN Xi
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024*, China.*
Email: chenxi@dlut.edu.cn.
YANG Arthur Li Bo
Center for Combinatorics, LPMC, Nankai University, Tianjin 300071*, China.* Email: yang@nankai.edu.cn.
ZHAO James Jing Yu (Corresponding author)
School of Mathematics, Tianjin University, Tianjin 300350*, China.* Email: jjyzhao@tju.edu.cn.
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For more information on Catalan numbers, see [2, 3]. As pointed out by Bóna and Sagan[4], the Narayana numbers were first studied by MacMahon (see [5, Article 495]). These numbers were rediscovered by Narayana[6], and later extensively studied by the combinatorial community, see, for instance, [7–12].

This paper is mainly concerned with a generalization of Narayana numbers given by Callan[13]. For integers $m \geq 0$, $n \geq m+1$ and $0 \leq k \leq n-m-1$, let

$$N_m(n,k) = \frac{m+1}{n+1}\binom{n+1}{k+1}\binom{n-m-1}{k}, \tag{1}$$

which was called the m-th order Narayana number by Callan. It is clear that $N(n,k) = N_0(n,k)$. Let $P_{m,n}(x)$ denote the generating polynomial of the m-th order Narayana numbers, namely

$$P_{m,n}(x) = \sum_{k=0}^{n-m-1} N_m(n,k)x^k, \tag{2}$$

which we may call the m-th order Narayana polynomial.

As a generalization, it is natural to expect that the m-th order Narayana numbers and polynomials share many interesting properties with the classical ones. It is known that the classical Narayana number $N(n,k)$ counts the number of lattice paths of n up-steps $U = (1,1)$ and n down-steps $D = (1,-1)$ with $k+1$ peaks UD such that the path never goes below the horizontal line through its initial point. Similarly, $N_m(n,k)$ counts the number of lattice paths of n up-steps and $n-m$ down-steps with $k+1$ peaks such that the path never goes below the horizontal line through its initial point. Such a combinatorial interpretation of $N_m(n,k)$ is due to Werner Schulte, as mentioned in [13]. Chen, et al.[14] conjectured the total positivity of the Narayana triangle composed of $N(n,k)$, and later Wang and Yang[15] proved the total positivity of the triangle composed of $N_m(n,k)$ for any $m \geq 0$. For the classical Narayana polynomials, the following recurrence relation

$$(n+1)P_{0,n}(x) = (2n-1)(1+x)P_{0,n-1}(x) - (n-2)(x-1)^2P_{0,n-2}(x) \tag{3}$$

was combinatorially proved by Sulanke (see [11, Equation (2)]). Liu and Wang[16] pointed out that the real-rootedness of $P_{0,n}(x)$ can be easily derived from this recurrence relation besides some other proofs. Based on the real-rootedness of $P_{0,n}(x)$, Chen, et al.[17] further proved the asymptotic normality of its coefficients.

The aim of this paper is to give some recurrence relations of the m-th order Narayana polynomial for any fixed $m \geq 0$, and then to prove its real-rootedness as well as the asymptotic normality of its coefficients. This paper is organized as follows. In Section 2, we will give an overview of the extended Zeilberger algorithm established by Chen, et al.[18], and then use their algorithm to give some recurrence relations of $P_{m,n}(x)$. In Section 3, we will recall a criterion for determining the real-rootedness of polynomials due to Liu and Wang[16], and then use their criterion to prove the real-rootedness of $P_{m,n}(x)$. Finally, in Section 4 we will apply some results due to Bender[19] and Harper[20] to show the asymptotic normality of $N_m(n,k)$ for any fixed $m \geq 0$. We would like to point out that the asymptotic normality of classical Narayana numbers was proposed by Shapiro[21] in 2001 as an open problem and recently proved by Chen, et al.[17].

2 Recurrence Relations

In this section we aim to give some recurrence relations of the m-th order Narayana polynomial $P_{m,n}(x)$ for any fixed $m \geq 0$. The main result of this section is as follows.

