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Adaptive Output-Feedback Stabilization for PDE-ODE Cascaded Systems with Unknown Control Coefficient and Spatially Varying Parameter*

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Abstract This paper investigates the adaptive stabilization for a class of uncertain PDE-ODE cascaded systems. Remarkably, the PDE subsystem allows unknown control coefficient and spatially varying parameter, and only its one boundary value is measurable. This renders the system in question more general and practical, and the control problem more challenging. To solve the problem, an invertible transformation is first introduced to change the system into an observer canonical form, from which a couple of filters are constructed to estimate the unmeasurable states. Then, by adaptive technique and infinite-dimensional backstepping method, an adaptive controller is constructed which guarantees that all states of the resulting closed-loop system are bounded while the original system states converging to zero. Finally, a numerical simulation is provided to illustrate the effectiveness of the proposed method.

Keywords Adaptive stabilization, output-feedback, PDE-ODE cascaded systems, unknown control coefficient, unknown spatially varying parameter.

1 Introduction

Many dynamic processes in engineering can be described by partial differential equation (PDE) cascaded with ordinary differential equation (ODE). For example, the metal rolling processes are represented by first-order hyperbolic PDE-ODE cascaded systems[1], the fluid-structure interaction models are described by parabolic PDE-ODE cascaded systems[2] and

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the flexible air-breathing hypersonic vehicles are described by the cascaded systems consisting of Euler-Bernoulli beam equations and ODEs[3]. Roughly speaking, due to the presence of both PDE and ODE subsystems, the feedback control design of PDE-ODE cascaded systems is more difficult than that of pure PDE or ODE systems, and hence attracts lots of attention over the last two decades (see, e.g., [4–18] and references therein). Several control methods for this kind of systems have been developed, for example, the method of infinite-dimensional backstepping[9, 14], active disturbance rejection control[4, 5] and sliding mode control[16, 17].

Recently, the stabilization and tracking control problems of first-order hyperbolic PDE-ODE cascaded systems have been investigated under different assumptions on system uncertainties[5–9]. Specifically, stabilization has been addressed in [5, 9], but all the system parameters in [5, 9] are required to be exactly known. Moreover, by modelling unknown time-delay as a first-order hyperbolic PDE, tracking and stabilizing controllers have been respectively designed for uncertain first-order hyperbolic PDE-ODE cascaded systems in [6] and [7, 8], but the unknown parameters are constants rather than spatially varying functions. It is necessary to point out that the actual values of spatially varying parameters in PDE systems are difficult (even impossible) to obtain in most cases. Hence, the output-feedback control of first-order hyperbolic PDE-ODE cascaded systems with unknown spatially varying parameters deserves further investigation.

In this paper, we consider the stabilization for the following PDE-ODE cascaded system with unknown control coefficient and spatially varying parameter:

$$\begin{cases} \dot{X}(t) \;=\; AX(t) + Bv(0,t), \\ v_t(x,t) \;=\; v_x(x,t) + \lambda(x)v(x,t), \\ v(1,t) \;=\; kU(t), \end{cases} \tag{1}$$

where $X : \boldsymbol{R}_+ \to \boldsymbol{R}^n$ and $v : [0,1] \times \boldsymbol{R}_+ \to \boldsymbol{R}$ with initial values $X(0) = X_0$ and $v(x,0) = v_0(x)$ are the states of ODE and PDE subsystems, respectively; $U(t)$ is the input to the entire system; $v_t = \frac{\partial v}{\partial t}$ and $v_x = \frac{\partial v}{\partial x}$; the pair (A, B) with $A \in \boldsymbol{R}^{n\times n}$ and $B \in \boldsymbol{R}^{n\times 1}$ is stabilizable; k is an unknown nonzero constant, called control coefficient and its sign (i.e., $\mathsf{sign}(k)$) is known; $\lambda(x)$ is an unknown continuous function defined on $[0,1]$. In the system (1), only $X(t)$ and $v(0,t)$ are the measurable information which are available for feedback.

System (1) can describe the dynamics of distributed chemical reaction processes, in which the evolution of substance in the transport pipe is represented by PDE while the evolution of substance in the reactor being described by ODE (see, e.g., [19, 20] and references therein). In such dynamic process, control coefficient k is used to denote the deviation between the input acting on the plant and the designed one, which always exists and is unknown in practice. Although the first-order hyperbolic PDE-ODE cascaded systems similar as (1) have been studied in [5, 9], the system parameters in [5, 9] are known and all the system states are required to be available for feedback.

The control objective of this paper is to design an adaptive controller to guarantee that all states of the resulting closed-loop system are bounded, meanwhile the original system states converge to zero. To achieve the objective, the following assumptions are respectively imposed on the unknown control coefficient k and spatially varying parameter $\lambda(x)$ of the system (1).

Assumption 1 There exist known nonzero constants $\underline{k}$ and $\overline{k}$ with the same sign such that

$$\underline{k} \leq k \leq \overline{k}.$$

Assumption 2 There exist known constants $\underline{\lambda}$ and $\overline{\lambda}$ such that

$$\underline{\lambda} \leq \lambda(x) \leq \overline{\lambda}, \quad \forall x \in [0,1].$$

To solve the control problem under investigation, an adaptive controller is designed in this paper. Specifically, by an invertible transformation, the system (1) is first transformed into an observer canonical form. Then, based on the obtained observer canonical form, some filters are constructed to estimate the unmeasurable states. Finally, by adaptive technique based on gradient algorithm and projection operator, an adaptive controller is designed which guarantees the desired performance of the resulting closed-loop system. It is worthwhile emphasizing that, the designed controller depends on ODE state and only one boundary value rather than all states of the PDE subsystem. Moreover, unknown control coefficient is also allowed in this paper, which makes the existing methods ineffective for the control problem under investigation.

