Stability Analysis of Event-Triggered Networked Control Systems with Time-Varying Delay and Packet Loss*

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Abstract Networked control systems (NCSs) are facing a great challenge from the limitation of network communication resources. Event-triggered control (ETC) is often used to reduce the amount of communication while still keeping a satisfactory performance of the system, by transmitting the measurements only when an event-triggered condition is satisfied. However, some network-induced problems would happen inevitably, such as communication delay and packet loss, which can degrade the control performance significantly and can even lead to instability. In this paper, a periodic event-triggered NCS considering both time-varying delay and packet loss is studied. The system is discretized into a piecewise linear system with uncertainty. Then the model is handled by a polytopic over-approximation method to be more suitable for stability analysis. Finally, stability conditions are obtained and presented in terms of linear matrix inequalities (LMIs). The result is illustrated by a numerical example.

Keywords Delay, event-triggered control, networked control systems, packet loss, stability analysis.

1 Introduction

In the past decades there has been a widespread attention in networked control systems (NCSs), in which the control loop is closed over a digital communication network, see [1, 2]. Compared to the traditional control systems, NCSs offer enormous benefits with respect to lower costs, simplified installation and maintenance.

However, the communication network has a considerable influence on the control performance since its load affects the quality of service by inducing communication delays or packet losses, which may degrade the system performance and may even cause instability of the system.

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The essential reason for the problems is the limitation of network communication resources. Actually, we should make efforts to use as few transmissions as possible to ensure a satisfactory performance of the system. To do so, a technique named event-triggered control (ETC) was proposed, see [3, 4]. "Event-triggered" means that measurements are transmitted not periodically any more, but only when an event generated by some well-designed event-triggered conditions occurs. Thus, by designing a proper event-triggered condition, we can decrease the amount of transmissions without making too much influence on the performance of the system.

In some results about ETC, such as [5], the event-triggered conditions are monitored continuously, which is actually inapplicable due to the limitation of the hardware. While in [6, 7], a so-called periodically event-triggered control (PETC) strategy was proposed where the event-triggered conditions are only verified periodically, right at the sampling instant.

Although the amount of transmissions can be decreased by using ETC, some network-induced problems would happen inevitably, such as delay and packet loss, which can degrade the control performance significantly and even lead to instability. The delay and packet loss also make obstacles for stability analysis of the NCSs. However, only a few works about stabilizing the system consider ETC, time-varying delay and packet loss at the same time, as shown below.

In [7], a delay system method was proposed to design event-triggered controllers of NCSs. [8] studied quantized control for stochastic Markov jump systems with interval time-varying delays and bounded system noise under the event-triggered mechanism. [9] formulated the NCSs with time delay and uncertainty into an aperiodic sampled data system. [10] investigated eventtriggered output feedback $\mathcal{H}_\infty$ control for a networked control system with transmission delays. Besides, [11] proposed a predictor-based controller combined with event-triggering mechanisms in order to control an LTI system with large input and output delays. Some other works, disposed the continuous system with time-varying delay by discretization and used polytopic over-approximation methods to handle the uncertainty, see [12–14].

In [15], the authors proposed a structure of NCSs with a communication logic, which incorporates model-based NCSs, predictive control, and an event-triggered communication scheme into a unified framework to consider the packet loss. [16] focused on a special Lyapunov-based ETC design with consideration of packet dropouts. [17] investigated the event-triggered predictive control problem for networked nonlinear systems where the data dropout induced by the networks can be actively compensated by a fuzzy predictive controller. Besides, [18] investigated an NCS whose sensor can choose different power levels at which it can transmit its measurement to the controller. The level of transmission power determines the probability of packet loss. The objective of this study is to find an appropriate transmission power probability distribution and a system controller jointly such that NCSs can be exponentially stabilized within a given energy budget. Moreover, [19] provided a probabilistic characterization for the packet losses and investigated almost sure stabilization under an event-triggered control law. The closed-loop system with ETC and packet dropouts can be rewritten as a switched system, switching between the normal transmission and the communication dropout.

In [20], a distributed event-triggered NCS with communication delay and packet loss was studied. However it dealt with the normal ETC in which the event-triggered conditions are monitored continuously. Furthermore, an improved event-triggered communication scheme and a sampled-state-error dependent tracking model for NCSs with communication delay and packet loss were presented in [21].

The purpose of this paper is to analyse the stability of an NCS with PETC strategy, time-varying delay and packet loss. Instead of using a zero-order hold for the control input generation, the input signal is determined by a dynamic model of the continuous-time state-feedback loop. The system is discretized into a piecewise linear model with exponential uncertainty caused by the time-varying delay. Then a newly proposed method of polytopic over-approximation in our previous work [14] is used to make the model suitable for stability analysis. The packet loss is handled by means of a switched system approach, see [19]. Finally we obtain LMIs conditions for the globally asymptotic stability (GAS) of the system.

The rest of the paper is as follows. In Section 2, description of an NCS with PETC, time-varying delay and packet loss is given. In Section 3, we discretize the system into a piecewise linear model and give the main points of the polytopic over-approximation method to deal with the uncertainty. Section 4 gives the stability conditions in terms of LMIs for GAS of the system. Section 5 uses a numerical example to illustrate the results. Conclusions are presented in Section 6.