Theorem 2.1 *Fixing $m \geq 0$, the polynomial sequence $\{P_{m,n}(x)\}_{n \geq m+1}$ satisfies the following two recurrence relations:*

$$P_{m,n}(x) = \frac{(2n-m-1)\Big(\big(2n^2-2n-m(2n-m-2)\big)x + 2n(n-m-1)\Big)}{(n+1)(n-m)(2n-m-2)} P_{m,n-1}(x) - \frac{(2n-m)(n-m-2)(n-1)(x-1)^2}{(n+1)(n-m)(2n-m-2)} P_{m,n-2}(x), \quad \text{for } n \geq m+3 \tag{4}$$

and

$$(n+1)(n-m)P_{m,n}(x) = \Big((n+1)(n-m) + \big(m^2+2m+3n(n-m-1)\big)x\Big)P_{m,n-1}(x) - (2n-m)(x^2-x)P'_{m,n-1}(x), \quad \text{for } n \geq m+2 \tag{5}$$

with initial values $P_{m,m+1}(x) = m+1$ and $P_{m,m+2}(x) = m+1+\binom{m+2}{2}x$.

Instead of providing elementary proofs of (4) and (5), we will show how to use the symbolic method to derive these recurrences from the expression of $P_{m,n}(x)$. We refer the reader to Chen and Kauers[22] for a nice survey on the method of creative telescoping and related open problems. We find that the extended Zeilberger algorithm developed by Chen, et al.[18] is a powerful tool for dealing with such issues.

Let us give a brief review of the extended Zeilberger algorithm. We adopt the notation and terminology of [18]. A function $f(k)$ is called a hypergeometric term if $f(k+1)/f(k)$ is a rational function of k. Given two hypergeometric terms $f(k)$ and $g(k)$, they are said to be similar if $f(k)/g(k)$ is a rational function of k. The extended Zeilberger algorithm developed by Chen, et al. is applicable to ℓ hypergeometric terms $f_1(k, p_1, p_2, \cdots, p_s)$, $\cdots$, $f_\ell(k, p_1, p_2, \cdots, p_s)$ of k with parameters $p_1, p_2, \cdots, p_s$ such that both

$$\frac{f_i(k, p_1, p_2, \cdots, p_s)}{f_j(k, p_1, p_2, \cdots, p_s)}$$

and

$$\frac{f_i(k+1, p_1, p_2, \cdots, p_s)}{f_i(k, p_1, p_2, \cdots, p_s)}$$

are all rational functions of k and $p_1, p_2, \cdots, p_s$ for any $1 \leq i, j \leq \ell$. The extended Zeilberger algorithm is devised to find a hypergeometric term $g(k, p_1, p_2, \cdots, p_s)$, namely $g(k+1, p_1, p_2, \cdots, p_s)/g(k, p_1, p_2, \cdots, p_s)$ is a rational function of k and $p_1, p_2, \cdots, p_s$, and polynomial coefficients $a_1(p_1, p_2, \cdots, p_s)$, $\cdots$, $a_\ell(p_1, p_2, \cdots, p_s)$ which are independent of k such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_\ell f_\ell(k) = g(k+1) - g(k), \tag{6}$$

where a_i stands for a_i $(p_1, p_2, \cdots, p_s)$, $f_i(k)$ stands for $f_i(k, p_1, p_2, \cdots, p_s)$, and $g(k)$ stands for $g(k, p_1, p_2, \cdots, p_s)$ for brevity. For $1 \le i \le \ell$, let

$$F_i = \sum_k f_i(k).$$

Summing the telescoping relation (6) over k usually leads to a homogeneous relation

$$a_1 F_1 + a_2 F_2 + \cdots + a_\ell F_\ell = 0. \tag{7}$$

The extended Zeilberger algorithm was implemented as the function Ext_Zeil in the Maple package APCI by Hou[23]. The calling sequence of this function is of the form **Ext_Zeil(**$[f_1, f_2, \cdots, f_\ell]$**,** k**)**. If the algorithm is applicable, it gives the output $[C, Ca_2/a_1, Ca_3/a_1, \cdots, Ca_\ell/a_1]$, where C is a k-free non-zero constant. In the following we shall take (3) as an example to illustrate the use of this package. The first step is to import this package in Maple in the following way.

In[1]:= [> **with(APCI);**

[AbelZ, Ext_Zeil, Gosper, MZeil, Zeil, hyper_simp, hyperterm, poch, qExt_Zeil, qGosper, qZeil, qbino, qhyper_simp, qhyperterm, qpoch]

Note that the recurrence (3) is of the form (7) with

$$f_1 = N_0(n,k)x^k, \quad f_2 = N_0(n-1,k)x^k, \quad f_3 = N_0(n-2,k)x^k.$$

To prove (3), we continue to set the values of f_i as follows.