The rest of this paper is organized as follows. Section 2 gives the controller design procedure. Section 3 presents the stability analysis of the resulting closed-loop system. Section 4 provides a numerical simulation to illustrate the effectiveness of the proposed method. Section 5 gives some concluding remarks. This paper ends with an Appendix which collects some useful criteria and the proofs of an important proposition and some inequalities.

Notation The following notation will be used in the paper. For any symmetric matrix Q, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ denote its minimum and maximum eigenvalues, respectively. For the vector or matrix X, $\|X\|$ means the Euclidean norm for vectors or the corresponding induced norm for matrices. Let $\|u(\cdot,t)\| = \sqrt{\int_0^1 u^2(x,t)dx}$ denote the L_2-norm of $u(x,t)$ defined on $[0,1] \times [0,+\infty)$.

2 Adaptive Control Design

In this section, the detailed controller design procedure is given. First, System (1) is transformed into an observer canonical form (i.e., System (4) below) by an invertible transformation. Then, certain proper filters are designed to estimate the unmeasurable states of the system. Finally, an adaptive controller is constructed by adaptive technique based on gradient algorithm and projection operator.

2.1 Construction of Observer Canonical Form

Since not all the states of System (1) are measurable, proper filters should be designed to estimate the unmeasurable states. To make the construction of filters convenient, an invertible transformation is introduced to change System (1) into an observer canonical form. In addition, the states of observer canonical form will be estimated in the next subsection.

First, we introduce the following transformation:

$$w(x,t) = \mu(x)v(x,t) - K\mathrm{e}^{(A+BK)x}X(t), \tag{2}$$

where $\mu(x) = \exp\left(\int_0^x \lambda(s)ds\right)$ and $K \in \boldsymbol{R}^{1\times n}$ is a row vector such that $A+BK$ is Hurwitz. It is not difficult to obtain the inverse transformation of (2) as follows:

$$v(x,t) = \frac{1}{\mu(x)}\left(w(x,t) + K\mathrm{e}^{(A+BK)x}X(t)\right). \tag{3}$$

Then, System (1) is transformed into the following new one under transformation (2), which is referred to as the observer canonical form of System (1):

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t), \\ w_t(x,t) = w_x(x,t) + \theta(x)w(0,t), \\ w(1,t) = \rho U(t) - K\mathrm{e}^{A+BK}X(t), \end{cases} \tag{4}$$

where $\rho = k\,\mu(1)$ and $\theta(x) = -K\mathrm{e}^{(A+BK)x}B$.

From (4), it can be seen that ρ is the new unknown control coefficient formed by the original control coefficient k and parameter $\lambda(x)$. Then, we only need to compensate the unknown parameter ρ in control design. Moreover, by Assumptions 1 and 2, we know that ρ belongs to a known bounded interval, i.e.,

$$\underline{\rho} \ \le\ \rho\ \le\ \overline{\rho}, \tag{5}$$

where $\underline{\rho} = \underline{k}\ \mathrm{e}^{\underline{\lambda}}$, $\overline{\rho} = \overline{k}\ \mathrm{e}^{\overline{\lambda}}$ if $\underline{k} > 0$ and $\underline{\rho} = \underline{k}\ \mathrm{e}^{\overline{\lambda}}$, $\overline{\rho} = \overline{k}\ \mathrm{e}^{\underline{\lambda}}$ if $\overline{k} < 0$.

2.2 Estimations of System States

To estimate the unmeasurable states of System (4), by the measurable information $X(t)$ and $v(0,t)$, we design the following filters:

$$\begin{cases} \phi_t(x,t) = \phi_x(x,t), \quad \phi(1,t) = U(t), \\ \varphi_t(x,t) = \varphi_x(x,t), \quad \varphi(1,t) = v(0,t), \\ N_t(x,t) = N_x(x,t), \quad N(1,t) = X(t), \end{cases} \tag{6}$$

with the corresponding initial values $\phi(x,0) = \phi_0(x)$, $\varphi(x,0) = \varphi_0(x)$ and $N(x,0) = N_0(x)$ being continuous functions defined on $[0,1]$.

For the aim of estimating $w(x,t)$, a nonadaptive estimation for $w(x,t)$ in the following form is constructed:

$$\begin{aligned} \overline{w}(x,t) = \rho\phi(x,t) - K\mathrm{e}^{A+BK}N(x,t) + \int_x^1 \theta(\xi)\varphi(1+x-\xi,t)d\xi \\ - \int_x^1 \theta(\xi)KN(1+x-\xi,t)d\xi. \end{aligned} \tag{7}$$

Define nonadaptive estimation error $\overline{e}(x,t)=w(x,t)-\overline{w}(x,t)$. From this, the second equation of (4), the last two equations of (6) and (7), and noting $v(0,t)=w(0,t)+KX(t)$, we obtain

$$\begin{aligned}\overline{e}_t(x,t)-\overline{e}_x(x,t)&=w_t(x,t)-w_x(x,t)+\overline{w}_x(x,t)-\overline{w}_t(x,t)\\&=\theta(x)w(0,t)-\theta(x)\varphi(1,t)+\theta(x)KN(1,t)\\&=\theta(x)w(0,t)-\theta(x)v(0,t)+\theta(x)KX(t)\\&=0.\end{aligned}\tag{8}$$

Then, from the third equation of (4), the first equation of (6) and (7), we have

$$\begin{aligned}\overline{e}(1,t)&=w(1,t)-\overline{w}(1,t)\\&=\rho U(t)-K\mathrm{e}^{A+BK}X(t)-\rho\phi(1,t)+K\mathrm{e}^{A+BK}N(1,t)\\&=0.\end{aligned}\tag{9}$$

Equations (8) and (9) imply that $\overline{e}(x,t)=0$ on $[0,1]\times[1,+\infty)$, that is, nonadaptive estimation $\overline{w}(x,t)$ can estimate $w(x,t)$ in finite time.