2 System Description

In this paper, we study a periodically event-triggered NCS with time-varying delay and packet loss. The structure of the system is shown in Figure 1.

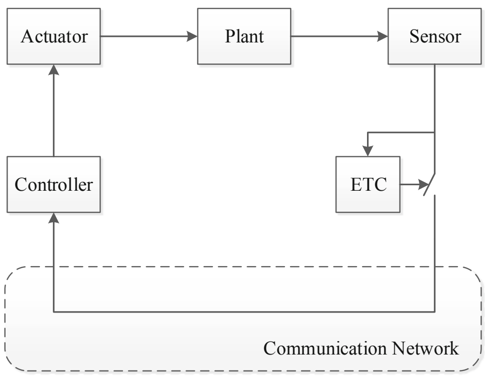

Figure 1 Structure of the NCS

A linear continuous-time plant is considered

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $A \in \mathbb{R}^{n_p \times n_p}$ and $B \in \mathbb{R}^{n_p \times m_p}$ are system matrices. It is controlled through a communication network.

The state is measured periodically by sensor at each sampling time nh, $n \in \mathbb{N}$, for some properly chosen sampling interval $h > 0$. By means of PETC, the state measurements are transmitted over the network only when a certain event-triggered condition is satisfied. We define the kth sampling time that triggers a transmission by t_k. Clearly, t_k should be an integral multiple of h. Suppose $t_k = n_k h$, $n_k \in \mathbb{N}$.

In this system, the data is transmitted with a chance of failure caused by the packet loss. In this paper the packet loss only in the communication path from sensor to controller is considered. We model the packet loss by the dynamics of its total number in the whole process of the system, which is presented by $\{L(k) \in \mathbb{N}\}$, $k \in \mathbb{N}$. $L(k)$ denotes the total number of the packet loss among the time $t \in [0, t_{k-1}]$ and is calculated by

$$L(k) = \sum_{i=0}^{k-1} l(i), \tag{2}$$

where $l(i) \in \{0, 1\}$ denotes whether the transmission at trigger time t_i is successful or not. In detail, $l(i) = 0$ stands for success, while $l(i) = 1$ indicates failure. About the packet loss, we make an assumption as follow.

Assumption 2.1 There exists a real number $\rho \in [0, 1]$ satisfying

$$\sum_{k=1}^{\infty} P[L(k) > \rho k] < \infty. \tag{3}$$

If the data is transmitted successfully, time-varying communication delay is involved. In this paper, we only consider delays in the feedback path from sensor to controller. About the system delay another assumption is made.

Assumption 2.2 For any $k \in \mathbb{N}$, $\tau_k \in [\tau_{\min}, \tau_{\max}]$ with $0 \le \tau_{\min} < \tau_{\max} < h$.

The time schedule of the network transmission is shown in Figure 2. The upper circles stand for the sensor side at each sampling time, and the lower ones present the controller side. Circles in gray show that the event trigger condition is satisfied and therefore transmission occurs at this point. Arrows from top to bottom show the successful transmissions with some delay, while at t_{k+1} the transmission is failed.

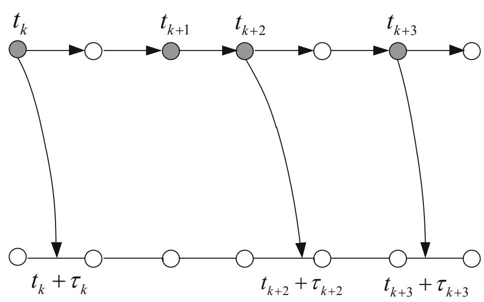

Figure 2 Time schedule of the NCS

The state-feedback controller of the system can be described as

$$u(t) = K\widehat{x}(t), \tag{4}$$

where $K \in \mathbb{R}^{m_p \times n_p}$ and $\widehat{x}(t)$ is a right-continuous signal, given by

$$\begin{aligned} &\dot{\widehat{x}}(t) = (A + BK)\widehat{x}(t), \\ &\widehat{x}(t_k + \tau_k) = x(t_k) \text{ if } l(k) = 0. \end{aligned} \tag{5}$$

At triggering time t_k, only if $l(k) = 0$ which means no packet loss occurs, the measured state of plant $x(t_k)$ is transmitted to the controller through the network and $\widehat{x}$ is then updated at time $t_k + \tau_k$. Otherwise, $\widehat{x}$ keeps following its own dynamic rule, as shown in (5). In this dynamic model, $\widehat{x}$ can be seen as the state of the controller.

According to the periodic event-triggered strategy, the triggering times t_k, $k \in \mathbb{N}$, are determined by the event-triggered condition. In this paper, we focus on an important class of event-triggered conditions, which has been applied in [22], given by

$$\|\widehat{x}(t_k) - x(t_k)\| > \sigma\|x(t_k)\|, \tag{6}$$

where $\sigma > 0$.

