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Bootstrap Inference on the Variance Component Functions in the Two-Way Random Effects Model with Interaction*

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Abstract In this paper, using the Bootstrap approach and generalized approach, the authors consider the one-sided hypothesis testing problems for variance component functions in the two-way random effects model. Firstly, the test statistics and confidence intervals for the sum of variance components are constructed. Next, the one-sided hypothesis testing problems for the ratio of variance components are also discussed. The Monte Carlo simulation results indicate that the Bootstrap approach is better than the generalized approach in most cases. Finally, the above approaches are applied to the real data examples of mice blood pH and molded plastic part's dimensions.

Keywords Bootstrap, generalized approach, two-way random effects model, variance component function.

1 Introduction

The two-way random effects model is widely used in quality control, experimental design, biomedical research, econometric modelling, market analysis and many other practical fields. For example, Wang, et al.[1] studied the impact of the plate material and ambient temperature of a specific type battery on its maximum output voltage using the two-way random effects model. Thompson[2] used this model to analyse the fuze burning time data with a view of estimating

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the precisions of the instruments. Cheng and Shao[3] considered five clinical trials and focused on the treatment-by-center interaction among them to discuss whether the treatment effect is significant or not.

In view of the wide applications of variance component, research has been done about its parameter estimation problems. Some estimation methods has been established including analysis of variance, maximum likelihood, restricted maximum likelihood, spectral decomposition, and minimum norm quadratic unbiased estimation. See [4–10] for more details. These abovementioned methods, however, are adequate for single variance component in general, but not for direct application for statistical inference on variance component function. For this reason, Gilder, et al.[11] studied the properties on intraclass correlation coefficients based on the modified large sample approach and generalized approach in the balanced two-way random effects model. Further, Ye and Wang[12] extended the conclusions of Gilder, et al.[11] to unbalanced scenarios. Li[13] derived the generalized confidence interval for the ratio of variance components in the unbalanced two-fold nested designs. Recent studies indicate that, the generalized approach performs satisfactorily in controlling the Type I error probabilities in most scenarios, but lacks robustness with small sample size[14–16].

Since it is difficult to directly apply the traditional test approaches to construct exact test statistics, Efron[17] proposed the Bootstrap approach based on computer numerical algorithm. This approach has been widely used for statistical inference problems such as error estimation, hypothesis testing and interval estimation. Ma, et al.[18] studied homogeneous testing problems of inverse Gaussian means under heterogeneity, whose results showed that the Bootstrap approach is better than the generalized approach in large sample cases. Yue, et al.[19] constructed the Bootstrap test statistics of regression coefficients for two-way error component regression model. Ye and Jiang[20] applied the Bootstrap approach into panel data model and discussed the hypothesis testing problems of regression coefficients and variance components. In this paper, for one-sided hypothesis testing and interval estimation problems of variance component functions in the two-way random effects model, the Bootstrap approach and generalized approach are established, and the excellent statistical properties of the Bootstrap approach are verified by the Monte Carlo simulation.

This paper is organized as follows. In Section 2, the two-way random effects model is introduced. In Section 3, using the Bootstrap approach and generalized approach, the test statistics and pivot quantities for the sum of variance components are constructed. In Section 4, the one-sided hypothesis testing and interval estimation problems for the ratio of variance components are analyzed. In Section 5, the Monte Carlo simulation results are presented to verify the statistical excellent properties of the proposed approaches. In Section 6, the proposed approaches are applied to the real data examples of mice blood pH and molded plastic part's dimensions. In Section 7, the summary of this paper is given.

2 Preliminaries

In this paper, we consider the two-way random effects model with interaction

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \tag{1}$$

where $i = 1, 2, \cdots, a$, $j = 1, 2, \cdots, b$, $k = 1, 2, \cdots, c$, μ is the fixed effect, α_i, β_j and γ_{ij} are random effects, ε_{ijk} is the random error. Assume that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_j \sim N(0, \sigma_\beta^2)$, $\gamma_{ij} \sim N(0, \sigma_\gamma^2)$, $\varepsilon_{ijk} \sim N(0, \sigma_\varepsilon^2)$, and all random variables are mutually independent.

Denote $y = (y_{111}, y_{112}, \cdots, y_{11c}, \cdots, y_{ab1}, y_{ab2}, \cdots, y_{abc})'$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_a)'$, $\beta = (\beta_1, \beta_2, \cdots, \beta_b)'$, $\gamma = (\gamma_{11}, \gamma_{12} \cdots, \gamma_{ab})'$, $\varepsilon = (\varepsilon_{111}, \varepsilon_{112} \cdots, \varepsilon_{abc})'$. Besides, 1_T is a $T \times 1$ vector with every element unity, I_T is an identity matrix of order T. $\otimes$ denotes the Kronecker product, rank(A) is the rank of matrix A, and $n = abc$. The corresponding design matrices of random effects α_i, β_j and γ_{ij} are

$$Z_\alpha = I_a \otimes 1_b \otimes 1_c, \quad Z_\beta = 1_a \otimes I_b \otimes 1_c, \quad Z_\gamma = I_a \otimes I_b \otimes 1_c.$$

Then the model (1) can be rewritten in matrix form as

$$y = 1_n\mu + Z_\alpha\alpha + Z_\beta\beta + Z_\gamma\gamma + \varepsilon. \tag{2}$$

The covariance matrix of y is

$$\begin{aligned}\Sigma &= \mathrm{Cov}(y)\\ &= \sigma_\alpha^2 Z_\alpha Z_\alpha{}' + \sigma_\beta^2 Z_\beta Z_\beta{}' + \sigma_\gamma^2 Z_\gamma Z_\gamma{}' + \sigma_\varepsilon^2 I_n\\ &= \sigma_\alpha^2(I_a \otimes J_b \otimes J_c) + \sigma_\beta^2(J_a \otimes I_b \otimes J_c) + \sigma_\gamma^2(I_a \otimes I_b \otimes J_c) + \sigma_\varepsilon^2 I_n,\end{aligned}$$

where $J_a = 1_a 1_a'$. Define $\overline{J}_a = J_a/a$. It can be concluded that Σ is as follows after spectral decomposition

$$\Sigma = \sum_{i=1}^{4} \sigma_i^2 M_i + \sigma_0^2 \overline{J}_n, \tag{3}$$

where $\sigma_0^2 = bc\sigma_\alpha^2 + ac\sigma_\beta^2 + c\sigma_\gamma^2 + \sigma_\varepsilon^2$, $\sigma_1^2 = bc\sigma_\alpha^2 + c\sigma_\gamma^2 + \sigma_\varepsilon^2$, $\sigma_2^2 = ac\sigma_\beta^2 + c\sigma_\gamma^2 + \sigma_\varepsilon^2$, $\sigma_3^2 = c\sigma_\gamma^2 + \sigma_\varepsilon^2$, $\sigma_4{}^2 = \sigma_\varepsilon^2$. Accordingly, $M_1 = (I_a - \overline{J}_a) \otimes \overline{J}_b \otimes \overline{J}_c$, $M_2 = \overline{J}_a \otimes (I_b - \overline{J}_b) \otimes \overline{J}_c$, $M_3 = (I_a - \overline{J}_a) \otimes (I_b - \overline{J}_b) \otimes \overline{J}_c$, $M_4 = I_a \otimes I_b \otimes (I_c - \overline{J}_c)$.