Since ρ is unknown, the nonadaptive estimation $\overline{w}(x,t)$ given by (7) is inapplicable. By replacing unknown parameter ρ by its dynamic compensation, an adaptive state estimation of $w(x,t)$ is obtained:

$$\begin{aligned}\widehat{w}(x,t)=\widehat{\rho}(t)\phi(x,t)-K\mathrm{e}^{A+BK}N(x,t)+\int_x^1\theta(\xi)\varphi(1+x-\xi,t)d\xi\\-\int_x^1\theta(\xi)KN(1+x-\xi,t)d\xi,\end{aligned}\tag{10}$$

where $\widehat{\rho}(t)$ is the dynamic compensation to ρ whose updating law will be given later.

For the adaptive state estimation $\widehat{w}(x,t)$ of $w(x,t)$, we present the following proposition to give the equations that $\widehat{w}(x,t)$ satisfies.

Proposition 2.1 *The adaptive state estimation $\widehat{w}(x,t)$ of $w(x,t)$ satisfies the following equations:*

$$\begin{cases}\widehat{w}_t(x,t)=\widehat{w}_x(x,t)+\dot{\widehat{\rho}}(t)\phi(x,t)+\theta(x)\left(v(0,t)-KX(t)\right),\\\widehat{w}(1,t)=\widehat{\rho}(t)U(t)-K\mathrm{e}^{A+BK}X(t).\end{cases}\tag{11}$$

Proof First, letting $x=1$ in (10) and using (6), we have that the second equation of (11) holds. Then, calculating the first order partial derivatives of $\widehat{w}(x,t)$ with respect to t and x, respectively, it yields

$$\begin{aligned}\widehat{w}_t(x,t)=\widehat{\rho}(t)\phi_t(x,t)+\dot{\widehat{\rho}}(t)\phi(x,t)-K\mathrm{e}^{A+BK}N_t(x,t)+\int_x^1\theta(\xi)\varphi_t(1+x-\xi,t)d\xi\\-\int_x^1\theta(\xi)KN_t(1+x-\xi,t)d\xi,\end{aligned}\tag{12}$$

$$\begin{aligned}\widehat{w}_x(x,t)=\widehat{\rho}(t)\phi_x(x,t)-K\mathrm{e}^{A+BK}N_x(x,t)+\int_x^1\theta(\xi)\varphi_x(1+x-\xi,t)d\xi\\-\int_x^1\theta(\xi)KN_x(1+x-\xi,t)d\xi-\theta(x)\left(v(0,t)-KX(t)\right).\end{aligned}\tag{13}$$

Subtracting both sides of (12) and (13), and by (6), we obtain the first equation of (11). ∎

2.3 Design of the Adaptive Controller

In this subsection, we will first design the updating law of $\widehat{\rho}(t)$ by gradient algorithm, and then derive the explicit controller by infinite-dimensional backstepping method.

Define $\widehat{e}(x,t) = w(x,t) - \widehat{w}(x,t)$, and hence there holds $\widehat{e}(0,t) = w(0,t) - \widehat{w}(0,t)$. Then, by letting $x = 0$ in (7) and (10) and noting that $\overline{e}(0,t) = w(0,t) - \overline{w}(0,t)$, we have

$$\widehat{e}(0,t) = \overline{e}(0,t) + \overline{w}(0,t) - \widehat{w}(0,t) = \overline{e}(0,t) + (\rho - \widehat{\rho}(t))\,\phi(0,t).$$

By this, we define the following normalized performance index:

$$J(\widehat{\rho}(t)) = \frac{\widehat{\mathrm{e}}^2(0,t)}{2(1+\phi^2(0,t))}.$$

Note that $\overline{e}(0,t) = 0$ on $[1, +\infty)$, and hence there holds $\widehat{e}(0,t) = (\rho - \widehat{\rho}(t))\,\phi(0,t)$ on $[1, +\infty)$. Then, by gradient algorithm, we choose the following updating law:

$$\dot{\widehat{\rho}}(t) = \begin{cases} 0, & 0 \le t < 1, \\ \gamma_1 \mathrm{Proj}_{[\underline{\rho},\overline{\rho}]}\left\{ \dfrac{\widehat{e}(0,t)}{1+\phi^2(0,t)}\phi(0,t), \widehat{\rho}(t) \right\}, & t \ge 1, \end{cases} \tag{14}$$

where γ_1 is a positive constant; $\widehat{\rho}(0) \in [\underline{\rho}, \overline{\rho}]$ and the projection operator $\mathrm{Proj}_{[\cdot,\cdot]}\{\cdot,\cdot\}$ is defined by

$$\mathrm{Proj}_{[\underline{\rho},\overline{\rho}]}\{\varepsilon, \widehat{\rho}\} \;=\; \begin{cases} 0, & \text{if } \widehat{\rho} = \underline{\rho} \text{ and } \varepsilon \;<\; 0, \\ 0, & \text{if } \widehat{\rho} = \overline{\rho} \text{ and } \varepsilon \;>\; 0, \\ \varepsilon, & \text{else}. \end{cases} \tag{15}$$

To give the explicit form of controller $U(t)$, the following invertible transformation for System (11) is introduced:

$$\zeta(x,t) \;=\; \widehat{w}(x,t) - \int_0^x g(x-\xi)\widehat{w}(\xi,t)d\xi, \tag{16}$$

where

$$g(x) \;=\; \sum_{i=1}^{+\infty} g_i(x), \tag{17}$$

with

$$\begin{cases} g_1(x) = -\theta(x), \\ g_i(x) = \displaystyle\int_0^x g_{i-1}(x-\xi)\theta(\xi)d\xi, \quad i \ge 2. \end{cases} \tag{18}$$

It is necessary to point out that the above infinite series is convergent on $[0,1]$ by the method of successive approximation[21]. Then, there exists a constant M_g such that $|g(x)| \le M_g$,

$\forall x \in [0,1]$. Moreover, it can be verified from (17) and (18) that $g(x)$ satisfies the following equality:

$$g(x) \;=\; \int_0^x g(x-\xi)\theta(\xi)d\xi - \theta(x). \tag{19}$$

Then, the inverse transformation of (16) is given by

$$\widehat{w}(x,t) \;=\; \zeta(x,t) - \int_0^x \theta(x-\xi)\zeta(\xi,t)d\xi, \tag{20}$$

which can be verified by inserting (20) into (16) and using (19).