Define $\xi := [x^{\mathrm{T}} \quad \widehat{x}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n_\xi}$. Clearly, (6) is equivalent to a quadratic event-triggered condition, i.e.,

$$\xi^{\mathrm{T}}(t_k)Q\xi(t_k) > 0, \tag{7}$$

where

$$Q = \begin{bmatrix} (1-\sigma^2)I & -I \\ -I & I \end{bmatrix}. \tag{8}$$

Also, a simple condition is adopted to make sure the interval between two adjacent triggering times is not bigger than a certain value θh, $\theta \in \mathbb{N}^+$.

Thus, the triggering time t_k can be described as

$$t_{k+1} := \min\{t = nh, n > n_k | \xi^{\mathrm{T}}(t)Q\xi(t) > 0 \vee n - n_k \geq \theta\}. \tag{9}$$

Here, $n \in \mathbb{N}$, $n_k = t_k/h \in \mathbb{N}$ and we suppose $n_0 = 0$ and $t_0 = n_0 h = 0$.

3 System Formulation

3.1 System Discretization

In order to obtain a condition for globally asymptotic stability (GAS) of the system, a piecewise linear system (PLS) approach is applied. This approach is based on a discrete-time PLS model which is obtained by discretizing the system at sampling times nh, $n \in \mathbb{N}$. Define discrete state $x_n := x(nh)$, $\widehat{x}_n := \widehat{x}(nh)$ and $\xi_n := [x_n^{\mathrm{T}} \widehat{x}_n^{\mathrm{T}}]^{\mathrm{T}}$.

If $l(k) = 1$, which means the measurement supposed to be transmitted is lost, according to (5), the controller state $\widehat{x}$ follows its dynamic rule for $t \in [t_k,\ t_{k+1})$,

$$\dot{\widehat{x}}(t) = (A + BK)\widehat{x}(t). \tag{10}$$

Thus the discretized controller state is

$$\widehat{x}_{n+1} = \mathrm{e}^{(A+BK)h}\widehat{x}_n, \tag{11}$$

for $n \in [n_k, n_{k+1}) \cap \mathbb{N}$. Next from (1) and (4), we have

$$\dot{x}(t) = Ax(t) + BK\widehat{x}(t), \tag{12}$$

for $t \in [t_k,\ t_{k+1})$. Then from (10) and (12), it can be inferred that for $n \in [n_k, n_{k+1}) \cap \mathbb{N}$,

$$x_{n+1} = \mathrm{e}^{Ah}x_n + \int_0^h \mathrm{e}^{As}BK\widehat{x}((n+1)h - s)ds, \tag{13}$$

$$\widehat{x}_{n+1} = \mathrm{e}^{Ah}\widehat{x}_n + \int_0^h \mathrm{e}^{As}BK\widehat{x}((n+1)h - s)ds. \tag{14}$$

Therefore,

$$x_{n+1} - \mathrm{e}^{Ah}x_n = \widehat{x}_{n+1} - \mathrm{e}^{Ah}\widehat{x}_n. \tag{15}$$

Thus, the discretized plant state is

$$x_{n+1} = \mathrm{e}^{Ah}x_n + (\mathrm{e}^{(A+BK)h} - \mathrm{e}^{Ah})\widehat{x}_n \tag{16}$$

for $n \in [n_k, n_{k+1}) \cap \mathbb{N}$.

If $l(k) = 0$, which means no packet loss occurs, the controller refresh its state $\widehat{x}$ at the time $t = t_k + \tau_k$. After that it still follows the dynamic rule (10) for $t \in (t_k + \tau_k, t_{k+1})$. From Assumption 2.2, we know $t_k + \tau_k < t_k + h$. So the discretized controller state should be

$$\widehat{x}_{n+1} = \begin{cases} \mathrm{e}^{(A+BK)(h-\tau_k)}x_n, & \text{if } n = n_k, \\ \mathrm{e}^{(A+BK)h}\widehat{x}_n, & \text{if } n \in [n_k + 1, n_{k+1}) \cap \mathbb{N}. \end{cases} \tag{17}$$

As for the plant state x, (12) still holds for $t \in [t_k,\ t_{k+1})$. Therefore, similar to (15), it can be obtained that

$$x(t_k + \tau_k) - \mathrm{e}^{A\tau_k}x(t_k) = \widehat{x}(t_k + \tau_k) - \mathrm{e}^{A\tau_k}\widehat{x}(t_k), \tag{18}$$

$$x(t_k + h) - \mathrm{e}^{A(h-\tau_k)}x(t_k + \tau_k) = \widehat{x}(t_k + h) - \mathrm{e}^{A(h-\tau_k)}\widehat{x}(t_k + \tau_k), \tag{19}$$

and

$$x(t_k+(i+1)h)-\mathrm{e}^{Ah}x(t_k+ih)=\widehat{x}(t_k+(i+1)h)-\mathrm{e}^{Ah}\widehat{x}(t_k+ih), \tag{20}$$

for $i\in(0,n_{k+1}-n_k)\cap\mathbb{N}$. Thus, the discretized state of plant is

$$x_{n+1}=\begin{cases}(\mathrm{e}^{Ah}+\mathrm{e}^{(A+BK)(h-\tau_k)}-\mathrm{e}^{A(h-\tau_k)})x_n+(\mathrm{e}^{Ah+BK\tau_k}-\mathrm{e}^{Ah})\widehat{x}_n, & \text{if } n=n_k,\\ \mathrm{e}^{Ah}x_n+(\mathrm{e}^{(A+BK)h}-\mathrm{e}^{Ah})\widehat{x}_n, & \text{if } n\in[n_k+1,n_{k+1})\cap\mathbb{N}.\end{cases} \tag{21}$$

