J Syst Sci Complex (2020) 33: 1485–1496

# Controllability of Singular Distributed Parameter Systems in the Sense of Mild Solution*

**GE Zhaoqiang · GE Xiaochi**

DOI: 10.1007/s11424-020-9012-4
Received: 8 January 2019 / Revised: 27 February 2019


**Abstract** Necessary and sufficient conditions for the exact controllability and approximate controllability of a singular distributed parameter system are obtained. These general results are used to examine the exact controllability and approximate controllability of the Dzektser equation in the theory of seepage.

**Keywords** Approximate controllability, exact controllability, Hilbert space, mild solution, singular distributed parameter systems.

## 1 Introduction

A large number of works deal with control systems described by abstract Sobolev type evolution equations (e.g., [1–8]). Below we shall consider the problems of exact controllability and approximate controllability for control systems described by abstract singular distributed parameter systems.

Let $X, Y, U$ be Hilbert spaces. Also let $L(X,Y)$ be the space of all bounded linear operators from $X$ into $Y, L(X) = L(X,Y)$. The singular distributed parameter system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad 0 \le t \le T, \quad x(0) = x_0, \tag{1}$$

where $E \in L(X,Y), A$ is a closed linear operator from $X$ into $Y$ whose domain is dense in $X$ and $B \in L(U,Y)$, is an abstract form of various partial differential equations and systems of equations which occur in many applications (see, e.g., [5–8]).

For the sake of convenience, we introduce the following definition.

GE Zhaoqiang
*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an* 710049, *China.*
Email: gezqjd@mail.xjtu.edu.cn.
GE Xiaochi
*Cass Business School, City, University of London, London,* EC1Y 8TE, *UK.*
Email: Xiaochi.Ge@cass.city.ac.uk.
*This research was supported by the National Natural Science Foundation of China under Grant Nos. 61174081 and 61273135.
◇ *This paper was recommended for publication by Editor HONG Yiguang.*

**Definition 1.1** System (1) is called a regular system with order $n$ (positive integer or infinite) if there exist Hilbert spaces $X_1, X_2$ and $P \in L(Y, X_1 \times X_2), Q \in L(X_1 \times X_2, X)$, where $P$ is injective and $Q$ is bijective, such that

$$PEQ = \begin{bmatrix} I_1 & 0 \\ 0 & N \end{bmatrix} \in L(X_1 \times X_2), \quad PAQ = \begin{bmatrix} K & 0 \\ 0 & I_2 \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in L(U, X_1 \times X_2),$$

where $N$ is a nilpotent operator with order $n$ (see [9]), $K$ is the generator of the strongly continuous semigroup $\mathrm{e}^{Kt}$ (see e.g., [10]), $PAQ$ is a closed linear operator on $X_1 \times X_2$ whose domain is dense in $X_1 \times X_2$, $I_k \in L(X_k)$ is the identical operator ($k = 1, 2$).

In this case, the operators $P$ and $Q$ transfer (1) into the following decoupled system on Hilbert space $X_1 \times X_2$

$$\dot{x}_1(t) = Kx_1(t) + B_1u(t), \quad 0 \le t \le T, \quad x_1(0) = x_{10}, \tag{2}$$

$$N\dot{x}_2(t) = x_2(t) + B_2u(t), \quad 0 \le t \le T, \quad x_2(0) = x_{20}, \tag{3}$$

where

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q^{-1}x, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = Q^{-1}x(0).$$

The system represented by (2)–(3) is called the standard form of regular system (1).

It was proved that many systems are regular system for example, Navier-Stokes systems, robotic system, the system modelling the free surface evolution of filtered fluid, the system modelling the moisture transfer in soil, the modeling of multi-body mechanisms, finance system, input-output economics system, the problem of protein folding and so on (see, e.g., [9, 11–15]). Subsystem (2) is a classical system in control theory. The properties of (3) determine the peculiarities of (1). For example, it is known that controls from the class $C^{n-1}([0, +\infty), U)$ must be used to solve (3) in the weak sense; the system may fail to have solution for controls with less smoothness (see, e.g., [9, 13]). The exact null controllability of the system (2)–(3) with scalar control has been studied in [11].

This paper investigate the exact controllability and approximate controllability of the singular system (1) under some additional hypotheses, or, equivalently, of (2)–(3). The general results obtained are used to examine the exact controllability and approximate controllability of the Dzektser equation in the theory of seepage.

Throughout the paper, $R$ denotes the set of real numbers; $C$ denotes the set of complex numbers; $C^n(J, X)$ denotes the set of $n$ times continuously differentiable $X$-valued functions on interval $J$; $\ker A$ denotes the kernel of $A$; $\mathrm{ran}A$ denotes the range of $A$; $\overline{\mathrm{ran}A}$ denotes the closure of $\mathrm{ran}A$; $\mathrm{dom}A$ denotes the domain of $A$; $A^*$ denotes the dual operator of $A$; $\langle \cdot, \cdot \rangle_X$ denotes the inner product on the space $X$; $\| \cdot \|_X$ denotes the norm induced by the inner product on the space $X$; $L^2([0, T], X)$ denotes the class of Lebesgue measurable function from $[0, T]$ to $X$ with $\int_0^T \|f(t)\|_X^2 dt < \infty$.