Under transformation (16), System (11) is transformed into a new one, from which the explicit controller is more convenient to design.

Proposition 2.2 *Under transformation* (16)*, System* (11) *is transformed into*

$$\begin{cases} \zeta_t(x,t)=\zeta_x(x,t)-g(x)\widehat{e}(0,t) + \dot{\widehat{\rho}}(t)\phi(x,t) - \dot{\widehat{\rho}}(t)\displaystyle\int_0^x g(x-\xi)\phi(\xi,t)\mathrm{d}\xi, \\ \zeta(1,t) = \widehat{\rho}(t)U(t) - \displaystyle\int_0^1 g(1-\xi)\widehat{w}(\xi,t)\mathrm{d}\xi - K\mathrm{e}^{A+BK}X(t). \end{cases} \tag{21}$$

Moreover, there exist positive constants M_1 and M_2 such that

$$\|\zeta(\cdot,t)\| \le M_1\|\widehat{w}(\cdot,t)\|, \quad \|\widehat{w}(\cdot,t)\| \le M_2\|\zeta(\cdot,t)\|. \tag{22}$$

Proof See Appendix A.1. ∎

Letting $\zeta(1,t)=0$ in the second equation of (21), we obtain the following explicit controller:

$$U(t) = \frac{1}{\widehat{\rho}(t)}\left(\int_0^1 g(1-\xi)\widehat{w}(\xi,t)d\xi + K\mathrm{e}^{A+BK}X(t)\right). \tag{23}$$

It is worth pointing out that controller (23) depends on $X(t)$, $\widehat{w}(x,t)$ and $\widehat{\rho}(t)$ which are all available for feedback, and hence is implementable although the foregoing transformation (2) depends on unknown parameter $\lambda(x)$.

3 Stability Analysis

Before giving the main results, we present the following proposition to collect two properties of updating law (14), which will be frequently used in the later stability analysis.

Proposition 3.1 *For updating law* (14) *with projection operator* (15)*, the following claims hold:*

(i) $\frac{|\widehat{e}(0,t)|}{\sqrt{1+\phi^2(0,t)}}$ *is bounded and square integrable on* $[1,+\infty)$*,*

(ii) $\left|\dot{\widehat{\rho}}(t)\right|$ *is bounded and square integrable on* $[0,+\infty)$*.*

Proof To prove claim (i), the following Lyapunov function is chosen:

$$V(t) \;=\; \frac{1}{2\gamma_1}\widetilde{\rho}^2(t),$$

where $\widetilde{\rho}(t) = \rho - \widehat{\rho}(t)$. Calculating the time derivative of $V(t)$, and then using (14) and claim (ii) of Proposition 5.1 in Appendix A.3, we obtain

$$\begin{cases} \dot{V}(t) = 0, & 0 \le t < 1, \\ \dot{V}(t) \le -\dfrac{\widehat{e}(0,t)}{1+\phi^2(0,t)}\widetilde{\rho}(t)\phi(0,t), & t \ge 1. \end{cases} \tag{24}$$

Noting that $\widehat{e}(0,t) = \widetilde{\rho}(t)\phi(0,t)$ on $[1, +\infty)$, there holds

$$\dot{V}(t) \le -\frac{\widehat{e}^2(0,t)}{1+\phi^2(0,t)}, \quad \forall t \ge 1, \tag{25}$$

which, together with the first equality of (24), implies that $V(t)$ is nonincreasing and hence bounded on $[0, +\infty)$. Then, integrating (25) with respect to t over $[1, +\infty)$ gives $\int_1^{+\infty} \frac{\widehat{e}^2(0,t)}{1+\phi^2(0,t)} dt < +\infty$, which implies that $\frac{|\widehat{e}(0,t)|}{\sqrt{1+\phi^2(0,t)}}$ is square integrable on $[1, +\infty)$.

Moreover, by (5) and claim (i) of Proposition 5.1 in Appendix A.3, we conclude that $\widetilde{\rho}(t)$ is bounded on $[0, +\infty)$. Then, from the fact that $\widehat{e}(0,t) = \widetilde{\rho}(t)\phi(0,t)$ on $[1, +\infty)$, we have

$$\frac{|\widehat{e}(0,t)|}{\sqrt{1+\phi^2(0,t)}} \le |\widetilde{\rho}(t)| \frac{|\phi(0,t)|}{\sqrt{1+\phi^2(0,t)}} \le |\widetilde{\rho}(t)|, \quad \forall t \ge 1.$$

This, together with the boundedness of $\widetilde{\rho}(t)$, gives that $\frac{|\widehat{e}(0,t)|}{\sqrt{1+\phi^2(0,t)}}$ is bounded on $[1, +\infty)$.

We finally prove claim (ii). Using claim (i) of Proposition 5.1 in Appendix A.3, we obtain

$$\begin{aligned} |\dot{\widehat{\rho}}(t)| &\le \gamma_1 \frac{|\widehat{e}(0,t)|}{\sqrt{1+\phi^2(0,t)}} \frac{|\phi(0,t)|}{\sqrt{1+\phi^2(0,t)}} \\ &\le \gamma_1 \frac{|\widehat{e}(0,t)|}{\sqrt{1+\phi^2(0,t)}}, \quad \forall t \ge 1, \end{aligned}$$

which, together with claim (i) and noting $\dot{\widehat{\rho}}(t) = 0$ on $[0, 1)$, implies that claim (ii) holds. ▮

Now we are in a position to present the main results of this paper which are summarized in the following theorem.