Lemma 2.1 (a) M_i $(i = 1, 2, 3, 4)$ *and* $\overline{J}_n$ *are symmetric idempotent matrix and mutually orthogonal.*

(b) rank(M_i)=n_i(i= 1,2,3,4), rank($\overline{J}_n$)=1, *where* $n_1 = a-1$, $n_2 = b-1$, $n_3 = (a-1)(b-1)$, $n_4 = ab(c-1)$.

(c) $\sum_{i=1}^4 M_i + \overline{J}_n = I_n$.

From the definitions of M_i $(i = 1, 2, 3, 4)$ and $\overline{J}_n$, Lemma 2.1 is easy to be proved.

By premultipling (2) by M_i $(i = 1, 2, 3, 4)$, the transformed model has the form of

$$y_i = e_i, \quad E(e_i) = 0, \quad \mathrm{Cov}(e_i) = \sigma_i^2 M_i, \quad i = 1, 2, 3, 4, \tag{4}$$

where $y_i = M_i y$, $e_i = M_i e$, $e = Z_\alpha \alpha + Z_\beta \beta + Z_\gamma \gamma + \varepsilon$. Because M_i is a singular matrix, the model (4) is a singular linear model. By the unified theory of least square, we have

$$\widehat{\sigma}_i^2 = \frac{y' M_i y}{n_i}, \quad i=1,2,3,4,$$

where $n_i \geq 1$. According to the definition of χ^2 distribution, it is clear that

$$V_i = \frac{n_i \widehat{\sigma}_i^2}{{\sigma_i}^2} \sim \chi^2_{n_i}, \quad i = 1, 2, 3, 4,$$

and V_i $(i = 1, 2, 3, 4)$ are mutually independent.

3 Inference on the Sum of Variance Components

In practical applications, the sum of variance components is used to characterize the variability of several random effects, which is common in improving the accuracy of the gun tube. See [10, 21, 22] for more details. For this, the Bootstrap approach and generalized approach are applied into hypothesis testing problems for the sum of two variance components in Model (1). The hypotheses of interest are

$$H_0 : \sigma^2_\alpha + \sigma^2_\varepsilon \leq c_0 \ \text{ versus } \ H_1 : \sigma^2_\alpha + \sigma^2_\varepsilon > c_0, \tag{5}$$

$$H_0 : \sigma^2_\beta + \sigma^2_\varepsilon \leq c_0 \ \text{ versus } \ H_1 : \sigma^2_\beta + \sigma^2_\varepsilon > c_0, \tag{6}$$

$$H_0 : \sigma^2_\gamma + \sigma^2_\varepsilon \leq c_0 \ \text{ versus } \ H_1 : \sigma^2_\gamma + \sigma^2_\varepsilon > c_0, \tag{7}$$

where c_0 is a specified value.

3.1 The Bootstrap Approach

For the hypothesis testing problem (5), we have

$$T_1 = \frac{(bc)^{-1}(\widehat{\sigma}_1^2 - \widehat{\sigma}_3^2) + \widehat{\sigma}_4^2 - c_0}{\sqrt{(bc)^{-2}\left(\frac{2\sigma_1^4}{n_1} + \frac{2\sigma_3^4}{n_3}\right) + \frac{2\sigma_4^4}{n_4}}}. \tag{8}$$

If σ_1^2, σ_3^2 and σ_4^2 are known, T_1 in (8) will be the test statistic of hypothesis testing problem (5). However, σ_1^2, σ_3^2 and σ_4^2 are often unknown in practical application. In such cases, the parameters might be replaced by their estimators $\widehat{\sigma}_1^2$, $\widehat{\sigma}_3^2$ and $\widehat{\sigma}_4^2$. The test statistic is given by

$$T_1 = \frac{(bc)^{-1}(\widehat{\sigma}_1^2 - \widehat{\sigma}_3^2) + \widehat{\sigma}_4^2 - c_0}{\sqrt{(bc)^{-2}\left(\frac{2\widehat{\sigma}_1^4}{n_1} + \frac{2\widehat{\sigma}_3^4}{n_3}\right) + \frac{2\widehat{\sigma}_4^4}{n_4}}}. \tag{9}$$

It is difficult to obtain the exact distribution of T_1, then the Bootstrap approach is used to construct the test statistic. Thus, the Bootstrap test statistic based on (9) is expressed as

$$TB_1 = \frac{(bc)^{-1}(\widehat{\sigma}_{1B}^2 - \widehat{\sigma}_{3B}^2) + \widehat{\sigma}_{4B}^2 - c_0}{\sqrt{(bc)^{-2}\left(\frac{2\widehat{\sigma}_{1B}^4}{n_1} + \frac{2\widehat{\sigma}_{3B}^4}{n_3}\right) + \frac{2\widehat{\sigma}_{4B}^4}{n_4}}}, \tag{10}$$

where $\widehat{\sigma}_{1B}^2 \sim \frac{\widehat{\sigma}_1^2}{n_1}\chi_{n_1}^2$, $\widehat{\sigma}_{3B}^2 \sim \frac{\widehat{\sigma}_3^2}{n_3}\chi_{n_3}^2$, $\widehat{\sigma}_{4B}^2 \sim \frac{c_0-(bc)^{-1}(\widehat{\sigma}_1^2-\widehat{\sigma}_3^2)}{n_4}\chi_{n_4}^2$. Further, the Bootstrap p-value is computed by TB_1 in (10) as

$$p_1{=}P(TB_1 \geq t_1|H_0), \tag{11}$$

where t_1 denotes the observed value of T_1. The calculation of p_1 can be realized later by Algorithm 1 in Section 5. Let δ be the nominal significance level. The null hypothesis H_0 in (5) is rejected whenever the above p-value is less than the nominal significance level of δ.

Similarly, the Bootstrap test statistics for the hypothesis testing problems (6) and (7) are respectively defined as

$$TB_2{=}\frac{(ac)^{-1}(\widehat{\sigma}_{2B}^2-\widehat{\sigma}_{3B}^2)+\widehat{\sigma}_{4B}^2-c_0}{\sqrt{(ac)^{-2}\left(\frac{2\widehat{\sigma}_{2B}^4}{n_2}+\frac{2\widehat{\sigma}_{3B}^4}{n_3}\right)+\frac{2\widehat{\sigma}_{4B}^4}{n_4}}},$$

$$TB_3{=}\frac{(c)^{-1}(\widehat{\sigma}_{3B}^2-\widehat{\sigma}_{4B}^2)+\widehat{\sigma}_{4B}^2-c_0}{\sqrt{(c)^{-2}\left(\frac{2\widehat{\sigma}_{3B}^4}{n_3}+\frac{2\widehat{\sigma}_{4B}^4}{n_4}\right)+\frac{2\widehat{\sigma}_{4B}^4}{n_4}}},$$

where $\widehat{\sigma}_{2\mathrm{B}}^2 \sim \frac{\widehat{\sigma}_2^2}{n_2}\chi_{n_2}^2$. Then the Bootstrap p-values based on TB_2 and TB_3 are respectively computed as

$$p_2{=}P(TB_2 \geq t_2|H_0), \quad p_3{=}P(TB_3 \geq t_3|H_0).$$

Similar to t_1 in (11), t_2 and t_3 are observed values.