In[2]:= [> $\boldsymbol{f_1 := \frac{1}{n+1}\binom{n+1}{k+1}\binom{n-1}{k}x^k}$ **:**

In[3]:= [> $\boldsymbol{f_2 := \frac{1}{n}\binom{n}{k+1}\binom{n-2}{k}x^k}$ **:**

In[4]:= [> $\boldsymbol{f_3 := \frac{1}{n-1}\binom{n-1}{k+1}\binom{n-3}{k}x^k}$ **:**

Then we run the following command

In[5]:= [> **Ext_Zeil(**$[\boldsymbol{f_1, f_2, f_3}], \boldsymbol{k}$**);**

$$\left[k\_free_1,\ -\frac{k\_free_1(x+1)(2n-1)}{n+1},\ \frac{k\_free_1(n-2)(x-1)^2}{n+1}\right].$$

The above output implies that there exists some nonzero constant C independent of k such that

$$C \cdot P_{0,n}(x) + \left(-\frac{C\cdot(x+1)(2n-1)}{n+1}\right) \cdot P_{0,n-1}(x) + \left(\frac{C\cdot(n-2)(x-1)^2}{n+1}\right) \cdot P_{0,n-2}(x) = 0,$$

which simplifies to (3).

Now we are able to prove Theorem 2.1.

Proof of Theorem 2.1 Let us first prove (4), which is of the form (7) with

$$f_1 = N_m(n,k)x^k, \quad f_2 = N_m(n-1,k)x^k, \quad f_3 = N_m(n-2,k)x^k.$$

To this end, we input the following commands:

In[6]:= $[> \boldsymbol{f_1 := \frac{m+1}{n+1}\binom{n+1}{k+1}\binom{n-m-1}{k}x^k :}$

In[7]:= $[> \boldsymbol{f_2 := \frac{m+1}{n}\binom{n}{k+1}\binom{n-m-2}{k}x^k :}$

In[8]:= $[> \boldsymbol{f_3 := \frac{m+1}{n-1}\binom{n-1}{k+1}\binom{n-m-3}{k}x^k :}$

In[9]:= $[>$ **Ext_Zeil**$(\boldsymbol{[f_1, f_2, f_3], k});$

$$\left[k\_free_1, \frac{(m-2n+1)(m^2x-2mnx+2n^2x-2mn+2mx+2n^2-2nx-2n)k\_free_1}{(n+1)(-n+m)(m-2n+2)},\right.$$
$$\left.\frac{(-n+2+m)(n-1)(x-1)^2(m-2n)k\_free_1}{(n+1)(-n+m)(m-2n+2)}\right].$$

The above output implies that there exists some nonzero constant C such that

$$C \cdot P_{m,n}(x) + C \cdot \frac{a_2}{a_1} \cdot P_{m,n-1}(x) + C \cdot \frac{a_3}{a_1} \cdot P_{m,n-2}(x) = 0,$$

where

$$\begin{cases} a_1 = (n+1)(-n+m)(m-2n+2), \\ a_2 = (m-2n+1)(m^2x-2mnx+2n^2x-2mn+2mx+2n^2-2nx-2n), \\ a_3 = (-n+2+m)(n-1)(x-1)^2(m-2n). \end{cases}$$

This relation is equivalent to (4).

We proceed to prove (5), which is of the form (7) with

$$f_1 = N_m(n,k)x^k, \quad f_2 = N_m(n-1,k)x^k, \quad f_3 = (N_m(n-1,k)x^k)' = k \cdot N_m(n-1,k)x^{k-1}.$$

Hence, we only need to reset the value of f_3 and rerun the command **Ext_Zeil(**$[f_1, f_2, \cdots, f_\ell]$**,** k**)**.

In[10]:= $[> \boldsymbol{f_3 := k\frac{m+1}{n}\binom{n}{k+1}\binom{n-m-2}{k}x^{k-1} :}$

In[11]:= $[>$ **Ext_Zeil**$(\boldsymbol{[f_1, f_2, f_3], k});$

$$\left[k\_free_1, \frac{(m^2x-3mnx+3n^2x-mn+2mx+n^2-3nx-m+n)k\_free_1}{(n+1)(-n+m)}, \frac{x(m-2n)k\_free_1(x-1)}{(n+1)(-n+m)}\right].$$

Using the above output one can readily verify the validity of (5). This completes the proof. ∎

By comparing the coefficients of the powers of x on both sides of (4) and (5), we immediately obtain the following two recurrence relations for the m-th order Narayana numbers.