From (11), (16), (17) and (21), the discretized lifted state ξ_n, $n\in[n_k,n_{k+1})\cap\mathbb{N}$ can be described by

$$\xi_{n+1}=\begin{cases}A_0(\tau_k)\xi_n, & \text{if } n=n_k\wedge l(k)=0,\\ A_1\xi_n, & \text{otherwise},\end{cases} \tag{22}$$

where

$$A_0(\tau_k)=\begin{bmatrix}\mathrm{e}^{Ah}+\mathrm{e}^{(A+BK)(h-\tau_k)}-\mathrm{e}^{A(h-\tau_k)} & \mathrm{e}^{Ah+BK\tau_k}-\mathrm{e}^{Ah}\\ \mathrm{e}^{(A+BK)(h-\tau_k)} & 0\end{bmatrix} \tag{23}$$

and

$$A_1=\begin{bmatrix}\mathrm{e}^{Ah} & \mathrm{e}^{(A+BK)h}-\mathrm{e}^{Ah}\\ 0 & \mathrm{e}^{(A+BK)h}\end{bmatrix}. \tag{24}$$

Therefore, the PLS model (22) is essentially an uncertain system with the uncertain parameter $\tau_k\in[\tau_{\min},\tau_{\max}]$. It can be noticed that the uncertainty appears in an exponential fashion in $A_0(\tau_k)$, which is an obstacle to apply existing robust stability analysis techniques directly. A polytopic over-approximation method is used to deal with the uncertainty.

3.2 Polytopic Over-Approximation

The exponential uncertainty in (22) can be entirely represented by the matrix $A_0(\tau)$ satisfying $A_0(\tau)\in\boldsymbol{A_0}$, with

$$\boldsymbol{A_0}:=\{A_0(\tau)\mid\tau\in[\tau_{\min},\tau_{\max}]\}. \tag{25}$$

To perform the robust stability analysis we approximate the set of matrices in (25) in a fashion of polytope as

$$\boldsymbol{A_0}\subseteq\left\{\sum_{i=1}^{N}\alpha_iA_{0i}\;\middle|\;\alpha=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_N\end{bmatrix}\in\mathcal{A}\right\}, \tag{26}$$

where $A_{0i} \in \mathbb{R}^{n\times n}$, $i \in [1, N] \cap \mathbb{N}$ are suitably constructed matrices, $N \in \mathbb{N}$ denotes the number of vertices in the polytopic over-approximation and

$$\mathcal{A} = \left\{ \alpha \in \mathbb{R}^N \,\middle|\, \alpha_i \geq 0,\ i = 1, 2, \cdots, N \text{ and } \sum_{i=1}^{N} \alpha_i = 1 \right\}. \tag{27}$$

In the literature, different ways of constructing such polytopic embeddings of the uncertain system as in (26) were proposed, for example, the Jordan form method[12, 23, 24], the Cayley-Hamilton method[25] and the Taylor series method[13]. These methods are evaluated in two criteria: Computing complexity (related to N) and conservatism (related to the approximation error), see [26]. In this paper, we use a new method of polytopic over-approximation which performs relatively better in both two criteria, see our work [14]. The approach to get A_{0i}, $i \in [1, N] \cap \mathbb{N}$ can be seen in [14] and is omitted here.

According to (26) and $A_0(\tau_k) \in \boldsymbol{A_0}$, $k \in \mathbb{N}$, we know that there exists $\alpha^{(k)} \in \mathcal{A}$ satisfying

$$A_0(\tau_k) = \sum_{i=1}^{N} \alpha_i^{(k)} A_{0i}, \tag{28}$$

for $k \in \mathbb{N}$.

Consequently the PLS model (22) can be written as

$$\xi_{n+1} = \begin{cases} \sum_{i=1}^{N} \alpha_i^{(k)} A_{0i}\xi_n, & \text{if } n = n_k \wedge l(k) = 0, \\ A_1\xi_n, & \text{otherwise.} \end{cases} \tag{29}$$

4 Stability Analysis

In this section we use a piecewise quadratic Lyapunov function of the form

$$V(\xi_n) = \begin{cases} \xi_n^{\mathrm{T}} P_0 \xi_n, & \text{if } n = n_k, \\ \xi_n^{\mathrm{T}} P_1 \xi_n, & \text{if } n \in (n_k, n_{k+1}) \cap \mathbb{N}, \end{cases} \tag{30}$$

where $P_0,\ P_1 \in \mathbb{R}^{2n_p\times 2n_p}$. Stability conditions are presented in the theorem below.