Remark 3.1 The Bootstrap pivot quantity of $\sigma_\alpha^2{+}\sigma_\varepsilon^2$ can be constructed as TB_1^* based on TB_1 (see [14]). Suppose that $TB_1^*(\eta)$ is the 100η empirical percentile of TB_1^*. Then the approximate $100(1-\delta)\%$ Bootstrap confidence interval for $\sigma_\alpha^2{+}\sigma_\varepsilon^2$ is given by

$$\left[(bc)^{-1}(s_1^2-s_3^2)+s_4^2-TB_1^*(1-\delta/2)\sqrt{(bc)^{-2}\left(\frac{2s_1^4}{n_1}+\frac{2s_3^4}{n_3}\right)+\frac{2s_4^4}{n_4}},\right.$$
$$\left.(bc)^{-1}(s_1^2-s_3^2)+s_4^2-TB_1^*(\delta/2)\sqrt{(bc)^{-2}\left(\frac{2s_1^4}{n_1}+\frac{2s_3^4}{n_3}\right)+\frac{2s_4^4}{n_4}}\right],$$

where s_i^2 is the observed value of $\widehat{\sigma}_i^2$ $(i=1,2,3,4)$. In the same way, the approximate $100(1-\delta)\%$ Bootstrap confidence intervals based on TB_2^* and TB_3^* for $\sigma_\beta^2{+}\sigma_\varepsilon^2$ and $\sigma_\gamma^2{+}\sigma_\varepsilon^2$ respectively are also available.

3.2 The Generalized Approach

Firstly, considering the hypothesis testing the problem (5), the generalized test variable has the form of

$$\begin{aligned}T_4 &= \frac{1}{bc}\left(\frac{\sigma_1^2 s_1^2}{\widehat{\sigma}_1^2}-\frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2}\right)+\frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2}-(\sigma_\alpha^2+\sigma_\varepsilon^2)\\ &= \frac{1}{bc}(n_1 s_1^2 V_1^{-1}-n_3 s_3^2 V_3^{-1})+n_4 s_4^2 V_4^{-1}-(\sigma_\alpha^2+\sigma_\varepsilon^2).\end{aligned} \tag{12}$$

It is apparent that t_4, the observed value of T_4, is free of any unknown parameters. Note that $V_i = n_i\widehat{\sigma}_i^2/{\sigma_i}^2 \sim \chi^2_{n_i}$ $(i = 1,3,4)$, thus the distribution of T_4 is free of the nuisance parameters. From the expression in (12), T_4 is stochastically decreasing in $\sigma_\alpha^2 + \sigma_\varepsilon^2$. In other words, T_4 is a generalized test variable. Then, based on T_4, the generalized p-value for the hypothesis testing problem (5) can be computed as

$$\begin{aligned} p_4 &= P(T_4 \le 0|H_0)\\ &= P(V_1 \ge n_1 s_1^2(n_3 s_3^2 V_3^{-1} - bcn_4 s_4^2 V_4^{-1} + bcc_0)^{-1})\\ &= 1 - E_{V_3,V_4}\left[F_{\chi^2_{n_1}}(n_1 s_1^2 (n_3 s_3^2 V_3^{-1} - bcn_4 s_4^2 V_4^{-1} + bcc_0)^{-1})\right], \end{aligned} \tag{13}$$

where $F_{\chi^2_{n_1}}$ is the cumulative distribution function (cdf) of χ^2 distribution with n_1 degrees of freedom, and the expectation of (13) is taken with respect to V_3 and V_4. The null hypothesis H_0 in (5) will be rejected if p_4 is less than the nominal significance level of δ.

Next, we prove that the test based on (13) is p-invariant test. Regard to the following scale transformations

$$\begin{aligned} &(\sigma_1^2, \sigma_3^2, \sigma_4^2, \sigma_\alpha^2 + \sigma_\varepsilon^2) \to (a\sigma_1^2, a\sigma_3^2, a\sigma_4^2, a(\sigma_\alpha^2 + \sigma_\varepsilon^2)),\\ &(\widehat{\sigma}_1^2, \widehat{\sigma}_3^2, \widehat{\sigma}_4^2) \to (a\widehat{\sigma}_1^2, a\widehat{\sigma}_3^2, a\widehat{\sigma}_4^2), \quad a > 0, \end{aligned} \tag{14}$$

the parameter space of the hypothesis testing problem (5) has changed under the scale transformations (14). So this hypothesis testing problem is not invariant. Hence, consider the equivalent hypothesis

$$H_0 : \theta \le \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0, \tag{15}$$

where $\theta = \frac{\sigma_\alpha^2 + \sigma_\varepsilon^2}{s_1^2}$ and $\theta_0 = \frac{c_0}{s_1^2}$. By the scale transformations

$$\begin{aligned} &(\sigma_1^2, \sigma_3^2, \sigma_4^2, \theta) \to (a\sigma_1^2, a\sigma_3^2, a\sigma_4^2, \theta),\\ &(\widehat{\sigma}_1^2, \widehat{\sigma}_3^2, \widehat{\sigma}_4^2) \to (a\widehat{\sigma}_1^2, a\widehat{\sigma}_3^2, a\widehat{\sigma}_4^2), \quad a > 0, \end{aligned} \tag{16}$$

the hypothesis testing problem (5) is invariant under the scale transformations (16). However, the test variable T_4 is not an invariant test variable. To obtain the invariant test variable, we define

$$\widetilde{T}_4 = \left[\frac{1}{bc}\left(\frac{\sigma_1^2}{\widehat{\sigma}_1^2} - \frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2 s_1^2}\right) + \frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2 s_1^2}\right] - \theta. \tag{17}$$

The corresponding generalized p-value is $\widetilde{p}_4 = P(\widetilde{T}_4 \le 0\,|\theta = \theta_0)$. Therefore, for the hypothesis testing problem (5), the test based on $\widetilde{p}_4$ is p-invariant under the scale transformations (16).

Then, to obtain the confidence interval of $\sigma_\alpha^2 + \sigma_\varepsilon^2$, we define

$${T_4}^* = \frac{1}{bc}\left(\frac{\sigma_1^2 s_1^2}{\widehat{\sigma}_1^2} - \frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2}\right) + \frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2}.$$

Obviously, the observed value $t_4^* = \sigma_\alpha^2 + \sigma_\varepsilon^2$ of T_4^* is free of nuisance parameters, and the distribution of T_4^* is free of any unknown parameters. Thus, T_4^* is a generalized pivot quantity.

According to the quantile of T_4^*, the generalized upper confidence limit and lower confidence limit of $\sigma_\alpha^2+\sigma_\varepsilon^2$ are obtained at the confidence level of $1-\delta$, which are written as $T_4^*(1-\delta/2)$ and $T_4^*(\delta/2)$ respectively. Further, to obtain the invariant confidence interval for $\sigma_\alpha^2+\sigma_\varepsilon^2$, the invariant confidence interval of θ is considered under the scale transformations (16). Define

$$\widehat{T}_4^* = \frac{1}{bc}\left(\frac{\sigma_1^2}{\widehat{\sigma}_1^2} - \frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2 s_1^2}\right) + \frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2 s_1^2}.$$

It is easy to verify $\widehat{T}_4^*$ is a generalized pivot quantity for θ and invariant under the scale transformations (16). Hence, the invariant confidence interval of θ can be constructed by the quantiles of $\widehat{T}_4^*$.