Here we give several auxiliary results. In the following, we assume that $n$ is a positive integer, i.e., $N$ is a nilpotent operator with order $n$ (see, e.g., [13]), and the system (1) is of the form (2)–(3).

**Theorem 1.2** (see [9]) *Suppose* $u \in C^n([0,T],U)$. *Then, for each initial value*

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \left\{ \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} : \eta_1 \in \mathrm{dom} K, \eta_2 = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(0) \right\},$$

*there exists a unique solution*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in C^1([0,T], X_1 \times X_2) \cap C\left([0,T], \mathrm{dom} \begin{bmatrix} K & 0 \\ 0 & I_2 \end{bmatrix}\right)$$

*of* (2)–(3). *Furthermore*

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{Kt} x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)} B_1 u(\tau) d\tau \\ -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(0)\} \end{bmatrix}. \tag{4}$$

In view of this result, it seems natural to extend the concept of mild solution to singular distributed parameter system of the form (2)–(3).

**Definition 1.3** For $\left[\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}\right] \in X_1 \times X_2$, a function $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in C([0,T], X_1 \times X_2)$ is called a mild solution of (2)–(3) if it has the form (4) and satisfies $\left[\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}\right]$.

**Theorem 1.4** (see [11]) *Assume* $u \in C^{n-1}([0,T],U)$. *Then, for any initial value*

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in \left\{ \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} : \eta_1 \in X_1, \eta_2 = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(0) \right\},$$

*there exists a unique mild solution* $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in C([0,T], X_1 \times X_2)$ *of* (2)–(3).

It is well known that a mild solution $x_1(t)$ of (2) is expressible for $x_{10} \in X_1, u \in L^2([0,T],U)$ by the formula

$$x_1(t) = \mathrm{e}^{Kt} x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)} B_1 u(\tau) d\tau,$$

where the integral is understood in the sense of Bochner (see, e.g., [10]).

Note that the first line of the matrix in (4), which gives a solution of (2)–(3), is a mild solution of (2), while the second line, which is a sum over $k$, is a solution of (3). Hence, in accordance with Definition 1.3, it is natural, for $x_{20} \in X_2, u \in C^{n-1}([0,T],U)$, to regard the function

$$x_2(t) = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(t) \tag{5}$$

as a mild solution of (3). It follows from (5) that (3) is solvable if and only if the compatibility condition between the right-hand side and the initial data

$$x_{20} = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(0) \tag{6}$$

is satisfied.

In the following definitions we shall assume by default that the solutions are mild.

**Definition 1.5** System (2)–(3) is called exactly controllable on $[0,T]$ if, for any $x_{1T} \in X_1, x_{2T} \in X_2$, any $x_{10} \in X_1, x_{20} \in X_2$, there exists a control $u \in C^{n-1}([0,T],U)$ such that (6) is satisfied and $x_1(T) = x_{1T}, x_2(T) = x_{2T}$.

**Definition 1.6** System (2)–(3) is called approximately controllable on $[0,T]$ if, for any $x_{1T} \in X_1, x_{2T} \in X_2$, any $x_{10} \in X_1, x_{20} \in X_2$, and any $\varepsilon > 0$, there exists a control $u \in C^{n-1}([0,T],U)$ such that (6) is satisfied and

$$\|x_1(T) - x_{1T}\|_{X_1} < \varepsilon, \quad \|x_2(T) - x_{2T}\|_{X_2} < \varepsilon.$$

Our purpose here is to establish necessary and sufficient conditions for the exact controllability and approximate controllability of (2)–(3) with bounded operators $B_1$ and $B_2$.

The next definition will be used in the next section.

**Definition 1.7** A number $\lambda \in C$ is called the $E$-eigenvalue of the operator $A$ if there exists a vector $x \neq 0$ such that $\lambda Ex = Ax$. Such a vector $x$ is called the $E$-eigenvector of the operator $A$ corresponding to the $E$-eigenvalue $\lambda$.

It is easily verified that the $E$-eigenvectors corresponding to the same $E$-eigenvalue form a subspace of $X$.

## 2 Controllability of (2)

As for the exact controllability of (2), we have the following results.

**Theorem 2.1** (see [10]) *Subsystem* (2) *is exactly controllable on* $[0,T]$ *if and only if the following condition holds for some* $\gamma > 0$ *and for all* $z \in X_1$*:*

$$\left\langle \int_0^T \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} z d\tau, z \right\rangle_{X_1} \geq \gamma \|z\|_{X_1}^2. \tag{7}$$

According to Theorem 2.1, we obtain the following Theorem.