Theorem 3.2 *Consider System* (1) *under Assumptions* 1 *and* 2*. The proposed adaptive controller consisting of* (6), (10), (14) *and* (23) *guarantees that all the resulting closed-loop system states* $X(t)$, $v(x,t)$, $\phi(x,t)$, $\varphi(x,t)$, $N(x,t)$, $\widehat{w}(x,t)$ *and* $\widehat{\rho}(t)$ *are bounded on their separate domains of definition while the original system states* $X(t)$ *and* $v(x,t)$ *converging to zero, i.e.,*

$$\lim_{t\to+\infty} X(t) = 0, \quad \lim_{t\to+\infty} \sup_{x\in[0,1]} |v(x,t)| = 0.$$

Proof The proof is divided into two parts. The first part shows the boundedness of all states of the resulting closed-loop system and the second one shows the convergence of the original system states.

(i) Proof of the boundedness of all states of the resulting closed-loop system.

First, we choose the following four Lyapunov functions:

$$\begin{aligned}
V_1(t) =& \ X^{\mathrm{T}}(t)PX(t),\\
V_2(t) =& \ \frac{1}{2}\int_0^1 (1+x)\zeta^2(x,t)dx,\\
V_3(t) =& \ \frac{1}{2}\int_0^1 (1+x)\phi^2(x,t)dx,\\
V_4(t) =& \ \frac{1}{2}\int_0^1 (1+x)\varphi^2(x,t)dx,
\end{aligned}$$

where $P = P^{\mathrm{T}} > 0$ is the solution of $(A+BK)^{\mathrm{T}}P + P(A+BK) = -Q$ for some $Q = Q^{\mathrm{T}} > 0$. Calculating the time derivatives of $V_i(t)$, $i = 1,2,3,4$ (see Appendix A.2 for the detailed derivation), the following four inequalities are obtained:

$$\dot V_1(t) \le -\frac{\lambda_{\min}(Q)}{2}\|X(t)\|^2 + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\zeta^2(0,t) + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t) + l_1(t), \tag{26}$$

$$\dot V_2(t) \le 8M_g^2\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t) - \frac{1}{2}\zeta^2(0,t) - \frac{1}{4}V_2(t) + l_2(t)V_3(t) + l_3(t), \tag{27}$$

$$\dot V_3(t) \le h_1V_2(t) + h_2\|X(t)\|^2 - \frac{1}{2}\phi^2(0,t) - \frac{1}{2}V_3(t), \tag{28}$$

$$\dot V_4(t) \le 3\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t) + 3\zeta^2(0,t) - \frac{1}{2}V_4(t) + 3\|K\|^2\|X(t)\|^2 - \frac{1}{2}\varphi^2(0,t) + l_4(t), \tag{29}$$

where h_1 and h_2 are positive constants, $l_i(t),\ i = 1,2,3,4$ are bounded and integrable functions defined on $[1,+\infty)$ whose specified forms are given in Appendix A.2.

Then, with $V_i(t)$'s in hand, we define

$$V_5(t) = k_1V_1(t) + k_2V_2(t) + V_3(t) + V_4(t),$$

where $k_1 > \frac{2}{\lambda_{\min}(Q)}\left(h_2 + 3\|K\|^2\right)$ and $k_2 > \max\left\{k_1\frac{8\|PB\|^2}{\lambda_{\min}(Q)} + 6, 4h_1\right\}$. Then, using (26)–(29), we obtain

$$\begin{aligned}
\dot V_5(t) \le& \left(k_1\frac{4\|PB\|^2}{\lambda_{\min}(Q)} + 8k_2M_g^2 + 3\right)\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t)\\
&-\frac{1}{2}\phi^2(0,t) - c_1V_5(t) + l_5(t)V_5(t) + l_6(t),
\end{aligned} \tag{30}$$

where $c_1 = \frac{1}{\max\{k_1,k_2,1\}}\min\left\{\frac{k_1\frac{\lambda_{\min}(Q)}{2} - 3\|K\|^2 - h_2}{\lambda_{\max}(P)}, \frac{k_2}{4} - h_1, \frac{1}{4}\right\}$, $l_5(t) = k_2l_2(t)$ and $l_6(t) = k_1l_1(t) + k_2l_3(t) + l_4(t)$.

To prove the boundedness of the resulting closed-loop system states, we first show that $V_5(t)$ is bounded on $[0,+\infty)$ by a contradiction argument. Suppose that $V_5(t)$ is unbounded on $[1,+\infty)$, then there exists a set $S(t) \subset [1,+\infty)$ whose measure increases unboundedly as $t \to +\infty$ such that for $s \in S(t)$,

$$\frac{\widehat{\mathrm{e}}^2(0,s)}{1+\phi^2(0,s)} > \frac{1}{2}\left(k_1\frac{4\|PB\|^2}{\lambda_{\min}(Q)} + 8k_2M_g^2 + 3\right)^{-1}.$$

This, together with (25), implies that $\lim_{t\to+\infty} V(t) = -\infty$, which contradicts the definition of $V(t)$. Thus, $V_5(t)$ is bounded on $[1, +\infty)$. From this and the continuity of $V_5(t)$ on $[0, 1]$, it follows that $V_5(t)$ is bounded on $[0, +\infty)$.

By the definition of $V_5(t)$ and its boundedness, we see that $X(t)$, $\|\zeta(\cdot, t)\|$, $\|\phi(\cdot, t)\|$ and $\|\varphi(\cdot, t)\|$ are all bounded on $[0, +\infty)$. From the third equation of (6) and the boundedness of $X(t)$ on $[0, +\infty)$, it follows that $N(x, t)$ is bounded on $[0, 1] \times [0, +\infty)$. In addition, by (22) and the fact that $\|\zeta(\cdot, t)\|$ is bounded on $[0, +\infty)$, we have that $\|\widehat{w}(\cdot, t)\|$ is bounded on $[0, +\infty)$. This, together with (23) and the boundedness of $\widehat{\rho}(t)$, $g(x)$ and $X(t)$ on their separate domains of definition, gives that $U(t)$ is bounded on $[0, +\infty)$. Then, the first equation of (6) indicates the boundedness of $\phi(x, t)$ on $[0, 1] \times [0, +\infty)$. Subsequently, (7) and (10) respectively imply that $\overline{w}(x, t)$ and $\widehat{w}(x, t)$ are bounded on $[0, 1] \times [0, +\infty)$ by noting the boundedness of $\widehat{\rho}(t)$, $\phi(x, t)$, $N(x, t)$, $\theta(x)$ and $\|\varphi(\cdot, t)\|$.