Similarly, the generalized test variables of the hypothesis testing problems (6) and (7) are expressed respectively as

$$T_5 = \frac{1}{ac}(n_2 s_2^2 V_2^{-1} - n_3 s_3^2 V_3^{-1}) + n_4 s_4^2 V_4^{-1} - (\sigma_\beta^2 + \sigma_\varepsilon^2),$$
$$T_6 = \frac{1}{c}(n_3 s_3^2 V_3^{-1} - n_4 s_4^2 V_4^{-1}) + n_4 s_4^2 V_4^{-1} - (\sigma_\gamma^2 + \sigma_\varepsilon^2).$$

Based on T_5 and T_6, the generalized p-values for the hypothesis testing problems (6) and (7) are computed as follows

$$p_5 = 1 - E_{V_3,V_4}\left[F_{\chi^2_{n_2}}(n_2 s_2^2 (n_3 s_3^2 V_3^{-1} - acn_4 s_4^2 V_4^{-1} + acc_0)^{-1})\right],$$
$$p_6 = 1 - E_{V_4}\left[F_{\chi^2_{n_3}}(n_3 s_3^2 (cc_0 + (1-c) n_4 s_4^2 V_4^{-1})^{-1})\right].$$

Further, the generalized pivot quantities for $\sigma_\beta^2+\sigma_\varepsilon^2$ and $\sigma_\gamma^2+\sigma_\varepsilon^2$ are

$$T_5{}^* = \frac{1}{ac}\left(\frac{\sigma_2^2 s_2^2}{\widehat{\sigma}_2^2} - \frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2}\right) + \frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2}, \quad T_6{}^* = \frac{1}{c}\left(\frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2} - \frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2}\right) + \frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2}.$$

Similar to $\sigma_\alpha^2+\sigma_\varepsilon^2$, the invariant tests and invariant confidence intervals for $\sigma_\beta^2+\sigma_\varepsilon^2$ and $\sigma_\gamma^2+\sigma_\varepsilon^2$ can be obtained easily.

4 Inference on the Ratio of Variance Components

Using the Bootstrap approach and generalized approach, the hypothesis testing problems for the ratio of two variance components in the model (1) are also discussed. To be specific, the hypotheses of interest are

$$H_0: \sigma_\alpha^2/\sigma_\varepsilon^2 \le c_1 \quad \text{versus} \quad H_1: \sigma_\alpha^2/\sigma_\varepsilon^2 > c_1, \tag{18}$$
$$H_0: \sigma_\beta^2/\sigma_\varepsilon^2 \le c_1 \quad \text{versus} \quad H_1: \sigma_\beta^2/\sigma_\varepsilon^2 > c_1, \tag{19}$$
$$H_0: \sigma_\gamma^2/\sigma_\varepsilon^2 \le c_1 \quad \text{versus} \quad H_1: \sigma_\gamma^2/\sigma_\varepsilon^2 > c_1, \tag{20}$$

where c_1 is a specified value.

4.1 The Bootstrap Approach

For the hypothesis testing problem (18), we have

$$T_7 = \frac{n_1 \widehat{\sigma}_1^2}{bcc_1 \sigma_4^2 + \sigma_3^2}. \tag{21}$$

If σ_3^2 and σ_4^2 are already known, T_7 in (21) can be used for the hypothesis testing problem (18). However, σ_3^2 and σ_4^2 are often unknown in real application, so the test statistic can be obtained by replacing the parameters by their estimators $\widehat{\sigma}_3^2$ and $\widehat{\sigma}_4^2$. The test statistic has the form of

$$T_7 = \frac{n_1 \widehat{\sigma}_1^2}{bcc_1 \widehat{\sigma}_4^2 + \widehat{\sigma}_3^2}. \tag{22}$$

Similar to T_1, the exact distribution of T_7 is difficult to obtain, then the Bootstrap approach is used to construct the test statistic. Therefore, the Bootstrap test statistic based on (22) is defined as

$$TB_7 = \frac{n_1 \widehat{\sigma}_{1B}^2}{bcc_1 \widehat{\sigma}_{4B}^2 + \widehat{\sigma}_{3B}^2}, \tag{23}$$

where $\widehat{\sigma}_{1B}^2 \sim \frac{bcc_1 \widehat{\sigma}_4^2 + \widehat{\sigma}_3^2}{n_1} \chi_{n_1}^2$, $\widehat{\sigma}_{3B}^2 \sim \frac{(cc_1+1)\widehat{\sigma}_4^2}{n_3} \chi_{n_3}^2$, $\widehat{\sigma}_{4B}^2 \sim \frac{\widehat{\sigma}_4^2}{n_4} \chi_{n_4}^2$. Then, the Bootstrap p-value is computed by TB_7 in (23) as

$$p_7 = P(TB_7 \geq t_7 | H_0), \tag{24}$$

where t_7 denotes the observed value of T_7. The null hypothesis H_0 in (18) is rejected whenever the above p-value is less than the nominal significance level of δ.

Likewise, the Bootstrap test statistics for the hypothesis testing problems (19) and (20) can be represented as

$$TB_8 = \frac{n_2 \widehat{\sigma}_{2B}^2}{acc_1 \widehat{\sigma}_{4B}^2 + \widehat{\sigma}_{3B}^2}, \quad TB_9 = \frac{n_3 \widehat{\sigma}_{3B}^2}{(cc_1+1)\widehat{\sigma}_{4B}^2},$$

where $\widehat{\sigma}_{2B}^2 \sim \frac{acc_1 \widehat{\sigma}_4^2 + \widehat{\sigma}_3^2}{n_2} \chi_{n_2}^2$. Based on TB_8 and TB_9, the Bootstrap p-values are computed respectively as

$$p_8 = P(TB_8 \geq t_8 | H_0), \quad p_9 = P(TB_9 \geq t_9 | H_0).$$

Similar to t_7 in (24), t_8 and t_9 are observed values.

Remark 4.1 Similar to Remark 3.1, the Bootstrap pivot quantity of $\sigma_\alpha^2/\sigma_\varepsilon^2$ is constructed as TB_7^* based on TB_7. Let $TB_7^*(\eta)$ be the 100η empirical percentile of TB_7^*. The $100(1-\delta)\%$ Bootstrap confidence interval for $\sigma_\alpha^2/\sigma_\varepsilon^2$ is given by

$$\left[\frac{n_1 s_1^2}{bcs_4^2 TB_7^*(1-\delta/2)} - \frac{s_3^2}{bcs_4^2}, \frac{n_1 s_1^2}{bcs_4^2 TB_7^*(\delta/2)} - \frac{s_3^2}{bcs_4^2}\right].$$

Likewise, the $100(1-\delta)\%$ Bootstrap confidence intervals based on TB_8^* and TB_9^* for $\sigma_\beta^2/\sigma_\varepsilon^2$ and $\sigma_\gamma^2/\sigma_\varepsilon^2$ are also obtained.