**Theorem 2.2** *Subsystem* (2) *is exactly controllable on* $[0,T]$ *if and only if for some* $\gamma > 0$ *and all* $z \in X_1$*:*

$$\left\langle \int_0^T f^2(\tau) \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} z d\tau, z \right\rangle_{X_1} \geq \gamma \|z\|_{X_1}^2, \tag{8}$$

*where* $f(\tau) = \tau^n(\tau - T)^n$*. In this case* $\int_0^T f^2(\tau)\mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} d\tau$ *has bounded inverse.*

*Proof* Sufficiency. Since $\max_{\tau\in[0,T]} f^2(\tau) = (T/2)^{4n}$, by (8), we have that

$$(T/2)^{4n} \left\langle \int_0^T \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} z d\tau, z \right\rangle_{X_1} \geq \left\langle \int_0^T f^2(\tau) \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} z d\tau, z \right\rangle_{X_1} \geq \gamma \|z\|_{X_1}^2.$$

Therefore, (7) is true. By Theorem 2.1, the subsystem (2) is exactly controllable on $[0,T]$.

Necessity. Assume (7). If (8) is false, then for any positive integer $m$, there exists $x_m \in X_1$ and $\|x_m\|_{X_1} = 1$, such that

$$\left\langle \int_0^T f^2(\tau) \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m d\tau, x_m \right\rangle_{X_1} < \frac{1}{m}. \tag{9}$$

Since $f^2(\tau)$ is an increasing function, when $\tau \in [0, T/2]$ and $f^2(\tau)$ is a decreasing function, when $\tau \in [T/2, T]$, we have that

$$f^2(\tau) \geq (1/m)^{1/2} [(1/m)^{1/(4n)} - T]^{2n}, \quad \tau \in [(1/m)^{1/(4n)}, T - (1/m)^{1/(4n)}].$$

By (9), we get that

$$\begin{aligned}
\frac{1}{m} &> \left(\frac{1}{m}\right)^{1/2} [(1/m)^{1/(4n)} - T]^{2n} \int_{(1/m)^{1/(4n)}}^{T-(1/m)^{1/(4n)}} \left\langle \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m, x_m \right\rangle_{X_1} d\tau \\
&= \left(\frac{1}{m}\right)^{1/2} [(1/m)^{1/(4n)} - T]^{2n} \Bigg[ \int_0^T \left\langle \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m, x_m \right\rangle_{X_1} d\tau \\
&\quad - \int_0^{(1/m)^{1/(4n)}} \left\langle \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m, x_m \right\rangle_{X_1} d\tau \\
&\quad - \int_{T-(1/m)^{1/(4n)}}^T \left\langle \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m, x_m \right\rangle_{X_1} d\tau \Bigg].
\end{aligned}$$

By (7), we obtain that

$$\begin{aligned}
(\frac{1}{m})^{1/2} > [(1/m)^{1/(4n)} - T]^{2n} \Bigg[ \gamma &- \int_0^{(1/m)^{1/(4n)}} \left\langle \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m, x_m \right\rangle_{X_1} d\tau \\
&- \int_{T-(1/m)^{1/(4n)}}^T \left\langle \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} x_m, x_m \right\rangle_{X_1} d\tau \Bigg].
\end{aligned}$$

As $m \to +\infty$, we have that $0 \geq T^{2n}\gamma > 0$. This contradiction indicates that (8) is true. By [10], we have that $\int_0^T f^2(\tau) \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} d\tau$ has bounded inverse. ∎

As for the approximate controllability of (2), we have the following results.

**Theorem 2.3** (see [10]) *Subsystem* (2) *is approximately controllable on* $[0, T]$ *if and only if any one of the following conditions hold:*

(i) $\int_0^T \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} d\tau > 0$,

(ii) $B_1^* \mathrm{e}^{K^*\tau} z = 0$ *on* $[0, T] \Rightarrow z = 0$.

According to Theorem 2.3, we can obtain the following theorem.

**Theorem 2.4** *Subsystem* (2) *is approximately controllable on* $[0, T]$ *if and only if*

$$G(f, T) = \int_0^T f^2(\tau) \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} d\tau > 0$$

*for any polynomial* $f(\tau) \in R$ *not identically zero and* $\overline{\mathrm{ran} G(f, T)} = X_1$.

*Proof* The sufficiency is obvious. We only need to prove the necessity. If $z \in \ker G(f,T)$, then

$$0 = \langle G(f,T)z, z\rangle_{X_1} = \int_0^T \|f(\tau)B_1^* e^{K^*\tau} z\|_{X_1}^2 d\tau$$

and thus, $0 = f(\tau)B_1^* e^{K^*\tau} z$ for $0 \le \tau \le T$. Since $f(\tau)$ can have only finitely many zeros in the interval $0 \le \tau \le T$, it follows that $B_1^* e^{K^*\tau} z = 0$ on $[0,T]$. By Theorem 2.3, we have $z = 0$. Therefore

$$G(f,T) = \int_0^T f^2(\tau) e^{K\tau} B_1 B_1^* e^{K^*\tau} d\tau > 0$$

and $\ker G(f,T) = \{0\}$. By [10], we obtain $\overline{\mathrm{ran} G(f,T)} = X_1$. ∎

## 3 Controllability of (3)

As for the exact controllability of (3), we have the following results.