Corollary 2.2 *For $m \geq 0$ and $0 \leq k \leq n-m-1$, we have*

$$\begin{aligned} & N_m(n,k) \\ =& A(n,m)N_m(n-1,k) + B(n,m)N_m(n-1,k-1) \\ & - C(n,m)\big(N_m(n-2,k) - 2N_m(n-2,k-1) + N_m(n-2,k-2)\big), \quad \text{for } n \geq m+3, \end{aligned} \tag{8}$$

where

$$\begin{cases} A(n,m) = \dfrac{2n(2n-m-1)(n-m-1)}{(n+1)(n-m)(2n-m-2)}, \\ B(n,m) = \dfrac{(2n-m-1)\big(n^2+(n-m)(n-m-2)\big)}{(n+1)(n-m)(2n-m-2)}, \\ C(n,m) = \dfrac{(2n-m)(n-m-2)(n-1)}{(n+1)(n-m)(2n-m-2)}, \end{cases}$$

and

$$\begin{aligned} & (n+1)(n-m)N_m(n,k) \\ =& \big((2n-m)k + (n+1)(n-m)\big)N_m(n-1,k) \\ & + \big(m^2 + (3n-1)(n-m) - k(2n-m)\big)N_m(n-1,k-1), \quad \text{for } n \geq m+2, \end{aligned} \tag{9}$$

with initial values $N_m(m+1,0) = m+1$, $N_m(m+2,0) = m+1$ and $N_m(m+2,1) = \binom{m+2}{2}$.

Remark 2.3 It should be mentioned that (9) can also be obtained by using the Mathematica package HolonomicFunctions programmed by Koutschan[24, 25]. Letting $m = 0$ in (8) leads to

$$\begin{aligned} N_0(n,k) = & \frac{2n-1}{n+1}\big(N_0(n-1,k) + N_0(n-1,k-1)\big) \\ & - \frac{n-2}{n+1}\big(N_0(n-2,k) - 2N_0(n-2,k-1) + N_0(n-2,k-2)\big), \end{aligned}$$

a known recurrence relation satisfied by the classical Narayana polynomials, see Sulanke (see [26, Equation (8)]). It would also be interesting to give combinatorial proofs of (8) and (9) based on Schulte's Dyck path interpretation of the m-th order Narayana numbers.

3 Real Zeros

This section is devoted to proving the real-rootedness of the m-th order Narayana polynomials. The main result of this section is as follows.

Theorem 3.1 *For any $m \geq 0$ and $n \geq m+1$, the generalized Narayana polynomials $P_{m,n}(x)$ have only real zeros.*

To prove the above theorem, we will use the theory of Sturm sequences. Let us first review some definitions and results on Sturm sequences. Following Liu and Wang[16], we use PF to represent the set of real-rooted polynomials with nonnegative real coefficients, including any nonnegative constant for convenience. Given two polynomials $f(x), g(x) \in \mathrm{PF}$, suppose $f(u_i) = 0$ and $g(v_j) = 0$. We say that $g(x)$ interlaces $f(x)$, denoted $g(x) \preceq f(x)$, if either $\deg f(x) = \deg g(x) = n$ and

$$v_n \le u_n \le v_{n-1} \le \cdots \le v_2 \le u_2 \le v_1 \le u_1,$$

or $\deg f(x) = \deg g(x) + 1 = n$ and

$$u_n \le v_{n-1} \le \cdots \le v_2 \le u_2 \le v_1 \le u_1.$$

Following Liu and Wang[16], we also let $a \preceq bx + c$ for any nonnegative a, b, c, and let $0 \preceq f$ and $f \preceq 0$ for any $f \in \mathrm{PF}$. Given a polynomial sequence $\{f_n(x)\}_{n\ge0}$, if each $f_n(x) \in \mathrm{PF}$ and moreover

$$f_0(x) \preceq f_1(x) \preceq \cdots \preceq f_{n-1}(x) \preceq f_n(x) \preceq \cdots,$$

then $\{f_n(x)\}_{n\ge0}$ is said to be a generalized Sturm sequence. The following result was given by Liu and Wang, see [16, Corollary 2.4].