Theorem 4.1 *The NCS described by* (1), (4), (5) *and* (9) *is GAS if there exist matrices* $P_0,\ P_1 \in \mathbb{R}^{2n_p\times 2n_p}$, *and scalars* $\phi > 1$, $\beta \in (0, 1)$, $k_1 \geq 0$, $k_2 \geq 0$, *and* $a_s \geq 0$, $b_s \geq 0$, $s \in \{1, 2, \cdots, 6\}$ *such that*

$$P_0 - k_1 Q > 0, \tag{31a}$$

$$P_1 + k_2 Q > 0, \tag{31b}$$

$$A_{0i}^{\mathrm{T}}(P_0 + a_1 Q)A_{0i} < \phi^{\theta-1}\beta P_0 - b_1 Q, \tag{32a}$$
$$A_1^{\mathrm{T}}(P_0 + a_2 Q)A_1 < \phi^{\theta} P_0 - b_2 Q, \tag{32b}$$
$$A_{0i}^{\mathrm{T}}(P_1 - a_3 Q)A_{0i} < \beta P_0 - b_3 Q, \tag{32c}$$
$$A_1^{\mathrm{T}}(P_1 - a_4 Q)A_1 < \phi P_0 - b_4 Q, \tag{32d}$$
$$A_1^{\mathrm{T}}(P_1 - a_5 Q)A_1 < \phi P_1 + b_5 Q, \tag{32e}$$
$$A_1^{\mathrm{T}}(P_0 + a_6 Q)A_1 < \phi P_1 + b_6 Q, \tag{32f}$$

for $i \in \{1, 2, \cdots, N\}$, *and*

$$(1-\rho)\ln\beta + (\theta - 1 + \rho)\ln\phi < 0. \tag{33}$$

Proof Firstly, we should prove $V(\xi_n)$ given by (30) can serve as a Lyapunov function. Given the event-triggered condition (9), when $n \in \{n_k | k \in \mathbb{N}\}$, we have $\xi_n^{\mathrm{T}} Q \xi_n > 0$, then

$$V(\xi_n) = \xi_n^{\mathrm{T}} P_0 \xi_n > \xi_n^{\mathrm{T}}(P_0 - k_1 Q)\xi_n > 0. \tag{34}$$

When $n \notin \{n_k | k \in \mathbb{N}\}$, we have $\xi_n^{\mathrm{T}} Q \xi_n \le 0$, then

$$V(\xi_n) = \xi_n^{\mathrm{T}} P_1 \xi_n > \xi_n^{\mathrm{T}}(P_1 + k_2 Q)\xi_n > 0. \tag{35}$$

Thus, $V(\xi_n)$ is always positive and can be used as a Lyapunov function.

Next we consider possible situations in the process for $n \in [n_k, n_{k+1}) \cap \mathbb{N}$.

When $n = n_k$, $n + 1 = n_{k+1}$, we have $\xi_n^{\mathrm{T}} Q \xi_n > 0$ and $\xi_{n+1}^{\mathrm{T}} Q \xi_{n+1} > 0$. So if $l(k) = 0$,

$$\begin{aligned} V(\xi_{n+1}) &= \xi_{n+1}^{\mathrm{T}} P_0 \xi_{n+1} \\ &< \xi_{n+1}^{\mathrm{T}}(P_0 + a_1 Q)\xi_{n+1} \\ &= \xi_n^{\mathrm{T}} \bigg(\sum_{i=1}^{N} \alpha_i^{(k)} A_{0i}^{\mathrm{T}}\bigg)(P_0 + a_1 Q)\bigg(\sum_{i=1}^{N} \alpha_i^{(k)} A_{0i}\bigg)\xi_n \\ &< \xi_n^{\mathrm{T}}(\phi^{\theta-1}\beta P_0 - b_1 Q)\xi_n \\ &< \phi^{\theta-1}\beta \xi_n^{\mathrm{T}} P_0 \xi_n \\ &= \phi^{\theta-1}\beta V(\xi_n), \end{aligned} \tag{36}$$

and if $l(k) = 1$,

$$\begin{aligned} V(\xi_{n+1}) &= \xi_{n+1}^{\mathrm{T}} P_0 \xi_{n+1} \\ &< \xi_{n+1}^{\mathrm{T}}(P_0 + a_2 Q)\xi_{n+1} \\ &= \xi_n^{\mathrm{T}} A_1^{\mathrm{T}}(P_0 + a_2 Q)A_1 \xi_n \\ &< \xi_n^{\mathrm{T}}(\phi^{\theta} P_0 - b_2 Q)\xi_n \\ &< \phi^{\theta} \xi_n^{\mathrm{T}} P_0 \xi_n \\ &= \phi^{\theta} V(\xi_n). \end{aligned} \tag{37}$$