**Theorem 3.1** *Subsystem* (3) *is exactly controllable on* $[0,T]$ *if and only if*

$$\mathrm{ran}[B_2 \quad NB_2 \quad \cdots \quad N^{n-1}B_2] = X_2. \tag{10}$$

*Proof* The exact controllability of (3) on $[0,T]$ implies, in particular, that it has a solution satisfying the initial condition $x_2(0) = x_{20}$ for any $x_{20} \in X_2$. By (6), we have $x_{20} \in \mathrm{ran}[B_2 \quad NB_2 \quad \cdots \quad N^{n-1}B_2]$. This proves that (10) is necessity. Now we prove the sufficiency. Since (10) holds, for any $x_{20} \in X_2$, there exist vectors $\alpha_k \in U, k = 0,1,\cdots,n-1$, such that $x_{20} = -\sum_{k=0}^{n-1} N^k B_2 \alpha_k$. Hence, the condition (6) is satisfied, proved that

$$u^{(k)}(0) = \alpha_k, \quad k = 0,1,\cdots,n-1. \tag{11}$$

By (5), it follows that, for any $t \ge 0$, the corresponding solution is determined only by the value $u^{(k)}(t), k = 0,1,\cdots,n-1$. Therefore if a control $u(t)$ satisfies (11) and if

$$u^{(k)}(T) = \beta_k, \quad k = 0,1,\cdots,n-1, \tag{12}$$

then (5) yields that $x_2(T) = -\sum_{k=0}^{n-1} N^k B_2 \beta_k$. In order to build a control $u \in C^{n-1}([0,T],U)$ satisfying (11) and (12) we can proceed as follows. Let

$$u_1(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\alpha_k + t^n \sum_{k=0}^{n-1} \frac{t^k}{k!} c_k, \quad c_k \in U, \quad k = 0,1,\cdots,n-1, \quad u_2(t) = \sum_{k=0}^{n-1} \frac{(t-T)^k}{k!}\beta_k.$$

We have

$$u_1^{(k)}(0) = \alpha_k, \quad u_2^{(k)}(T) = \beta_k, \quad k = 0,1,\cdots,n-1. \tag{13}$$

We choose a number $t_0 \in (0,T)$ and set

$$u(t) = u_1(t), \quad 0 \le t \le t_0; \quad u(t) = u_2(t), \quad t_0 \le t \le T. \tag{14}$$

We shall find coefficients $c_k, k = 0,1,\cdots,n-1$, so as to have

$$u_1^{(k)}(t_0) = u_2^{(k)}(t_0), \quad k = 0,1,\cdots,n-1. \tag{15}$$

Differentiating, this gives, for $j = 0, 1, \cdots, n-1$,

$$u_1^{(j)}(t_0) = \sum_{k=j}^{n-1} \frac{t_0^{k-j}}{(k-j)!}\alpha_k + \sum_{k=0}^{n-1} \frac{(n+k)!t_0^{k+n-j}}{(k+n-j)!k!}c_k.$$

Therefore, in order that (15) be valid for $j = 0, 1, \cdots, n-1$, the following equations must be satisfied

$$\sum_{k=j}^{n-1} \frac{t_0^{k-j}}{(k-j)!}\alpha_k + \sum_{k=0}^{n-1} \frac{(n+k)!t_0^{k+n-j}}{(k+n-j)!k!}c_k = u_2^{(j)}(t_0).$$

Consequently, the required coefficients $c_k, k = 0, 1, \cdots, n-1$, constitute a solution of the nonhomogeneous $n \times n$-system of linear algebraic equations. The determinant of the matrix of this system is the Wronskian of the following system of linearly independent functions $t^n, t^{n+1}, \frac{t^{n+2}}{2!}, \cdots, \frac{t^{2n-1}}{(n+1)!}$, at $t = t_0$. Therefore, the determinant does not vanish, and so the above nonhomogeneous system of linear algebraic equations has a unique solution $c_k \in U, k = 0, 1, \cdots, n-1$. Consequently, the control $u$, as defined by (14) with the coefficients $c_0, c_1, \cdots, c_{n-1}$ just obtained, lies in $C^{n-1}([0,T], U)$ and satisfies the condition (13). Hence, this proves Theorem 3.1. ▮

As for the approximate controllability of (3), we have the following theorem.