Theorem 3.2 (see [16]) *For a sequence $\{f_n(x)\}_{n\ge0}$ of polynomials with nonnegative coefficients, assume that $f_0(x), f_1(x) \in \mathrm{PF}$, $f_0(x) \preceq f_1(x)$ and*

- *there exist polynomials $a_n(x), b_n(x), c_n(x)$ with real coefficients such that*

$$f_{n+1}(x) = a_n(x)f_n(x) + b_n(x)f'_n(x) + c_n(x)f_{n-1}(x)$$

  and $\deg f_{n+1}(x) = \deg f_n(x)$ or $\deg f_n(x) + 1$; and

- *for $x \le 0$ we have $b_n(x) \le 0$ and $c_n(x) \le 0$.*

Then $\{f_n(x)\}_{n\ge0}$ is a generalized Sturm sequence.

We proceed to prove Theorem 3.1.

Proof of Theorem 3.1 Let us fix $m \ge 0$. It suffices to show that $\{P_{m,k}(x)\}_{k\ge m+1}$ is a generalized Sturm sequence. To use Theorem 3.2, set $f_n(x) = P_{m,m+1+n}(x)$ for each $n \ge 0$. By (1) each polynomial $f_n(x) = P_{m,m+1+n}(x)$ has only nonnegative coefficients. Note that

$$f_0(x) = P_{m,m+1}(x) = m + 1, \quad f_1(x) = P_{m,m+2}(x) = m + 1 + \binom{m+2}{2}x,$$

and hence $f_0(x), f_1(x) \in \mathrm{PF}$, $f_0(x) \preceq f_1(x)$. By the recurrence relation (5) of Theorem 2.1, we find that

$$f_{n+1}(x) = a_n(x)f_n(x) + b_n(x)f'_n(x) + c_n(x)f_{n-1}(x), \quad \text{for } n \ge 1$$

with

$$a_n(x) = 1 + \frac{3n^2 + (9 + 3m)n + (m^2 + 5m + 6)}{(n + m + 3)(n + 2)} \cdot x,$$

$$b_n(x) = -\frac{(2n + m + 4)}{(n + m + 3)(n + 2)} \cdot x(x - 1),$$

$$c_n(x) = 0.$$

It is easy to verify that the conditions of Theorem 3.2 are satisfied, and thus $\{f_n(x)\}_{n\geq 0}$, that is $\{P_{m,k}(x)\}_{k\geq m+1}$, is a generalized Sturm sequence. This completes the proof. ∎

Brändén[27] pointed out a connection between the classical Narayana polynomials and the Jacobi polynomials. Recall that the classical hypergeometric function ${}_2F_1$[28] with parameters a, b, c is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n, $$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$. The Jacobi polynomial $J_n^{(\alpha,\beta)}(x)$ can be expressed as (see [29, P. 254])

$$ J_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -\beta-n; 1+\alpha; \frac{x-1}{x+1}\right). \tag{10} $$

Brändén[27] noted that

$$ P_{0,n+1}(x) = \frac{1}{n+1}(1-x)^n J_n^{(1,1)}\left(\frac{1+x}{1-x}\right). $$

It is well known that for $\alpha, \beta > -1$ the Jacobi polynomial sequence $\{J_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ is a generalized Sturm sequence. We have the following result, from which Theorem 3.1 also follows.

Theorem 3.3 *For any $m \geq 0$ and $n \geq 0$, we have*

$$ P_{m,n+m+1}(x) = \frac{m+1}{n+1}(1-x)^n J_n^{(1,m+1)}\left(\frac{1+x}{1-x}\right). $$

Proof By (1) it is routine to verify that

$$ P_{m,n+m+1}(x) = (m+1)\cdot {}_2F_1(-n, -n-m-1; 2; x). $$

Then we immediately obtain the desired result from (10). ∎

4 Asymptotic Normality

This section is devoted to the study of the asymptotic property of generalized Narayana numbers $N_m(n,k)$.