Since both $\overline{w}(x, t)$ and $\overline{e}(x, t)$ are bounded on $[0, 1] \times [0, +\infty)$, we obtain that $w(x, t)$ is bounded on $[0, 1] \times [0, +\infty)$. Then, by transformation (3) and noting the boundedness of $\mu(x)$ and $X(t)$, we conclude that $v(x, t)$ is bounded on $[0, 1] \times [0, +\infty)$. This, together with the second equation of (6), gives the boundedness of $\varphi(x, t)$ on $[0, 1] \times [0, +\infty)$.

In summary, we obtain the boundedness of all the resulting closed-loop system states $X(t)$, $v(x, t)$, $\phi(x, t)$, $\varphi(x, t)$, $N(x, t)$, $\widehat{w}(x, t)$ and $\widehat{\rho}(t)$ on their separate domains of definition.

(ii) Proof of the convergence of original system states $X(t)$ and $v(x, t)$.

From (30), we have

$$\dot{V}_5(t) \le -c_1 V_5(t) + l_5(t) V_5(t) + l_7(t), \tag{31}$$

with

$$l_7(t) = l_6(t) + \left(k_1 \frac{4\|PB\|^2}{\lambda_{\min}(Q)} + 8k_2 M_g^2 + 3\right) \frac{\widehat{\mathrm{e}}^2(0, t)}{1 + \phi^2(0, t)} \phi^2(0, t)$$

being bounded and integrable on $[1, +\infty)$. Then, by Lemma 5.2 in Appendix A.3, we obtain that $V_5(t)$ is bounded and integrable on $[1, +\infty)$. Thus, (31) gives that $\dot{V}_5(t)$ is bounded on $[1, +\infty)$. By Lemma 5.3 in Appendix A.3, it follows that

$$\lim_{t\to+\infty} V_5(t) \ = \ 0,$$

which gives

$$\lim_{t\to+\infty} \|\zeta(\cdot, t)\| = \lim_{t\to+\infty} \|\phi(\cdot, t)\| = 0, \qquad \lim_{t\to+\infty} X(t) = 0. \tag{32}$$

Then, (22) gives that $\lim_{t\to+\infty} \|\widehat{w}(\cdot, t)\| = 0$, and hence (23) indicates that $\lim_{t\to+\infty} U(t) = 0$ by noting the boundedness of $\widehat{\rho}(t)$ and $g(x)$. Thus, by the first and third equations of (6), we obtain

$$\lim_{t\to+\infty} \sup_{x\in[0,1]} |\phi(x, t)| \ = \ 0 \text{ and } \lim_{t\to+\infty} \sup_{x\in[0,1]} |N(x, t)| \ = \ 0. \tag{33}$$

By (7), (32), (33) and the boundedness of $\theta(x)$, we have $\lim_{t\to+\infty} \sup_{x\in[0,1]} |\overline{w}(x, t)| = 0$, and hence $\lim_{t\to+\infty} \sup_{x\in[0,1]} |w(x, t)| = 0$. Thus, by (3), (32) and the boundedness of $\mu(x)$, it follows that

$$\lim_{t\to+\infty}\sup_{x\in[0,1]}|v(x,t)|=0.$$

This completes the proof. ∎

4 Simulation Example

In this section, we illustrate the effectiveness of the proposed method by the following system:

$$\begin{cases}\dot{X}(t) &= 0.1X(t)+v(0,t),\\ v_t(x,t) &= v_x(x,t)+\lambda(x)v(x,t),\\ v(1,t) &= kU(t),\end{cases}$$

where $X(t)\in\boldsymbol{R}$; the initial values are $X(0)=1$ and $v(x,0)=3\sin(2\pi x)$; the actual values of system parameters are assumed as $k=0.9$ and $\lambda(x)=2.3+\frac{1}{10+x}$. Choose $\underline{k}=0.8$, $\overline{k}=1$, $\underline{\lambda}=2$ and $\overline{\lambda}=2.5$. Then, we have $\underline{\rho}=0.8\,\mathrm{e}^2$ and $\overline{\rho}=\mathrm{e}^{2.5}$.

Let $K=-1$, $\gamma_1=0.1$ and $\widehat{\rho}(0)=8$ in the controller consisting of (6), (10), (14) and (23). We implement the simulation by the explicit forward Euler method (see Page 406 of [22]) by dividing the spatial domain $[0,1]$ and the time domain $[0,15]$ into 30 sections and 15000 sections, respectively, and then obtain three simulation figures. Specifically, Figures 1 and 2 show that the ODE subsystem state $X(t)$ and the PDE subsystem state $v(x,t)$ all converge to zero ultimately. Figure 3 shows that the dynamic compensation $\widehat{\rho}(t)$ of parameter ρ always belongs to $[0.8\,\mathrm{e}^2,\mathrm{e}^{2.5}]$.