**Theorem 3.2** *Let $N \in L(X_2)$ be a nilpotent operator of order $n, B_2 \in L(U, X_2)$. Then the following statements are equivalent:*

(i) *subsystem* (3) *is approximately controllable on* $[0,T]$;

(ii)

$$\operatorname{ran}[B_2 \quad NB_2 \quad \cdots \quad N^{n-1}B_2] = X_2; \tag{16}$$

(iii) *subsystem* (3) *is exactly controllable on* $[0,T]$ .

*Proof* (i) $\Rightarrow$ (ii): The approximate controllability of (3) on $[0,T]$ implies, in particular, that it has a solution satisfying the initial condition $x_2(0) = x_{20}$ for any $x_{20} \in X_2$. By (6), we have $x_{20} \in \operatorname{ran}[B_2 \quad NB_2 \quad \cdots \quad N^{n-1}B_2]$. This proves that (ii) is true.

(ii) $\Rightarrow$ (iii): By Theorem 3.1, (iii) is true.

(iii) $\Rightarrow$ (i), obviously. ▮

**Remark 3.3** Note that approximate controllability and exact controllability are equivalent for (3).

## 4 Controllability of (2)–(3)

As for the exact controllability of (2)–(3), we have the following theorem.

**Theorem 4.1** *System* (2)–(3) *is exactly controllable on* $[0,T]$ *if and only if both* (2) *and* (3) *are exactly controllable on* $[0,T]$.

*Proof* The necessity is obvious. We only need to prove the sufficiency. Assume $x_{10}, x_{1T} \in$

$X_1$ and $x_{20}, x_{2T} \in X_2$. We have to find $u \in C^{n-1}([0,T],U)$ such that

$$x_1(t) = \mathrm{e}^{Kt}x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)}B_1u(\tau)d\tau, \quad x_2(t) = -\sum_{k=0}^{n-1} N^kB_2u^{(k)}(t),$$
$$x_1(T) = x_{1T}, \quad x_2(T) = x_{2T}. \tag{17}$$

We choose $u(t) = u_1(t) + u_2(t)$. Thus,

$$x_1(t) = \mathrm{e}^{Kt}x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)}B_1u_1(\tau)d\tau + \int_0^t \mathrm{e}^{K(t-\tau)}B_1u_2(\tau)d\tau,$$
$$x_2(t) = -\sum_{k=0}^{n-1} N^kB_2u_1^{(k)}(t) - \sum_{k=0}^{n-1} N^kB_2u_2^{(k)}(t).$$

We choose $u_1(t)$ to be of the form

$$u_1(t) = t^n(t-T)^nv(t) \tag{18}$$

for some $v \in C^n([0,T],U)$. Thus, $u_1^{(k)}(0) = u_1^{(k)}(T) = 0$, if $k < n$.By the proof of Theorem 3.1, there exists $u_2 \in C^{n-1}([0,T],U)$ such that $-\sum_{k=0}^{n-1} N^kB_2u_2^{(k)}(T) = x_2(T)$. From Theorem 2.2, for polynomial $f(\tau) = \tau^n(\tau-T)^n$, there exists $y \in X_1$, such that

$$\int_0^T f^2(\tau)\mathrm{e}^{K\tau}B_1B_1^*\mathrm{e}^{K^*\tau}yd\tau - \left[x_{1T} - \mathrm{e}^{KT}x_{10} - \int_0^T \mathrm{e}^{K(T-\tau)}B_1u_2(\tau)d\tau\right] = 0. \tag{19}$$

Let $v(\tau) = f(\tau)B_1^*\mathrm{e}^{K^*(T-\tau)}y$. Then, by (18) and (19), $u_1(\tau) = f^2(\tau)B_1^*\mathrm{e}^{K^*(T-\tau)}y$ and

$$\int_0^T \mathrm{e}^{K(T-\tau)}B_1u_1(\tau)d\tau - \left[x_{1T} - \mathrm{e}^{KT}x_{10} - \int_0^T \mathrm{e}^{K(T-\tau)}B_1u_2(\tau)d\tau\right] = 0.$$

Thus (17) is true. Therefore (2)–(3) is exactly controllable on $[0,T]$. ∎

As for the approximate controllability of (2)–(3), we have the following theorem.

**Theorem 4.2** *System* (2)–(3) *is approximately controllable on* $[0,T]$ *if and only if both* (2) *and* (3) *are approximately controllable on* $[0,T]$.