Let us first recall some concepts and results. Suppose that $\{f_n(x)\}_{n\geq 0}$ is a polynomial sequence with nonnegative coefficients given by

$$ f_n(x) = \sum_{k=0}^{n} a(n,k)x^k. \tag{11} $$

We say that the coefficient $a(n,k)$ is asymptotically normal with mean μ_n and variance σ_n^2 by a central limit theorem if

$$ \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\leq \mu_n + x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt \right| = 0, \tag{12} $$

where

$$p(n,k)=\frac{a(n,k)}{\sum_{j=0}^{n} a(n,j)}.$$

We say that $a(n,k)$ is asymptotically normal with mean μ_n and variance σ_n^2 by a local limit theorem on the real set $\mathbb{R}$ if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\sigma_n p(n,\lfloor\mu_n+x\sigma_n\rfloor)-\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)\right|=0. \tag{13}$$

It is known that (13) implies (12). However, (12) does not imply (13) in general. But if

$$(a(n,k))^2\geq a(n,k-1)\cdot a(n,k+1)$$

and

$$\{k\,:\,a(n,k)\neq 0\}=\{k\,:u_n\leq k\leq v_n\}$$

for some integers u_n, v_n, then (13) is valid in the presence of (12), see Bender[19] and Canfield[30].

The main tool we use here is the following criterion, see Bender[19] and Harper[20].

Theorem 4.1 (see [19, Theorem 2]) *Let $\{f_n(x)\}_{n\geq 0}$ be a real-rooted polynomial sequence with nonnegative coefficients as in* (11)*. Let*

$$\mu_n=\frac{f_n'(1)}{f_n(1)}, \tag{14}$$

$$\sigma_n^2=\frac{f_n''(1)}{f_n(1)}+\mu_n-\mu_n^2. \tag{15}$$

If $\sigma_n^2\to+\infty$ as $n\to+\infty$, then the coefficient of $f_n(x)$ is asymptotically normal with mean μ_n and variance σ_n^2 by local and central limit theorems.

Now we are in the position to state the main result of this section.

Theorem 4.2 *For any fixed integer $m\geq 0$, let $\{P_{m,n}(x)\}_{n\geq m+1}$ be defined as in* (2)*. Then the coefficient of $P_{m,n}(x)$, namely the generalized Narayana number $N_m(n,k)$, is asymptotically normal by local and central limit theorems with*

$$\mu_n=\frac{n(m+1+n)}{2n+m+2}, \tag{16}$$

$$\sigma_n^2=\frac{n(n+1)(m+1+n)(m+2+n)}{(2n+m+2)^2(2n+m+1)}. \tag{17}$$

Proof As in the proof of Theorem 3.1, let $f_n(x)=P_{m,m+1+n}(x)$ for each $n\geq 0$. By Theorem 3.1, we know that $\{f_n(x)\}_{n\geq 0}$ is a real-rooted polynomial sequence with nonnegative coefficients. Note that

$$\begin{aligned}P_{m,n}'(x)&=\sum_{k=0}^{n-m-1}\frac{m+1}{n+1}\binom{n+1}{k+1}\binom{n-m-1}{k}kx^{k-1}\\&=\frac{(m+1)(n-m-1)}{n+1}\sum_{k=0}^{n-m-1}\binom{n+1}{k+1}\binom{n-m-2}{k-1}x^{k-1},\end{aligned}$$

and

$$P''_{m,n}(x) = \sum_{k=0}^{n-m-1} \frac{m+1}{n+1}\binom{n+1}{k+1}\binom{n-m-1}{k}k(k-1)x^{k-2}$$
$$= \frac{(m+1)(n-m-1)(n-m-2)}{n+1}\sum_{k=0}^{n-m-1}\binom{n+1}{k+1}\binom{n-m-3}{k-2}x^{k-2}.$$

By the Chu-Vandermonde convolution formula (see [31] or [32, §5.1]), we obtain

$$P_{m,n}(1) = \frac{m+1}{n+1}\binom{2n-m}{n}, \tag{18}$$

$$P'_{m,n}(1) = \frac{(m+1)(n-m-1)}{n+1}\binom{2n-m-1}{n-1}, \tag{19}$$

$$P''_{m,n}(1) = \frac{(m+1)(n-m-1)(n-m-2)}{n+1}\binom{2n-m-2}{n-2}. \tag{20}$$

Thus, from (14), (15), and (18)–(20) it follows that

$$\mu_n = \frac{f'_n(1)}{f_n(1)} = \frac{P'_{m,m+1+n}(1)}{P_{m,m+1+n}(1)} = \frac{n(m+1+n)}{2n+m+2},$$

and

$$\sigma_n^2 = \frac{f''_n(1)}{f_n(1)} + \mu_n - \mu_n^2 = \frac{P''_{m,m+1+n}(1)}{P_{m,m+1+n}(1)} + \mu_n - \mu_n^2 = \frac{n(n+1)(m+1+n)(m+2+n)}{(2n+m+2)^2(2n+m+1)}.$$