4.2 The Generalized Approach

For the hypothesis testing problem (18), the generalized test variable is

$$\begin{aligned} T_{10} &= \frac{1}{bc}\left(\frac{\sigma_1^2 s_1^2}{\widehat{\sigma}_1^2} - \frac{\sigma_3^2 s_3^2}{\widehat{\sigma}_3^2}\right)\left(\frac{\sigma_4^2 s_4^2}{\widehat{\sigma}_4^2}\right)^{-1} - (\sigma_\alpha^2/\sigma_\varepsilon^2) \\ &= \frac{1}{bcs_4^2}(F_{n_4,n_1} s_1^2 - F_{n_4,n_3} s_3^2) - (\sigma_\alpha^2/\sigma_\varepsilon^2). \end{aligned} \tag{25}$$

It is obvious that t_{10}, the observed value of T_{10}, is free of any unknown parameters. From the expression of T_{10} in (25), it is easy to see that the distribution of T_{10} is free of nuisance parameters. In addition, T_{10} is stochastically decreasing in $\sigma_\alpha^2/\sigma_\varepsilon^2$. Therefore, T_{10} is a generalized test variable for the hypothesis testing problem (18) and the generalized p-value is computed as

$$\begin{aligned} p_{10} = P(T_{10} \le 0|H_0) &= P(F_{n_4,n_3} \ge s_3^{-2}(F_{n_4,n_1} s_1^2 - bcc_1 s_4^2)) \\ &= 1 - E_{F_{n_4,n_1}}\left[F_{F_{n_4,n_3}}(s_3^{-2}(F_{n_4,n_1} s_1^2 - bcc_1 s_4^2))\right], \end{aligned} \tag{26}$$

where $F_{F_{n_4,n_3}}$ is the CDF of F distribution with (n_4, n_3) degrees of freedom, and the expectation of (26) is taken with respect to F_{n_4,n_1}. The null hypothesis H_0 in (18) will be rejected if p_{10} is less than the nominal significance level of δ.

Next, we show that the test based on (26) is p-variant. Apparently, under the scale transformations

$$\begin{aligned} &(\sigma_1^2, \sigma_3^2, \sigma_4^2) \to (a\sigma_1^2, a\sigma_3^2, a\sigma_4^2), \\ &(\widehat{\sigma}_1^2, \widehat{\sigma}_3^2, \widehat{\sigma}_4^2) \to (a\widehat{\sigma}_1^2, a\widehat{\sigma}_3^2, a\widehat{\sigma}_4^2), \quad a > 0, \end{aligned} \tag{27}$$

the generalized test variable T_{10} and the hypothesis testing problem (18) are both invariant. Accordingly, for the hypothesis testing problem (18), the test based on p_{10} is p-invariant under the scale transformations (27).

To obtain the confidence interval of $\sigma_\alpha^2/\sigma_\varepsilon^2$, we define

$$T_{10}{}^* = \frac{1}{bcs_4^2}(F_{n_4,n_1} s_1^2 - F_{n_4,n_3} s_3^2).$$

Obviously, $T_{10}{}^*$ is an invariant generalized pivot quantity for $\sigma_\alpha^2/\sigma_\varepsilon^2$ under the scale transformations (27). Hence, the invariant generalized upper confidence limit and lower confidence limit of $\sigma_\alpha^2/\sigma_\varepsilon^2$ are constructed as $T_{10}^*(1-\delta/2)$ and $T_{10}^*(\delta/2)$, respectively.

Similar to (18), the invariant generalized test variables for the hypothesis testing problems (19) and (20) are

$$\begin{aligned} T_{11} &= \frac{1}{acs_4^2}(F_{n_4,n_2} s_2^2 - F_{n_4,n_3} s_3^2) - (\sigma_\beta^2/\sigma_\varepsilon^2), \\ T_{12} &= \frac{1}{c}\left(F_{n_4,n_3}\frac{s_3^2}{s_4^2} - 1\right) - (\sigma_\gamma^2/\sigma_\varepsilon^2). \end{aligned}$$

Thus, the generalized p-values based on T_{11} and T_{12} for the hypothesis testing problems (19) and (20) are computed as

$$\begin{aligned} p_{11} &= 1 - E_{F_{n_4,n_2}}\left[F_{F_{n_4,n_3}} s_3^{-2}(F_{n_4,n_2} s_2^2 - bcc_1 s_4^2)\right], \\ p_{12} &= F_{F_{n_4,n_3}}((cc_1{+}1)s_4^2/s_3^2). \end{aligned}$$

Further, the invariant generalized pivot quantities for $\sigma_\beta^2/\sigma_\varepsilon^2$ and $\sigma_\gamma^2/\sigma_\varepsilon^2$ are

$$T_{11}{}^* = \frac{1}{acs_4^2}(F_{n_4,n_2}s_2^2 - F_{n_4,n_3}s_3^2), \quad T_{12}{}^* = \frac{1}{c}\left(F_{n_4,n_3}\frac{s_3^2}{s_4^2} - 1\right).$$

5 Simulation Study

In this section, Type I error probabilities and powers of the above testing approaches are investigated from the numerical perspective by using the Monte Carlo simulation. For convenience, we only provide the algorithm of the Boostrap approach for the hypothesis testing problem (5) as follows.

Algorithm 1

Step 1 For a given $(a, b, c, \sigma_\alpha^2, \sigma_\gamma^2, c_0)$, generate $\widehat{\sigma}_i^2 \sim \frac{\widehat{\sigma}_i^2}{n_i}\chi_{n_i}^2, i = 1, 3, 4$, where σ_i^2 and n_i are given by (3) and Lemma 2.1, respectively.

Step 2 Compute T_1 in (9) and denote it as t_1.

Step 3 Generate $\widehat{\sigma}_{1B}^2 \sim \frac{\widehat{\sigma}_1^2}{n_1}\chi_{n_1}^2$, $\widehat{\sigma}_{3B}^2 \sim \frac{\widehat{\sigma}_3^2}{n_3}\chi_{n_3}^2$, $\widehat{\sigma}_{4B}^2 \sim \frac{c_0-(bc)^{-1}(\widehat{\sigma}_1^2-\widehat{\sigma}_3^2)}{n_4}\chi_{n_4}^2$, then compute TB_1 in (10).

Step 4 Repeat Step 3 m_1 times and compute p_1 by (11). If $p_1 \leq \delta$, then $Q = 1$. Otherwise, $Q = 0$.

Step 5 Repeat Steps 1–4 m_2 times and get $Q_1, Q_2, \cdots, Q_{m_2}$. Then Type I error probability is $\sum_{i=1}^{m_2} Q_i/m_2$.

Based on the above steps, the power of the hypothesis testing problem (5) under H_1 can be obtained similarly.

In this simulation, the parameters and sample sizes are set as follows. Firstly, let the nominal significance level δ be 0.025, 0.05, 0.075, and 0.1, the numbers of inner loops m_1 and outer loops m_2 both be 2500, and sample sizes (a, b, c) are $(3, 3, 3)$, $(3, 4, 5)$, $(4, 5, 6)$, $(6, 8, 10)$, $(8, 10, 12)$, and $(10, 12, 15)$. Secondly, for the hypothesis testing problem (5), we set $c_0 = 5$, $\sigma_\alpha^2 = 2, 2.5, 3, 3.5, 4.5$, and $\sigma_\gamma^2 = 0.5, 1, 1.5, 2, 3$. Finally, for the hypothesis testing problem (18), we suppose $c_1 = 1$, $\sigma_\alpha^2 = 1, 1.2, 1.5, 1.8, 2$, and $\sigma_\gamma^2 = 1, 1.1, 1.2, 1.5, 1.8$.