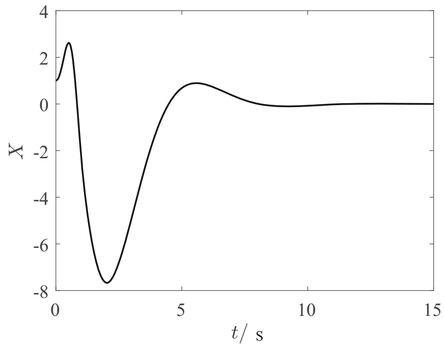

Figure 1 The trajectory of $X(t)$

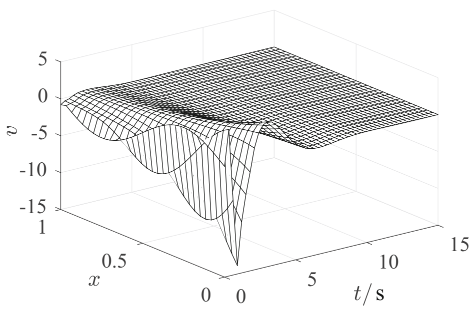

Figure 2 The trajectory of $v(x,t)$

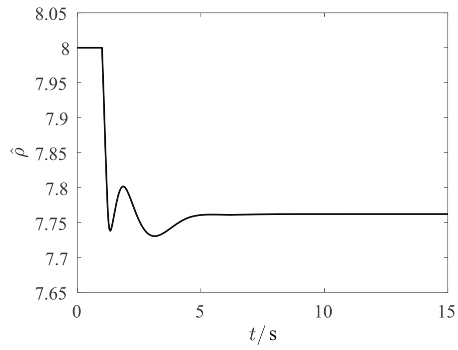

Figure 3 The trajectory of $\widehat{\rho}(t)$

5 Concluding Remarks

In this paper, the adaptive output-feedback stabilization has been investigated for a class of first-order hyperbolic PDE-ODE cascaded systems with unknown control coefficient and spatially varying parameter. The presence of unknown parameters makes the considered system more general and practical than those in the related literature. By combining infinite-dimensional backstepping method with adaptive dynamic compensation technique and the constructive methods of filters, an adaptive controller is constructed which guarantees the desirable performance of the resulting closed-loop system. It is necessary to point out that, the system investigated in this paper only has parametric uncertainties. However, the physical systems are often suffered from external disturbances. Therefore, it is meaningful to investigate the output-feedback stabilization for first-order hyperbolic PDE-ODE cascaded systems with both disturbances and unknown parameters in the future.

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Appendix

This section collects some useful criteria and the proofs of Proposition 2.2 and Inequalities (26)–(29) in this paper.

A.1 Proof of Proposition 2.2

First, setting $x = 1$ in (16) and using (11), we directly obtain the first equation of (21). Moreover, calculating the first order partial derivatives of $\zeta(x,t)$ with respect to t and x, respectively, we get

$$\begin{aligned}\zeta_t(x,t) &= \widehat{w}_t(x,t) - \int_0^x g(x-\xi)\widehat{w}_t(\xi,t)d\xi,\\ &= \widehat{w}_x(x,t) + \dot{\widehat{\rho}}(t)\phi(x,t) + \theta(x)(v(0,t) - KX(t)) - \int_0^x g(x-\xi)\widehat{w}_\xi(\xi,t)d\xi\\ &\quad -\dot{\widehat{\rho}}(t)\int_0^x g(x-\xi)\phi(\xi,t)d\xi - \int_0^x g(x-\xi)\theta(\xi)d\xi(v(0,t) - KX(t)), \qquad (A.1)\\ \zeta_x(x,t) &= \widehat{w}_x(x,t) - \int_0^x g'(x-\xi)\widehat{w}(\xi,t)d\xi - g(0)\widehat{w}(x,t)\\ &= \widehat{w}_x(x,t) - \int_0^x g(x-\xi)\widehat{w}_\xi(\xi,t)d\xi - g(x)\widehat{w}(0,t). \qquad (A.2)\end{aligned}$$

Then, noting $w(0,t) = v(0,t) - KX(t)$ and by (19), (A.1) and (A.2), the following equality is obtained:

$$\zeta_t(x,t) = \zeta_x(x,t) - g(x)\widehat{e}(0,t) + \dot{\widehat{\rho}}(t)\phi(x,t) - \dot{\widehat{\rho}}(t)\int_0^x g(x-\xi)\phi(\xi,t)d\xi.$$

By (16), (20) and the boundedness of $\theta(x)$ and $g(x)$ on $[0,1]$, we know that there exist positive constants M_1 and M_2 such that $\|\zeta(\cdot,t)\| \le M_1\|\widehat{w}(\cdot,t)\|$ and $\|\widehat{w}(\cdot,t)\| \le M_2\|\zeta(\cdot,t)\|$. ∎

A.2 Proofs of (26)–(29)

To prove (26), taking the time derivative of $V_1(t)$ and using Young's inequality, we get

$$\begin{aligned}\dot{V}_1(t) &= X^{\mathrm{T}}(t)\left((A+BK)^{\mathrm{T}}P + P(A+BK)\right)X(t) + 2X^{\mathrm{T}}(t)PBw(0,t)\\ &= -X^{\mathrm{T}}(t)QX(t) + 2X^{\mathrm{T}}(t)PB\zeta(0,t) + 2X^{\mathrm{T}}(t)PB\widehat{e}(0,t)\\ &\le -\lambda_{\min}(Q)\|X(t)\|^2 + \rho_1\|X(t)\|^2\|PB\|^2 + \frac{2}{\rho_1}\zeta^2(0,t) + \frac{2}{\rho_1}\widehat{e}^2(0,t),\end{aligned}$$

by which and choosing $\rho_1 = \frac{\lambda_{\min}(Q)}{2\|PB\|^2}$, we obtain

$$\dot{V}_1(t) \le -\lambda_{\min}(Q)\|X(t)\|^2 + \frac{\lambda_{\min}(Q)}{2}\|X(t)\|^2 + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\zeta^2(0,t) + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\widehat{e}^2(0,t)$$

$$\le -\frac{\lambda_{\min}(Q)}{2}\|X(t)\|^2 + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\zeta^2(0,t) + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}(1+\phi^2(0,t))$$

$$\le -\frac{\lambda_{\min}(Q)}{2}\|X(t)\|^2 + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\zeta^2(0,t) + \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t) + l_1(t),$$

with $l_1(t) = \frac{4\|PB\|^2}{\lambda_{\min}(Q)}\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}$ being bounded and integrable on $[1,+\infty)$ by claim (i) of Proposition 3.1.