*Proof* The necessity is obvious. We only need to prove the sufficiency. Assume

$$x_{10}, x_{1T} \in X_1, \alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \beta_0, \beta_1, \cdots, \beta_{n-1} \in U$$

and $\varepsilon > 0$. We have to find $u \in C^{n-1}([0,T],U)$ such that

$$x_1(t) = \mathrm{e}^{Kt}x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)}B_1u(\tau)d\tau, x_2(t) = -\sum_{k=0}^{n-1} N^kB_2u^{(k)}(t)$$

and

$$u^{(k)}(0) = \alpha_k, u^{(k)}(T) = \beta_k, \quad k = 0, 1, \cdots, n-1, \quad \|x_1(T) - x_{1T}\| < \varepsilon. \tag{20}$$

We choose $u(t) = u_1(t) + u_2(t)$. Thus

$$x_1(t) = \mathrm{e}^{Kt}x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)}B_1u_1(\tau)d\tau + \int_0^t \mathrm{e}^{K(t-\tau)}B_1u_2(\tau)d\tau,$$

$$x_2(t) = -\sum_{k=0}^{n-1} N^k B_2 u_1^{(k)}(t) - \sum_{k=0}^{n-1} N^k B_2 u_2^{(k)}(t).$$

We choose $u_1(t)$ to be of the form

$$u_1(t) = t^n (t-T)^n v(t) \tag{21}$$

for some $v \in C^n([0,T],U)$. Thus $u_1^{(k)}(0) = u_1^{(k)}(T) = 0$,if $k < n$.By the proof of Theorem 3.1, there exists $u_2 \in C^{n-1}([0,T],U)$ such that $u_2^{(k)}(0) = \alpha_k, u_2^{(k)}(T) = \beta_k, k = 0,1,\cdots,n-1$. From Theorem 2.4, for any polynomial $f \in R$ not identically zero, there exists $y \in X_1$, such that

$$\left\| \int_0^T f^2(\tau) \mathrm{e}^{K\tau} B_1 B_1^* \mathrm{e}^{K^*\tau} y d\tau - \left[ x_{1T} - \mathrm{e}^{KT} x_{10} - \int_0^T \mathrm{e}^{K(T-\tau)} B_1 u_2(\tau) d\tau \right] \right\| < \varepsilon. \tag{22}$$

Let $f(\tau) = \tau^n(\tau - T)^n, v(\tau) = f(\tau) B_1^* \mathrm{e}^{K^*(T-\tau)} y$. Then, by (21) and (22),

$$u_1(\tau) = f^2(\tau) B_1^* \mathrm{e}^{K^*(T-\tau)} y$$

and

$$\left\| \int_0^T \mathrm{e}^{K(T-\tau)} B_1 u_1(\tau) d\tau - \left[ x_{1T} - \mathrm{e}^{KT} x_{10} - \int_0^T \mathrm{e}^{K(T-\tau)} B_1 u_2(\tau) d\tau \right] \right\| < \varepsilon.$$

Thus, $x_1(t) = \mathrm{e}^{Kt} x_{10} + \int_0^t \mathrm{e}^{K(t-\tau)} B_1 u(\tau) d\tau, x_2(t) = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(t)$ satisfy (20). Therefore (2)–(3) is approximately controllable on $[0,T]$. ▮

## 5 Controllability of Dzektser Equation

Consider the Dzektser equation, which describes the evolution of the free surface of seepage liquid (see, e.g., [6]),

$$\left(1 + \frac{\partial^2}{\partial \xi^2}\right) \frac{\partial}{\partial t} x(\xi,t) = \left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^4}{\partial \xi^4}\right) x(\xi,t) + u(t), \quad (\xi,t) \in (0,\pi) \times [0,+\infty), \tag{23}$$

$$\begin{cases} x(0,t) = \dfrac{\partial^2 x(0,t)}{\partial \xi^2} = x(\pi,t) = \dfrac{\partial^2 x(\pi,t)}{\partial \xi^2} = 0, \\ t \in [0,+\infty), \quad x(\xi,0) = x_0(\xi), \quad \xi \in (0,\pi), \end{cases} \tag{24}$$

Let

$$\begin{aligned}
&X = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}, \quad Y = L^2([0,\pi],R), \\
&E = 1 + \frac{\partial^2}{\partial \xi^2}, \quad A = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^4}{\partial \xi^4}, \\
&\mathrm{dom} A = \{x \in H^4(0,\pi) : x(0) = x''(0) = x(\pi) = x''(\pi) = 0\}, \\
&(x(t))(\xi) = x(\xi,t), \quad (Bu(t))(\xi) = bu(t), \quad \xi \in (0,\pi), \quad u \in U = R, \quad b = 1 \in L^2([0,\pi],R),
\end{aligned}$$

where the meanings of $H^2(0,\pi)$ and $H^4(0,\pi)$ are the same as in [11]. Then Dzektser equation (23)–(24) can be reduced to the following system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{25}$$

It is easily checked that $\sin k\xi$ is the $E$-eigenvector of the operator $A$ corresponding to $E$-eigenvalue $-k^2(1+\frac{k^2}{k^2-1})$ of the operator $A\,(k=2,3,\cdots), E\sin\xi=0, A\sin\xi=\sin\xi$,

$$E\sin k\xi=(1-k^2)\sin k\xi\quad(k=2,3,\cdots),\qquad A\sin k\xi=(2k^4-k^2)\sin k\xi\quad(k=2,3,\cdots)$$

and

$$1=\sum_{k=1}^{+\infty}\frac{\langle 1,\sin k\xi\rangle_Y}{\langle \sin k\xi,\sin k\xi\rangle_Y}\sin k\xi=\frac{4}{\pi}\sin\xi+\sum_{k=2}^{+\infty}\frac{4\sin(2k-1)\xi}{(2k-1)\pi}.$$