When $n = n_k$, $n+1 \neq n_{k+1}$, we have $\xi_n^{\mathrm{T}} Q \xi_n > 0$ and $\xi_{n+1}^{\mathrm{T}} Q \xi_{n+1} \leq 0$. So if $l(k) = 0$,

$$\begin{aligned} V(\xi_{n+1}) &= \xi_{n+1}^{\mathrm{T}} P_1 \xi_{n+1} \\ &\leq \xi_{n+1}^{\mathrm{T}} (P_1 - a_3 Q) \xi_{n+1} \\ &= \xi_n^{\mathrm{T}} \bigg(\sum_{i=1}^{N} \alpha_i^{(k)} A_{0i}^{\mathrm{T}} \bigg) (P_1 - a_3 Q) \bigg(\sum_{i=1}^{N} \alpha_i^{(k)} A_{0i} \bigg) \xi_n \\ &< \xi_n^{\mathrm{T}} (\beta P_0 - b_3 Q) \xi_n \\ &< \beta \xi_n^{\mathrm{T}} P_0 \xi_n \\ &= \beta V(\xi_n), \end{aligned} \tag{38}$$

and if $l(k) = 1$,

$$\begin{aligned} V(\xi_{n+1}) &= \xi_{n+1}^{\mathrm{T}} P_1 \xi_{n+1} \\ &\leq \xi_{n+1}^{\mathrm{T}} (P_1 - a_4 Q) \xi_{n+1} \\ &= \xi_n^{\mathrm{T}} A_1^{\mathrm{T}} (P_1 - a_4 Q) A_1 \xi_n \\ &< \xi_n^{\mathrm{T}} (\phi P_0 - b_4 Q) \xi_n \\ &< \phi \xi_n^{\mathrm{T}} P_0 \xi_n \\ &= \phi V(\xi_n). \end{aligned} \tag{39}$$

When $n \neq n_k$, $n+1 \neq n_{k+1}$, we have $\xi_n^{\mathrm{T}} Q \xi_n \leq 0$ and $\xi_{n+1}^{\mathrm{T}} Q \xi_{n+1} \leq 0$. So

$$\begin{aligned} V(\xi_{n+1}) &= \xi_{n+1}^{\mathrm{T}} P_1 \xi_{n+1} \\ &\leq \xi_{n+1}^{\mathrm{T}} (P_1 - a_5 Q) \xi_{n+1} \\ &= \xi_n^{\mathrm{T}} A_1^{\mathrm{T}} (P_1 - a_5 Q) A_1 \xi_n \\ &< \xi_n^{\mathrm{T}} (\phi P_1 + b_6 Q) \xi_n \\ &\leq \phi \xi_n^{\mathrm{T}} P_1 \xi_n \\ &= \phi V(\xi_n). \end{aligned} \tag{40}$$

When $n \neq n_k$, $n+1 = n_{k+1}$, we have $\xi_n^{\mathrm{T}} Q \xi_n \leq 0$ and $\xi_{n+1}^{\mathrm{T}} Q \xi_{n+1} > 0$. So

$$\begin{aligned} V(\xi_{n+1}) &= \xi_{n+1}^{\mathrm{T}} P_0 \xi_{n+1} \\ &< \xi_{n+1}^{\mathrm{T}} (P_0 + a_6 Q) \xi_{n+1} \\ &= \xi_n^{\mathrm{T}} A_1^{\mathrm{T}} (P_0 + a_6 Q) A_1 \xi_n \\ &< \xi_n^{\mathrm{T}} (\phi P_1 + b_6 Q) \xi_n \\ &\leq \phi \xi_n^{\mathrm{T}} P_1 \xi_n \\ &= \phi V(\xi_n). \end{aligned} \tag{41}$$

Since θ is the largest number of samples between two trigger times, we have $n_{k+1} - n_k \leq \theta$. Above all, it can be inferred that if $l(k) = 0$,

$$V(\xi_{n_{k+1}}) < \phi^{\theta-1} \beta V(\xi_{n_k}), \tag{42}$$

and if $l(k) = 1$

$$V(\xi_{n_{k+1}}) < \phi^{\theta} V(\xi_{n_k}). \tag{43}$$

Combining (42) with (43) yields

$$V(\xi_{n_{k+1}}) < ((1 - l(k))\phi^{\theta-1}\beta + l(k)\phi^{\theta})V(\xi_{n_k}). \tag{44}$$

Thus, we have

$$V(\xi_{n_k}) < \eta(k)V(\xi_0), \tag{45}$$

where

$$\eta(k) = \prod_{i=0}^{k-1}((1 - l(i))\phi^{\theta-1}\beta + l(i)\phi^{\theta}). \tag{46}$$

On the one hand, from Assumption 2.1, we know there exists a real number $\rho \in [0, 1]$ such that (3) holds. According to [19], it can be proved that

$$\lim_{k\to\infty} \sup \frac{L(k)}{k} \le \rho. \tag{47}$$

The proof can be seen in [19] and is omitted.

