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General Decay Synchronization for Recurrent Neural Networks with Mixed Time Delays*

MUHAMMADHAJI Ahmadjan · TENG Zhidong

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Abstract This paper studies the general decay synchronization (GDS) of a class of recurrent neural networks (RNNs) with general activation functions and mixed time delays. By constructing suitable Lyapunov-Krasovskii functionals and employing useful inequality techniques, some sufficient conditions on the GDS of considered RNNs are established via a type of nonlinear control. In addition, one example with numerical simulations is presented to illustrate the obtained theoretical results.

Keywords General activation functions, general decay synchronization, mixed time delay, recurrent neural network.

1 Introduction

As is well known, in recent years neural networks have received much attentions due to their wide applications in a variety of areas such as signal processing, automatic control engineering, associative memories, parallel computation, combinatorial optimization and pattern recognition, and so on[1–4]. In particular, the recurrent neural networks (RNNs) systems are of great interest among the scholars. For example, in the past decades, there is a mass of literature concerned with the dynamical behaviors of recurrent neural networks have been studied by many researchers[5–15]. However, in mathematical modeling of real world problems, time delays are frequently encountered as a result of the inherent communication time between neurons and the finite switching speed of amplifiers. In particular, in hardware implementation, time delays usually causes oscillation, instability, divergence, chaos, or other bad performances of neural networks[16]. Thus, the study of dynamic behaviors for delayed recurrent neural networks has been one of the hot research topics in the past few decades[5–16].

It is worth noting that, a very important problem in neural networks system is the synchronization. Especially, when we investigate the dynamical behaviors of chaotic neural networks,

MUHAMMADHAJI Ahmadjan · TENG Zhidong
College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046*, China.*
Email: ahmatjanam@aliyun.com.
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the synchronization can play an extremely vital role. The so-called synchronization is that the orbits of two chaotic systems which start from different initial conditions gradually converge to the same equilibrium point[17]. In recent years, many researchers have focused on the synchronization for delayed neural networks (NNs) and obtained most significant results[12–16, 18–21]. Moreover, lots of synchronization results have been obtained under different synchronization methods, such as linear matrix inequality (LMI) technique[22], matrix measure strategy and Lyapunov approach[23, 24], Lyapunov theory and fractional-order differential inequalities[25], generalized Halanay inequalities and matrix measure approaches[26], Lyapunov function approach[27], and so on.

On the other hand, in the process of investigating neural networks the estimate of the convergent rate of synchronization is a very interesting and useful for studying the synchronization of chaotic systems. However, in some cases, the convergence rate of the synchronization can not be shown or it is very difficult to estimate. Recently, Wang, et al.[28, 29] investigated the synchronization problem for a classes of chaotic NNs with discontinuous and continuous activations by introducing new concept of synchronization, namely general decay synchronization (GDS). This motivates us to consider a new type of convergence rate, such as convergence with general decay. Moreover, studies on the general decay synchronization for recurrent neural networks with both time-varying and distributed delays are fairly rare. Therefore, based on the above analysis and reasons, in this paper we study the following n-dimensional RNNs with both time-varying and distributed delays

$$\begin{aligned}\dot{x}_i(t) = &-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_j(t)))\\ &+\sum_{j=1}^{n} d_{ij}\int_{t-\sigma_j(t)}^{t} h_j(x_j(s))ds + I_i,\end{aligned} \tag{1}$$

where $i \in \mathscr{I} \triangleq \{1,2,\cdots,n\}$, $n \geq 2$ denotes the number of neurons in the neural networks; $x_i(t)$ corresponds to the state variable of the ith unit at time t; $c_i > 0$ denotes the rate with which the ith neuron resets its potential to the resting state when isolated from the other neuron and inputs; a_{ij}, b_{ij} and d_{ij} are denote the connection weights between the ith neuron and the jth neuron at time t; $f_i(\cdot), g_i(\cdot)$ and $h_i(\cdot)$ are the nonlinear activation functions, I_i is the external input vector. $\tau_i(t)$ and $\sigma_i(t)$ are the transmission time-varying delays and satisfy $0 \leq \tau_i(t) \leq \tau_i$ and $0 \leq \sigma_i(t) \leq \sigma_i$.