Clearly, for any fixed nonnegative integer m, we have $\sigma_n^2 \to +\infty$ when $n \to \infty$. By applying Theorem 4.1, we immediately obtain the desired result. ∎

With the above theorem, we are now able to give an asymptotic formula for the m-th order Narayana numbers. Let us first recall some common notations for asymptotic estimation. Given two functions $f(n)$ and $g(n)$, if $\lim_{n\to\infty} f(n)/g(n) = 1$ then we denote $f(n) \sim g(n)$; if $\limsup_{n\to\infty} |f(n)|/g(n) < \infty$ then we denote $f(n) = O(g(n))$; and if $\lim_{n\to\infty} f(n)/g(n) = 0$ then we denote $f(n) = o(g(n))$. As a consequence of Theorem 4.2, we have the following result.

Corollary 4.3 *Let $P_{m,n}(1)$ be given by* (18), *μ_n be given by* (16), *and σ_n be given by* (17). *Then for $k = \lfloor \mu_n + x\sigma_n \rfloor$ and $x = O(1)$, there holds*

$$N_m(m+1+n,k) \sim \frac{P_{m,m+1+n}(1)}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{x^2}{2}\right), \qquad \text{as } n \to \infty. \tag{21}$$

In view of the explicit expression (1) of $N_m(n,k)$, it is desirable to give a direct approach to (21) as was done for the normal approximation of the binomial distribution. In the following we will give an alternative proof of Corollary 4.3.

The second proof of Corollary 4.3 By (16) and (17) we see that μ_n goes to $n/2$ and σ_n^2 goes to $n/8$ for fixed m as n goes to ∞. Thus, it suffices to estimate

$$\frac{N_m(n,k)}{P_{m,n}(1)} = \frac{\binom{n+1}{k+1}\binom{n-m-1}{k}}{\binom{2n-m}{n}} \tag{22}$$

for $|k-n/2| = o(n^{2/3})$. From Stirling's formula for factorials it follows that

$$\binom{n}{\frac{n}{2}} \sim \frac{2^n}{\sqrt{\pi n/2}} \tag{23}$$

and

$$\binom{n}{k} \sim \frac{2^n}{\sqrt{\pi n/2}} \exp\left(-\frac{(k-n/2)^2}{n/2}\right), \qquad \text{for } |n/2-k| = o(n^{2/3}), \tag{24}$$

as n goes to ∞, see [33, P. 66]. Hence, by (23) and (24) we get

$$\binom{n+1}{k+1} \sim \frac{2^{n+1}}{\sqrt{2(n+1)\pi/4}} \exp\left(-\frac{(k+1-(n+1)/2)^2}{2(n+1)/4}\right),$$

$$\binom{n-m-1}{k} \sim \frac{2^{n-m-1}}{\sqrt{2(n-m-1)\pi/4}} \exp\left(-\frac{(k-(n-m-1)/2)^2}{2(n-m-1)/4}\right),$$

$$\binom{2n-m}{n} \sim \frac{2^{2n-m}}{\sqrt{2(2n-m)\pi/4}} \exp\left(-\frac{(n-(2n-m)/2)^2}{2(2n-m)/4}\right).$$

Substituting the above approximations into (22) yields

$$\frac{N_m(n,k)}{P_{m,n}(1)} \sim \frac{1}{\sqrt{2\pi}\sigma'_n} \exp\left(-\frac{(k-\mu'_n)^2}{2{\sigma'_n}^2}\right),$$

where

$$\mu'_n = \frac{n(n-m-1)}{2n-m}, \qquad {\sigma'_n}^2 = \frac{(n+1)(n-m-1)}{4(2n-m)}.$$

By replacing n by $m+1+n$, we obtain

$$\frac{N_m(m+1+n,k)}{P_{m,m+1+n}(1)} \sim \frac{1}{\sqrt{2\pi}\sigma''_n} \exp\left(-\frac{(k-\mu''_n)^2}{2{\sigma''_n}^2}\right), \qquad \text{as } n \to \infty, \tag{25}$$

where

$$\mu''_n = \frac{n(m+1+n)}{2n+m+2}, \qquad {\sigma''_n}^2 = \frac{n(m+2+n)}{4(2n+m+2)}.$$

We would like to point out that the asymptotic formulas (21) and (25) have the same mean ($\mu_n = \mu''_n$) but different variances, and the error committed is negligible when n goes to ∞. ∎

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