On the other hand, from the equation

$$\ln((1 - l)\beta + l\phi) = (1 - l) \ln \beta + l \ln \phi, \tag{48}$$

for $l \in \{0, 1\}$, it can be inferred that

$$\begin{aligned}
\ln \eta(k) &= \sum_{i=0}^{k=1} \ln((1 - l(i))\phi^{\theta-1}\beta + l(i)\phi^{\theta}) \\
&= \sum_{i=0}^{k-1}(1 - l(i))((\theta - 1) \ln \phi + \ln \beta) + \sum_{i=0}^{k-1} l(i)\theta \ln \phi \\
&= (k - L(k))((\theta - 1) \ln \phi + \ln \beta) + L(k)\theta \ln \phi \\
&= (k - L(k)) \ln \beta + (k\theta - k + L(k)) \ln \phi.
\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty} \sup \frac{\ln \eta(k)}{k} &= \lim_{k\to\infty} \sup \left(\left(1 - \frac{L(k)}{k}\right) \ln \beta + \left(\theta - 1 + \frac{L(k)}{k}\right) \ln \phi\right) \\
&\le (1 - \rho) \ln \beta + (\theta - 1 + \rho) \ln \phi.
\end{aligned} \tag{50}$$

From (33), we know $\lim_{k\to\infty} \sup \frac{\ln \eta(k)}{k} < 0$ holds. That is to say, $\lim_{k\to\infty} \sup \ln \eta(k) = -\infty$, and consequently, $\lim_{k\to\infty} \sup \eta(k) = 0$.

Finally, according to (45), it can proved that $\lim_{k\to\infty} V(\xi_{n_k}) = 0$, which implies GAS of the system. ∎

5 Numerical Example

In this section, the presented method is illustrated by a numerical example based on a state-feedback controller.

5.1 Trajectories of the System

The example is taken from [27] with plant (1) given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{51}$$

and the state-feedback controller (4), where $K = [1\ -4]$. Here we take $h = 0.2$, $\tau_{\min} = 0$, $\tau_{\max} = 0.1$ and $\rho = 0.2$. The event-triggered condition is given by (9) with $\sigma = 0.1$ and $\theta = 6$.

Firstly, by using the polytopic over-approximation method presented in [14], we can get A_{0i}, $i \in [1, N] \cap \mathbb{N}$ that satisfy (26) as follows. Because $n_p = 2$ in this example, we have $N = n_p + 1 = 3$.

$$A_{01} = \begin{bmatrix} 0.9882 & 0.1215 & 0 & 0 \\ -0.2982 & 0.7918 & 0 & 0 \\ 0.9882 & 0.1215 & 0 & 0 \\ -0.2982 & 0.7918 & 0 & 0 \end{bmatrix}, \tag{52a}$$

$$A_{02} = \begin{bmatrix} 1.4244 & 0.1897 & -0.3135 & -0.0891 \\ -0.4540 & 1.2365 & 0.1783 & -0.5809 \\ 1.3339 & 0.1640 & 0 & 0 \\ -0.4025 & 1.0688 & 0 & 0 \end{bmatrix}, \tag{52b}$$

$$A_{03} = \begin{bmatrix} 0.9617 & 0.1600 & 0.0222 & -0.0891 \\ -0.5420 & 1.3283 & 0.1452 & -0.5809 \\ 0.9997 & 0.0058 & 0 & 0 \\ -0.2336 & 0.9037 & 0 & 0 \end{bmatrix}. \tag{52c}$$

Next choose β and ϕ properly to make (33) hold. For a given β, take

$$\phi = \mathrm{e}^{\frac{-0.01-(1-\rho)\ln\beta}{\theta+\rho-1}}. \tag{53}$$

Thus, we can make a rough traversal for $\beta \in (0, 1)$. For example, choose β arbitrarily from $\{0.01, 0.02, \cdots, 0.99\}$.

Finally with A_{0i}, β and ϕ determined, we use the LMIs toolbox in Matlab to check if the LMIs conditions in Theorem 4.1, i.e., (31) and (32) are satisfied. For this given example, a solution can be obtained which means the system is GAS.

Suppose that the original states are $x(0) = [1\ -3]^{\mathrm{T}}$ and $\widehat{x}(0) = [0\ 0]^{\mathrm{T}}$. Figure 3 shows the trajectories of x and $\widehat{x}$, and Figure 4 gives the random delay and packet loss. Note that when the measurement is not transmitted there are no delay and packet loss.