Let $X_1$ be the closure of the subspace $\mathrm{span}\{\sin k\xi: k=2,3,\cdots\}$ in the norm of the space $X$; $X_2=\mathrm{span}\{\sin\xi\}$. Then $X_2$ is one dimensional. Let $E_1$ and $A_2$ denote the restrictions of $E$ and $A$ on $X_1$ and $X_2$, respectively. Then

$$K\sin k\xi=E_1^{-1}A\sin k\xi=-k^2\left(1+\frac{k^2}{k^2-1}\right)\sin k\xi,\quad k=2,3,\cdots,$$
$$\mathrm{dom}K=\mathrm{span}\{\sin k\xi: k=2,3,\cdots\},$$
$$b_1=E_1^{-1}\left(1-\frac{4}{\pi}\sin\xi\right)=\sum_{k=2}^{+\infty}\frac{4\sin(2k-1)\xi}{[1-(2k-1)^2](2k-1)\pi},$$
$$b_2=A_2^{-1}\frac{\langle 1,\sin\xi\rangle_Y}{\langle\sin\xi,\sin\xi\rangle_Y}\sin\xi=\frac{4}{\pi}\sin\xi.$$

The regular standard form of (25) is

$$\dot{x}_1(t)=Kx_1(t)+B_1u(t),\quad x_1(0)=x_{10},\tag{26}$$
$$0=x_2(t)+B_2u(t),\quad x_2(0)=x_{20},\tag{27}$$

where $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}\in X_1\times X_2, K$ is the generator of the strongly continuous semigroup $\mathrm{e}^{Kt}$ on $X_1$, $N=0$ in (27), $B_1u=b_1u$ and $B_2u=b_2u$.

First of all we discuss the exact controllability of (23)–(24).

Since $X_2=\mathrm{ran}[B_2]$, by Theorem 3.1, subsystem (27) is exactly controllable on $[0,T]$ for any $T>0$. It is obviously that the semigroup associated with (26) is given by

$$\mathrm{e}^{Kt}y=\sum_{k=2}^{+\infty}\mathrm{e}^{-k^2\left(1+\frac{k^2}{k^2-1}\right)t}\frac{\langle y,\sin k\xi\rangle_X}{\|\sin k\xi\|_X^2}\sin k\xi.\tag{28}$$

Since $\mathrm{e}^{Kt}=\mathrm{e}^{K^*t}$, the condition for exact controllability of (26) is the existence of a $\gamma>0$ such that

$$\gamma\sum_{k=2}^{+\infty}\frac{|\langle y,\sin k\xi\rangle_X|^2}{\|\sin k\xi\|_X^2}=\gamma\|y\|_X^2\le\int_0^T\|B_1^*\mathrm{e}^{K^*\tau}y\|_U^2d\tau\le\|B_1\|_{L(U,X_1)}^2$$
$$\times\sum_{k=2}^{+\infty}\frac{1-\mathrm{e}^{-2k^2(1+\frac{k^2}{k^2-1})T}}{2k^2(1+\frac{k^2}{k^2-1})}\frac{|\langle y,\sin k\xi\rangle_X|^2}{\|\sin k\xi\|_X^2},$$

i.e.,

$$\sum_{k=2}^{+\infty}\left[\|B_1\|_{L(U,X_1)}^2\frac{1-\mathrm{e}^{-2k^2(1+\frac{k^2}{k^2-1})T}}{2k^2(1+\frac{k^2}{k^2-1})}-\gamma\right]\frac{|\langle y,\sin k\xi\rangle_X|^2}{\|\sin k\xi\|_X^2}\ge 0.\tag{29}$$

It is obviously that no $\gamma$ satisfying (29) will ever exist. Consequently, subsystem (26) is never exactly controllable on $[0,T]$ for any $T>0$. By Theorem 4.1, Dzektser equation (23)-(24) is not exactly controllable on $[0,T]$ for any $T>0$.