For the hypothesis testing problem (5), Table 1 presents the simulated Type I error probabilities of the Bootstrap approach (BA) and generalized approach (GA) at different nominal significance levels. When the sample size is small, Type I error probabilities of the BA is slightly liberal, while those of the GA is slightly conservative. With the increase of the sample size, the actual levels of the proposed two approaches are closer to the nominal significance levels. Table 2 presents the simulated powers of the BA and GA at different nominal significance levels. The powers of the BA are apparently better than those of the GA in most cases.

For the hypothesis testing problem (18), Tables 3 and 4 give the simulated Type I error probabilities and powers of the BA and GA at different nominal significance levels, respectively. As in Table 3, similar to (5), the BA is slightly liberal whereas the GA is slightly conservative, when the sample size is small. However, this result is improved significantly as the sample size increases. Namely, the above two approaches both can efficiently control Type I error probabilities. As in Table 4, it is clear that the powers of these two approaches both increase as

$\sigma_\alpha^2/\sigma_\varepsilon^2$ departs from the null hypothesis and as the sample size increases, but the BA is better than the GA in most cases.

Table 1 Type I error probabilities for the hypothesis testing problem (5) ($\sigma_\alpha^2 + \sigma_\varepsilon^2 = c_0$)

					δ							
a	b	c	σ_α^2	σ_γ^2	0.025		0.05		0.075		0.1	
					BA	GA	BA	GA	BA	GA	BA	GA
3	3	3	2	0.5	0.0552	0.0228	0.0776	0.054	0.1064	0.0804	0.1232	0.1104
			2.5	1	0.0432	0.0212	0.0712	0.0488	0.0952	0.0748	0.1188	0.1028
			3	1.5	0.0316	0.0208	0.0604	0.0456	0.0868	0.0688	0.11	0.0932
			3.5	2	0.026	0.018	0.056	0.0408	0.0824	0.064	0.1072	0.0908
			4.5	3	0.0192	0.0156	0.0448	0.0364	0.0752	0.0588	0.1024	0.0808
3	4	5	2	0.5	0.0444	0.0276	0.0668	0.0564	0.0944	0.0844	0.1156	0.1116
			2.5	1	0.0324	0.0244	0.0628	0.0512	0.0848	0.0784	0.1072	0.1032
			3	1.5	0.026	0.0216	0.0552	0.0472	0.0792	0.072	0.1056	0.098
			3.5	2	0.0228	0.0204	0.0512	0.0456	0.0768	0.07	0.1024	0.094
			4.5	3	0.022	0.0216	0.0492	0.0428	0.0752	0.0664	0.1	0.0924
4	5	6	2	0.5	0.0364	0.0292	0.06	0.0556	0.0836	0.0808	0.1096	0.1092
			2.5	1	0.0304	0.0248	0.0556	0.0524	0.0812	0.0788	0.1044	0.1032
			3	1.5	0.0272	0.0236	0.0512	0.0508	0.0772	0.0752	0.102	0.0988
			3.5	2	0.0264	0.0232	0.0512	0.048	0.0748	0.0744	0.1024	0.0964
			4.5	3	0.0264	0.0228	0.046	0.0468	0.0748	0.0744	0.0992	0.0948
6	8	10	2	0.5	0.0284	0.0264	0.0544	0.052	0.0788	0.0788	0.1036	0.104
			2.5	1	0.0244	0.0244	0.0532	0.0504	0.0764	0.076	0.1016	0.102
			3	1.5	0.0244	0.024	0.0512	0.0508	0.0752	0.0752	0.098	0.0984
			3.5	2	0.0252	0.0244	0.0496	0.0496	0.0736	0.074	0.1	0.0996
			4.5	3	0.0244	0.024	0.0484	0.0488	0.0744	0.072	0.1016	0.0988
8	10	12	2	0.5	0.0276	0.026	0.0532	0.0516	0.0756	0.0776	0.1036	0.1032
			2.5	1	0.0252	0.0248	0.0524	0.0504	0.0752	0.0752	0.0996	0.0996
			3	1.5	0.0256	0.0244	0.0516	0.05	0.0756	0.0752	0.0996	0.1004
			3.5	2	0.0244	0.0244	0.05	0.0496	0.0756	0.0744	0.0996	0.1
			4.5	3	0.0244	0.0252	0.05	0.0496	0.0752	0.0744	0.1	0.0996
10	12	15	2	0.5	0.0244	0.026	0.0512	0.0508	0.0772	0.076	0.1012	0.1
			2.5	1	0.0248	0.0248	0.052	0.05	0.0764	0.0752	0.0988	0.1004
			3	1.5	0.0248	0.0248	0.0516	0.0504	0.0744	0.0744	0.1	0.1004
			3.5	2	0.0256	0.0248	0.0512	0.0492	0.074	0.0744	0.0984	0.0996
			4.5	3	0.0256	0.0248	0.0508	0.0492	0.072	0.0748	0.0984	0.0992

Table 2 Powers for the hypothesis testing problem (5) ($\sigma_\gamma^2 = 1$)

a	b	c	σ_α^2	σ_γ^2	δ 0.025 BA	0.025 GA	0.05 BA	0.05 GA	0.075 BA	0.075 GA	0.1 BA	0.1 GA
3	3	3	3.5	3	0.1208	0.0728	0.162	0.1172	0.2092	0.164	0.2416	0.2052
			4	3.5	0.198	0.1164	0.2568	0.174	0.2972	0.2396	0.328	0.2884
			4.5	4	0.2928	0.1664	0.3436	0.2452	0.3976	0.3148	0.4316	0.372
			5	4.5	0.3836	0.2116	0.4388	0.3068	0.4804	0.3964	0.5132	0.452
			5.5	5	0.4648	0.268	0.5168	0.3816	0.5556	0.464	0.5896	0.522
3	4	5	3.5	3	0.114	0.0908	0.1676	0.1392	0.2172	0.198	0.2492	0.2384
			4	3.5	0.2208	0.1644	0.2748	0.2364	0.3224	0.2956	0.3656	0.346
			4.5	4	0.3408	0.256	0.4052	0.334	0.4516	0.4108	0.488	0.4656
			5	4.5	0.4752	0.3596	0.5256	0.4516	0.5704	0.5268	0.6068	0.5788
			5.5	5	0.5888	0.46	0.6372	0.5576	0.6788	0.6364	0.7052	0.6788
4	5	6	3.5	3	0.1376	0.1164	0.1996	0.1904	0.2396	0.2348	0.2792	0.2788
			4	3.5	0.2688	0.2316	0.3476	0.3216	0.4084	0.3924	0.4472	0.4432
			4.5	4	0.4456	0.3916	0.5184	0.4788	0.5724	0.554	0.612	0.6072
			5	4.5	0.6244	0.5516	0.6768	0.6372	0.7192	0.6964	0.7428	0.7376
			5.5	5	0.7556	0.6872	0.802	0.7632	0.828	0.8116	0.8484	0.8412
6	8	10	3.5	3	0.178	0.1784	0.2496	0.2396	0.308	0.308	0.352	0.3556
			4	3.5	0.3884	0.372	0.4816	0.4676	0.5472	0.542	0.598	0.6004
			4.5	4	0.646	0.6256	0.7076	0.698	0.7616	0.7532	0.7872	0.7868
			5	4.5	0.8516	0.8288	0.8832	0.8712	0.906	0.9016	0.9196	0.9188
			5.5	5	0.962	0.9488	0.9736	0.9688	0.9784	0.9772	0.982	0.9828
8	10	12	3.5	3	0.2188	0.2188	0.3008	0.2912	0.3492	0.3516	0.4104	0.4108
			4	3.5	0.488	0.472	0.586	0.5812	0.634	0.634	0.6756	0.6808
			4.5	4	0.7744	0.7512	0.8216	0.818	0.8552	0.8552	0.876	0.8772
			5	4.5	0.9452	0.9324	0.9612	0.9588	0.9704	0.9696	0.9752	0.9748
			5.5	5	0.996	0.9948	0.9972	0.9968	0.998	0.998	0.9992	0.9992
10	12	15	3.5	3	0.2548	0.26	0.3456	0.346	0.4128	0.4128	0.4696	0.4684
			4	3.5	0.5844	0.5796	0.66	0.6648	0.7216	0.7204	0.76	0.7588
			4.5	4	0.8576	0.8456	0.8972	0.8944	0.9188	0.9196	0.9352	0.9352
			5	4.5	0.9844	0.9768	0.99	0.988	0.9924	0.9924	0.9928	0.9928
			5.5	5	0.9984	0.9984	0.9992	0.9988	0.9996	0.9996	1	1