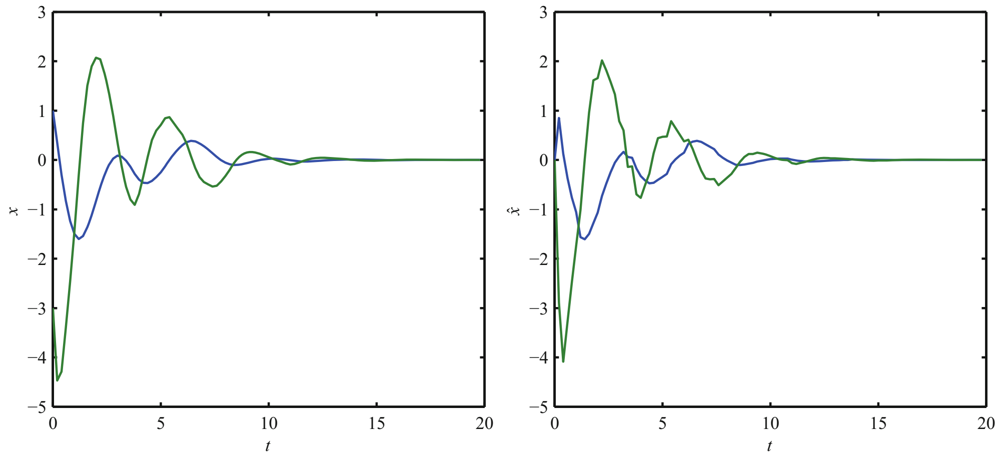

Figure 3 Trajectories of x and $\widehat{x}$

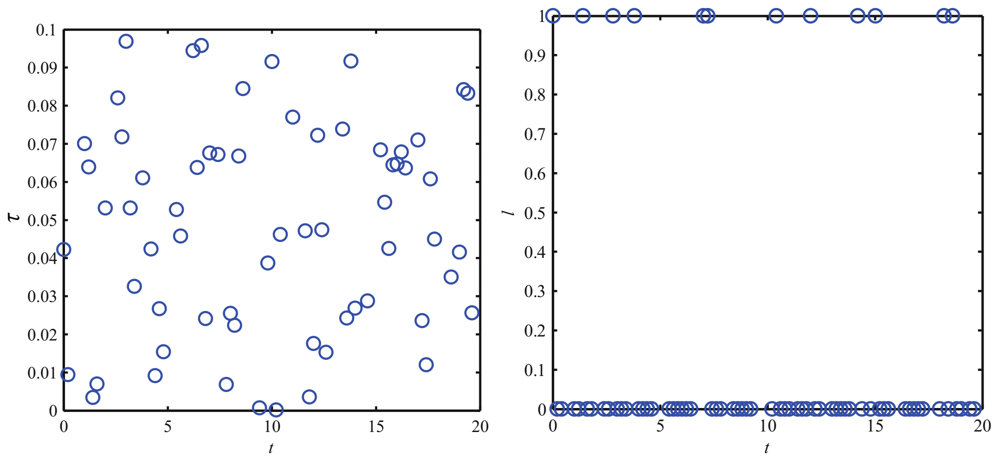

Figure 4 The random delay and packet loss

5.2 Stability Region

In this part, we employ the same example, however, make $\tau_{\max}$, ρ and σ be undetermined parameters.

On one hand, suppose $\rho = 0.2$, a stability region can be drawn by $\tau_{\max}$ and σ that make sure the GAS of the system according to Theorem 4.1, see Figure 5. The region is below the line in the picture. On the other hand, suppose $\tau_{\max} = 0.1$, similarly a stability region constructed by ρ and σ is presented in Figure 6.

It can be seen that the parameters given by former part is inside the regions, which implies the GAS of the system. In fact, some possible values of the parameters outside the regions also make the system stable, due to the conservativeness of the derived results.

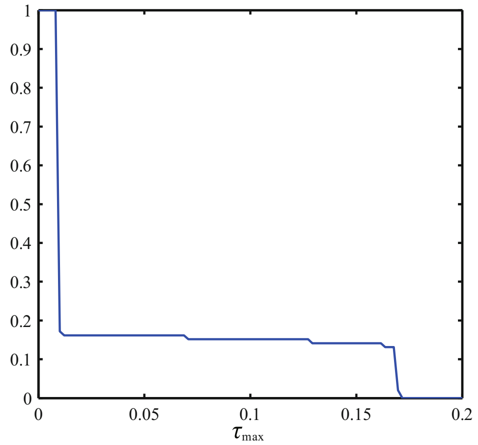

Figure 5 The stability region when $\rho = 0.2$

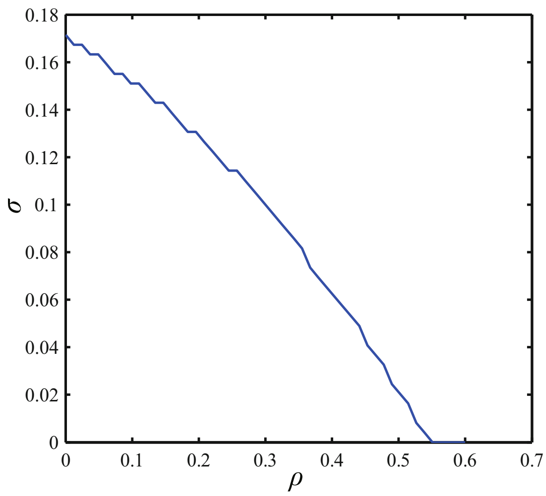

Figure 6 The stability region when $\tau_{\max} = 0.1$

6 Conclusion

In this paper, we focus on stability analysis of an NCS with PETC considering both time-varying delay and packet loss. Instead of using the tradition zero-hold fashion, the controller input follows its own dynamic rule when there is no measurement arrived. The system is discretized and formulated into a piecewise linear model with uncertainty caused by delay. Then the model is embedded in a polytopic over-approximation with a better structure suitable for stability analysis. A switched system method is used to deal with the stochastic packet loss. Finally LMIs stability conditions are obtained. A numerical example shows the effectiveness of our method for stability analysis.

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