To prove (27), computing the time derivative of $V_2(t)$ and noting $\zeta(1,t)=0$, we have

$$\begin{aligned}\dot{V}_2(t) = &-\frac{1}{2}\zeta^2(0,t) - \frac{1}{2}\int_0^1 \zeta^2(x,t)dx - \widehat{e}(0,t)\int_0^1 (1+x)\zeta(x,t)g(x)dx\\ &+\dot{\widehat{\rho}}(t)\int_0^1(1+x)\zeta(x,t)\phi(x,t)dx - \dot{\widehat{\rho}}(t)\int_0^1(1+x)\zeta(x,t)\int_0^x g(x-\xi)\phi(\xi,t)d\xi dx,\end{aligned}$$

which, together with Young's inequality, gives

$$\begin{aligned}\dot{V}_2(t) \le &-\frac{1}{2}\zeta^2(0,t) - \frac{1}{2}\int_0^1 \zeta^2(x,t)dx + (\rho_2+3\rho_3)\int_0^1(1+x)\zeta^2(x,t)dx\\ &+\frac{1}{4\rho_2}\widehat{\mathrm{e}}^2(0,t)\int_0^1(1+x)g^2(x)dx + \frac{1}{4\rho_3}|\dot{\widehat{\rho}}(t)|^2\int_0^1(1+x)\phi^2(x,t)dx\\ &+\frac{1}{4\rho_3}M_g^2|\dot{\widehat{\rho}}(t)|^2\int_0^1(1+x)\phi^2(x,t)dx\\ \le &-\frac{1}{2}\zeta^2(0,t) + \frac{1}{2\rho_2}M_g^2\widehat{\mathrm{e}}^2(0,t) + \frac{1}{2\rho_3}(1+M_g^2)|\dot{\widehat{\rho}}(t)|^2\int_0^1\phi^2(x,t)dx\\ &-\left(\frac{1}{2}-2\rho_2-6\rho_3\right)V_2(t).\end{aligned}$$

Then, by choosing $\rho_2 = \frac{1}{16}$ and $\rho_3 = \frac{1}{48}$, we get

$$\begin{aligned}\dot{V}_2(t) &\le 8M_g^2\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}(1+\phi^2(0,t)) - \frac{1}{2}\zeta^2(0,t) + 48(1+M_g^2)|\dot{\widehat{\rho}}(t)|^2V_3(t) - \frac{1}{4}V_2(t)\\ &\le 8M_g^2\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t) - \frac{1}{2}\zeta^2(0,t) - \frac{1}{4}V_2(t) + l_2(t)V_3(t) + l_3(t),\end{aligned}$$

with $l_2(t) = 48(1+M_g^2)|\dot{\widehat{\rho}}(t)|^2$, $l_3(t) = 8M_g^2\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}$ being bounded and integrable on $[1,+\infty)$ by Proposition 3.1.

Similar to the derivation of (27), for $V_3(t)$ and $V_4(t)$, there hold

$$\begin{aligned}&\dot{V}_3(t) \le h_1V_2(t) + h_2\|X(t)\|^2 - \frac{1}{2}\phi^2(0,t) - \frac{1}{2}V_3(t),\\ &\dot{V}_4(t) \le 3\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}\phi^2(0,t) + 3\zeta^2(0,t) - \frac{1}{2}V_4(t) + 3\|K\|^2\|X(t)\|^2 - \frac{1}{2}\varphi^2(0,t) + l_4(t),\end{aligned}$$

where $h_1 = 4M_2^2M_g^2M_3^2$, $h_2 = 2M_3^2\|K\mathrm{e}^{A+BK}\|^2$ with $M_3 = \max\left\{\frac{1}{|\underline{\rho}|}, \frac{1}{|\overline{\rho}|}\right\}$ and $l_4(t) = 3\frac{\widehat{\mathrm{e}}^2(0,t)}{1+\phi^2(0,t)}$ is bounded and integrable on $[1,+\infty)$. ∎

A.3 Some Useful Criteria

Proposition 5.1 (see [23]) *For the projection operator defined by* (15), *the following properties are guaranteed:*

(i) *For* $\widehat{\rho}(0) \in [\underline{\rho}, \overline{\rho}]$ *and any* ε, *the solution of* $\dot{\widehat{\rho}}(t) = \mathrm{Proj}_{[\underline{\rho},\overline{\rho}]}\{\varepsilon, \widehat{\rho}\}$ *remains in* $[\underline{\rho}, \overline{\rho}]$, *and moreover,* $\left|\mathrm{Proj}_{[\underline{\rho},\overline{\rho}]}\{\varepsilon, \widehat{\rho}\}\right| \leq |\varepsilon|$.

(ii) *If* $\underline{\rho} \leq \widehat{\rho}(t) \leq \overline{\rho}$ *and* $\underline{\rho} \leq \rho \leq \overline{\rho}$, *then* $-\widetilde{\rho}(t)\mathrm{Proj}_{[\underline{\rho},\overline{\rho}]}\{\varepsilon, \widehat{\rho}\} \leq -\widetilde{\rho}(t)\varepsilon$, *where* $\widetilde{\rho}(t) = \rho - \widehat{\rho}(t)$.

Lemma 5.2 (see [24]) *Let* $f(t)$, $l_1(t)$ *and* $l_2(t)$ *be real-valued functions defined on* $\boldsymbol{R}_+$ *and let* c *be a positive constant. If* $l_1(t)$ *and* $l_2(t)$ *are nonnegative and integrable on* $\boldsymbol{R}_+$, *and satisfy*

$$\dot{f}(t) \ \leq \ -cf(t) + l_1(t)f(t) + l_2(t), \quad f(0) \geq 0,$$

then $f(t)$ *is bounded and integrable on* $[0, +\infty)$.

Lemma 5.3 (see [25]) *If* $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ *is uniformly continuous and* $\lim_{t\to+\infty}\int_0^t f(\tau)d\tau$ *exists and is bounded, then there holds* $\lim_{t\to+\infty} f(t) = 0$.