Table 3 Type I error probabilities for the hypothesis testing problem (18) ($\sigma_\alpha^2/\sigma_\varepsilon^2 = c_1$)

a	b	c	σ_α^2	σ_γ^2	δ 0.025 BA	GA	0.05 BA	GA	0.075 BA	GA	0.1 BA	GA
3	3	3	1	1	0.0336	0.014	0.0576	0.0304	0.0848	0.0508	0.1048	0.0676
			1.2	1.1	0.034	0.014	0.06	0.0316	0.086	0.0492	0.1076	0.0676
			1.5	1.2	0.0344	0.014	0.0628	0.0312	0.0876	0.0512	0.1128	0.0692
			1.8	1.5	0.034	0.014	0.062	0.0312	0.0872	0.0504	0.112	0.0684
			2	1.8	0.0336	0.014	0.06	0.0316	0.0868	0.0496	0.1092	0.068
3	4	5	1	1	0.03	0.0188	0.0536	0.0404	0.078	0.0608	0.1092	0.0832
			1.2	1.1	0.0312	0.0192	0.0544	0.04	0.0792	0.0608	0.1116	0.084
			1.5	1.2	0.0316	0.0196	0.0576	0.042	0.0816	0.0604	0.1152	0.0836
			1.8	1.5	0.0316	0.0188	0.0568	0.042	0.0808	0.06	0.1128	0.084
			2	1.8	0.0316	0.0192	0.0552	0.0408	0.0796	0.0604	0.112	0.084
4	5	6	1	1	0.028	0.0224	0.0536	0.0432	0.0768	0.07	0.1036	0.0932
			1.2	1.1	0.0296	0.0232	0.0572	0.0432	0.0784	0.07	0.1072	0.092
			1.5	1.2	0.0296	0.0224	0.0596	0.0436	0.0824	0.0704	0.1112	0.092
			1.8	1.5	0.0296	0.0224	0.058	0.0432	0.0812	0.0704	0.11	0.0928
			2	1.8	0.0296	0.0224	0.0572	0.0428	0.0796	0.0696	0.1088	0.092
6	8	10	1	1	0.026	0.024	0.05	0.0488	0.0764	0.0728	0.102	0.0976
			1.2	1.1	0.026	0.024	0.0516	0.0492	0.0784	0.0732	0.1028	0.0976
			1.5	1.2	0.0284	0.0244	0.0556	0.0492	0.0816	0.0732	0.1068	0.0984
			1.8	1.5	0.028	0.024	0.0544	0.0492	0.0808	0.0732	0.1064	0.098
			2	1.8	0.026	0.024	0.0516	0.0492	0.0792	0.0732	0.1036	0.098
8	10	12	1	1	0.026	0.0244	0.0484	0.0484	0.076	0.0732	0.1008	0.0992
			1.2	1.1	0.026	0.0248	0.0504	0.0488	0.0796	0.0736	0.1036	0.0992
			1.5	1.2	0.0276	0.0244	0.0528	0.0492	0.0832	0.0736	0.1064	0.0984
			1.8	1.5	0.0268	0.0248	0.0524	0.0492	0.0828	0.0736	0.106	0.0988
			2	1.8	0.0264	0.0248	0.0512	0.0492	0.0804	0.0736	0.1036	0.0992
10	12	15	1	1	0.0236	0.0244	0.0496	0.0488	0.076	0.0748	0.1	0.0984
			1.2	1.1	0.0248	0.0248	0.0516	0.0496	0.0772	0.0744	0.1024	0.0992
			1.5	1.2	0.0284	0.0248	0.0532	0.0492	0.0812	0.0744	0.1072	0.0992
			1.8	1.5	0.028	0.0248	0.0528	0.0492	0.08	0.0744	0.106	0.0992
			2	1.8	0.026	0.0248	0.0516	0.0496	0.0772	0.0744	0.1036	0.0992

Table 4 Powers for the hypothesis testing problem (18) ($\sigma_\varepsilon^2 = 1$)

a	b	c	σ_α^2	σ_γ^2	δ							
					0.025		0.05		0.075		0.1	
					BA	GA	BA	GA	BA	GA	BA	GA
3	3	3	1.5	0.5	0.088	0.04	0.136	0.0764	0.1716	0.114	0.2068	0.1432
			2	0.6	0.1304	0.0704	0.1868	0.1244	0.2344	0.1608	0.2756	0.2012
			3	0.7	0.2152	0.14	0.2888	0.2072	0.3428	0.2636	0.3828	0.3104
			6	0.8	0.4172	0.3232	0.496	0.41	0.5468	0.4816	0.5844	0.5248
			10	1	0.5508	0.4696	0.6144	0.5624	0.66	0.6188	0.688	0.6556
3	4	5	1.5	0.5	0.0864	0.0592	0.1456	0.0988	0.1816	0.1448	0.2188	0.1772
			2	0.6	0.1472	0.108	0.2036	0.1596	0.252	0.206	0.2996	0.252
			3	0.7	0.2404	0.1956	0.316	0.268	0.37	0.3256	0.4156	0.3756
			6	0.8	0.46	0.4128	0.5392	0.4924	0.5936	0.554	0.6284	0.5972
			10	1	0.6056	0.5744	0.6672	0.636	0.6996	0.6844	0.7324	0.716
4	5	6	1.5	0.5	0.1064	0.082	0.1664	0.1332	0.2032	0.1696	0.2532	0.2092
			2	0.6	0.1784	0.1544	0.256	0.2172	0.3016	0.2692	0.35	0.316
			3	0.7	0.318	0.2928	0.4148	0.3792	0.474	0.4376	0.5248	0.4892
			6	0.8	0.6084	0.5884	0.6796	0.6616	0.718	0.7008	0.7464	0.7348
			10	1	0.7524	0.746	0.8032	0.7968	0.8288	0.822	0.8488	0.8444
6	8	10	1.5	0.5	0.1312	0.1084	0.1972	0.17	0.2592	0.2272	0.296	0.2696
			2	0.6	0.254	0.2228	0.34	0.3108	0.4116	0.3784	0.4544	0.4292
			3	0.7	0.47	0.4452	0.5728	0.5472	0.6312	0.6128	0.6708	0.6556
			6	0.8	0.7944	0.7884	0.8432	0.8344	0.8736	0.8636	0.892	0.8864
			10	1	0.9112	0.9116	0.9364	0.936	0.952	0.9504	0.9588	0.9584
8	10	12	1.5	0.5	0.1572	0.1388	0.228	0.2016	0.2856	0.256	0.334	0.2984
			2	0.6	0.308	0.2856	0.4064	0.3808	0.476	0.4432	0.5312	0.4968
			3	0.7	0.5904	0.5676	0.6716	0.6576	0.7232	0.7084	0.758	0.7436
			6	0.8	0.8964	0.8892	0.9228	0.92	0.9428	0.9396	0.9524	0.9496
			10	1	0.9728	0.9728	0.9824	0.982	0.9872	0.986	0.9888	0.9892
10	12	15	1.5	0.5	0.178	0.1544	0.2656	0.2328	0.3164	0.2872	0.374	0.3408
			2	0.6	0.3712	0.3444	0.4812	0.4552	0.5452	0.5196	0.6048	0.5764
			3	0.7	0.6828	0.6636	0.7636	0.746	0.7984	0.7828	0.8296	0.8152
			6	0.8	0.9496	0.9456	0.9668	0.9656	0.9752	0.9732	0.9812	0.9784
			10	1	0.992	0.9928	0.9948	0.9948	0.9952	0.9956	0.9964	0.9964