Now we discuss the approximate controllability of (23)–(24). As $N=0$ in (27), it follows from (ii) of Theorem 3.2 that $X_2=\text{span}\{\text{sin}\xi\}$. In this case condition of Theorem 4.2 guarantee that (23)–(24) is approximately controllable on $[0,T]$ for some $T>0$ if and only if (26) is approximately controllable. It is obviously that the strongly continuous semigroup associated with (26) is given by (28). Since $\mathrm{e}^{Kt}=\mathrm{e}^{K^*t}$, by (ii) of Theorem 2.3, the condition for approximate controllability is that there exists $T>0$ such that $B_1^*\mathrm{e}^{K^*t}y=0$ on $[0,T]\Rightarrow y=0$. In fact, for any $T>0$, if $0=B_1^*\mathrm{e}^{K^*t}y$ on $[0,T]$, then

$$\mathrm{e}^{K^*t}y=\sum_{k=2}^{+\infty}\mathrm{e}^{-k^2(1+\frac{k^2}{k^2-1})t}\frac{\langle y,\text{sin}k\xi\rangle_X}{\langle\text{sin}k\xi,\text{sin}k\xi\rangle_X}\text{sin}k\xi=0.$$

Since sine series $\sum_{k=2}^{+\infty}\mathrm{e}^{-k^2(1+\frac{k^2}{k^2-1})t}\frac{\langle y,\text{sin}k\xi\rangle_X}{\langle\text{sin}k\xi,\text{sin}k\xi\rangle_X}\text{sin}k\xi$ is uniformly convergent on $[0,\pi]$ for every $t\in(0,T]$, we have

$$\sum_{k=2}^{+\infty}\mathrm{e}^{-k^2(1+\frac{k^2}{k^2-1})t}\frac{\langle y,\text{sin}k\xi\rangle_X}{\langle\text{sin}k\xi,\text{sin}k\xi\rangle_X}\langle\text{sin}k\xi,\text{sin}m\xi\rangle_X=0.$$

By the orthogonality of the sine function system, we can get $\langle y,\sin m\xi\rangle_X=0, m=2,3,\cdots$, i.e., $y=0$. Hence, by Theorem 4.2, the Dzektser equation (23)–(24) is approximately controllable on $[0,T]$ for any $T>0$.

## 6 Conclusions

We have defined exact controllability and approximate controllability, and proved corresponding necessary and sufficient conditions for regular singular distributed parameter systems. An illustrative example was given. For a specific singular distributed parameter system, the exact controllability and approximate controllability can be tested according to the results of this paper.

### References


[1] Fang Z C, Li H, Liu Y, et al., An expanded mixed covolume element method for integro-differential equation of Sobolev type on triangular grids, *Advances in Difference Equations*, 2017, **143**: 1–22.

[2] Fedorov V E, Plekhanova M V, and Nazhimov R R, Degenerate linear evolution equations with the Riemann-Liouville fractional derivative, *Siberian Mathematics Journal*, 2018, **59**: 136–146.

[3] Gou H D and Li B L, Study on the mild solution of Sobolev type Hifer fractional evolution equations with boundary conditions, *Cahaos, Solitons and Fractals*, 2018, **112**: 168–179.

[4] Gou H and Li B, Existence of mild solutions for Sobolev-type Hilfer fractional evolution equations with boundary conditions, *Boundary Value Problems*, 2018, **48**: 1–25.

[5] Demidenko G V and Uspenskii S V, *Partial Differential Equations and Systems Not Solvable with Respect to the Highest-Order Derivative*, Nauchnaya Kniga, Novosibirst 1998; English Transl, Pure Appl. Math, vol. 256, Marcel Dekker, New York, 2003.

[6] Dzektser E S, Generalization of the equation of motion of ground waters with a free surface, *Dokl Akad Nauk SSSR*, 1972, **202**: 1031–1033; *English Transl. in Soviety Phys. Dokl.*, 1972, **17**: 108–110.

[7] Sveshnikov A G, Alhin A B, Korpusov M O, et al., *Linear and Nonlinear Equations of Sobolev Type*, Matematika. Prikladanaya Matematika, Fizmatlit, Moscow, 2007 (in Russian).

[8] Trzasska Z and Maszalak W, Singular distributed parameter systems, *IEE Control Theory Appl.*, 1993, **140**: 305–308.

[9] Ge Z Q, Impulse observability and impulse controllability of regular degenerate evolution systems, *Journal of Systems Science and Complexity*, 2016, **29**(4): 933–945.

[10] Curtain R F and Zwart H, *An Introduction to Infinite-Dimensional Linear System Theory*, Springer-Verlag, New York, 1995.

[11] Fedorov V E and Shklyar B, Exact null controllability of degenerate evolution equations with scalar control, *Sbornik: Mathematics*, 2012, **203**: 1817–1836.

[12] Fedorov V E, Degenerate strongly continuous semigroups of operators, *Algebrai Analiz*, 2000, **12**: 173–200; *English Transl. in St. Petersburg Math J.*, 2001, **12**: 471–489.

[13] Sviridyuk G A and Fedorov V E, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators, Inverse Ill-posed Probl, Ser.*, Utrecht, VSP, 2003.

[14] Leontif W, *Input-Output Economics*, Second Edition, Oxford University Press, Oxford, New York, 1986.

[15] Ge Z Q, Liu F, and Feng D X, Pulse controllability of singular distributed parameter systems, *Sci. China Inf. Sci.*, 2019, **62**: 049201:1–049201:3.