6 Illustrative Examples

To analyze the effectiveness of the proposed methods, we apply them to the examples of mice blood pH and molded plastic part's dimensions.

Example 1 The approaches above are applied to the study of mice blood pH of different strains. Firstly, the five strains used for the experiment are randomly selected from all available strains. Secondly, the interest is in the magnitude of fluctuation in daily blood pH rather than the comparison of daily average levels, so the time variable is also viewed as a random factor. Thus, this experiment can be fitted by the two-way random effects model with interaction. Let y_{ijk} denote the blood pH of the kth mouse of the ith strain in the jth day, then the statistical model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk},$$

where $i = 1, 2, \cdots, 5$, $j = 1, 2, \cdots, 6$, $k = 1, 2, \cdots, 5$, μ is the fixed effect, α_i is the effect of the ith strain, β_j is the effect of the jth day, γ_{ij} is the interaction between the ith strain and the jth day, and ε_{ijk} is the random error. Assume that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_j \sim N(0, \sigma_\beta^2)$, $\gamma_{ij} \sim N(0, \sigma_\gamma^2)$, $\varepsilon_{ijk} \sim N(0, \sigma_\varepsilon^2)$, and all random variables are mutually independent. Weir[23] presented the ANOVA results for this experiment (see Table 5).

Table 5 Analysis of variance

SV	DF	MS
Mouse strains	4	0.092
Days of test	5	0.0101
Interaction	20	0.0052
Random error	120	0.0034

Firstly, consider the hypothesis testing problem for the sum of variance components

$$H_0 : \sigma_\alpha^2 + \sigma_\varepsilon^2 \le 0.05 \quad \text{versus} \quad H_1 : \sigma_\alpha^2 + \sigma_\varepsilon^2 > 0.05. \tag{28}$$

Based on (11) and (13), the Bootstrap p-value and generalized p-value are 0.0980 and 0.0968 respectively by 10^4 loops. Hence, the null hypothesis H_0 in (28) is not rejected by the two approaches at the nominal significance level of 5%.

Next, consider the hypothesis testing problem for the ratio of variance components

$$H_0 : \sigma_\alpha^2/\sigma_\varepsilon^2 \le 10 \quad \text{versus} \quad H_1 : \sigma_\alpha^2/\sigma_\varepsilon^2 > 10. \tag{29}$$

The Bootstrap p-value based on (24) is 0.0170, and the generalized p-value based on (26) is 0.0337. Consequently, at the nominal significance level of 5%, the above two p-values indicate that these two approaches both reject the the null hypothesis H_0 in (29).

Example 2 The empirical data of molded plastic part's dimensions are analyzed in this example. In this experiment, two operators are randomly selected from those who often use coordinate measuring machine to measure the part's dimension. Each operator randomly tests ten parts in which each part is loaded twice into the fixture. Therefore, this data can be fitted by the two-way random effects model with interaction. Let y_{ijk} denote the dimension of the kth loading of the ith part by the jth operator. The statistical model for data analysis has the form of

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk},$$

where $i = 1, 2, \cdots, 10$, $j = 1, 2$, $k = 1, 2$, μ denotes the fixed effect, α_i and β_j are the effects of the ith part and the jth operator respectively, γ_{ij} is the interaction between the ith part and the jth operator, and ε_{ijk} is the random error. Assume that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_j \sim N(0, \sigma_\beta^2)$, $\gamma_{ij} \sim N(0, \sigma_\gamma^2)$, $\varepsilon_{ijk} \sim N(0, \sigma_\varepsilon^2)$, and all random variables are mutually independent. Gilder, et al.[11] gave the ANOVA results (See Table 6).

Table 6 Analysis of variance

SV	DF	MS
BArts	9	1.3055×10^{-3}
Operators	1	6.4803×10^{-4}
Interaction	9	2.8197×10^{-4}
Random error	20	1.6348×10^{-4}

Firstly, consider the hypothesis testing problem for the sum of variance components

$$H_0 : \sigma_\alpha^2 + \sigma_\varepsilon^2 \leq 0.005 \quad \text{versus} \quad H_1 : \sigma_\alpha^2 + \sigma_\varepsilon^2 > 0.005. \tag{30}$$

Based on (11) and (13), the Bootstrap p-value and generalized p-value are 0.9976 and 0.9766 respectively by 10^4 loops. Thus, the null hypothesis H_0 in (30) is not rejected by the two approaches at the nominal significance level of 5%.

Next, consider the hypothesis testing problem for the ratio of variance components

$$H_0 : \sigma_\alpha^2/\sigma_\varepsilon^2 \leq 2 \quad \text{versus} \quad H_1 : \sigma_\alpha^2/\sigma_\varepsilon^2 > 2. \tag{31}$$

The Bootstrap p-value based on (24) is 0.0103, and the generalized p-value based on (26) is 0.0217. Therefore, at the nominal significance level of 5%, the above two p-values indicate that these two approaches both reject the the null hypothesis H_0 in (31).

7 Conclusion

Using the Bootstrap approach and generalized approach, we study the one-sided hypothesis testing and interval estimation problems for the sum and ratio of variance components in the two-way random effects model with interaction. Firstly, the test statistics and confidence intervals for the sum of variance components are constructed. Secondly, the test statistics and confidence intervals for the ratio of variance components are established. Thirdly, the Monte Carlo simulation results show that the Bootstrap approach is better than the generalized approach in most cases, whether for the sum or ratio of variance components. Finally, the above approaches are applied to the examples of mice blood pH and molded plastic part's dimensions to verify the rationality and validity of the proposed approaches. In summary, for two-way random effects model with interaction, the Bootstrap approach is preferentially suggested to be used for hypothesis testing and interval estimation problems for the sum and ratio of variance components.

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