The main purpose of the paper is by constructing suitable Lyapunov-Krasovskii functionals and applying the method given in [28, 29] to establish some new sufficient conditions on the general decay synchronization for System (1).

2 Preliminaries

In this paper, we always use $\mathscr{I} \triangleq \{1,2,\cdots,n\}$ and $R^+ = [0,+\infty)$, unless otherwise stated. The initial conditions associated with the system (1) are given by

$$x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad i = 1,2,\cdots,n,$$

where $\tau = \max_{j\in\mathscr{I}}\{\tau_j, \sigma_j\}$ and $\varphi(s) = (\varphi_1(s), \varphi_2(s), \cdots, \varphi_n(s)) \in C([-\tau, 0], R^n)$, which denotes the Banach space of all continuous functions mapping $[-\tau, 0]$ into R^n with norm defined by

$$\|\varphi\| = \sup_{s\in[-\tau,0]} \|\varphi(s)\|$$

with $\|\varphi(s)\| = \max_{i\in\mathscr{I}} |\varphi_i(s)|$.

Throughout this paper, we assume that the following assumptions are satisfied.

H$_1$ Activation functions $f_j(u), g_j(u)$ and $h_j(u)$ are continuous and there exist nonnegative constants $L_j, H_j, K_j, N_j, M_j, O_j \geq 0$, such that for any $v_1, v_2 \in R$,

$$|f_j(v_1) - f_j(v_2)| \leq L_j|v_1 - v_2| + N_j, \quad |g_j(v_1) - g_j(v_2)| \leq H_j|v_1 - v_2| + M_j,$$
$$|h_j(v_1) - h_j(v_2)| \leq K_j|v_1 - v_2| + O_j.$$

H$_2$ Time-varying delays $\tau_j(t)$ and $\sigma_j(t)$ are differentiable, and there exist real numbers $0 \leq \zeta_j \leq 1$ and $0 \leq \gamma_j \leq \frac{1}{2}$, such that for any $t \in R^+$

$$0 \leq \dot{\tau}_j(t) \leq \zeta_j, \quad 0 \leq \dot{\sigma}_j(t) \leq \gamma_j.$$

In the paper, we consider the system (1) as the drive system, the response system is given as follows

$$\begin{aligned}\dot{y}_i(t) = & -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(y_j(t - \tau_j(t))) \\ & + \sum_{j=1}^{n} d_{ij} \int_{t-\sigma_j(t)}^{t} h_j(y_j(s))ds + u_i + I_i, \end{aligned} \tag{2}$$

where $u_i(t)$ is the controller to be designed.

Let $e_i(t) = y_i(t) - x_i(t)$, then from (1) and (2), the error dynamical system is expressed as

$$\begin{aligned}\dot{e}_i(t) = & -c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \widetilde{f}_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} \widetilde{g}_j\left(e_j(t - \tau_j(t))\right) \\ & + \sum_{j=1}^{n} d_{ij} \int_{t-\sigma_j(t)}^{t} \widetilde{h}_j(e_j(s))ds + u_i, \end{aligned} \tag{3}$$

where $\widetilde{f}_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t))$.

Now, we will give the definitions of ψ-type function and GDS.

Definition 2.1 (see [28, 29]) A function $\psi : R^+ \to [1, +\infty)$ is said to be ψ-type function if it satisfies the following conditions

1) It is differentiable and nondecreasing;

2) $\psi(0) = 1$ and $\psi(+\infty) = +\infty$;

3) $\widetilde{\psi}(t) = \dot{\psi}(t)/\psi(t)$ is nondecreasing and $\psi^* = \sup_{t\geq 0} \widetilde{\psi}(t) < +\infty$, where $\dot{\psi}(t)$ is the time derivative of $\psi(t)$;

4) For any $t,\ s \geq 0,\ \psi(t + s) \leq \psi(t)\psi(s)$.

It is not difficult to check that functions $\psi(t) = \mathrm{e}^{\alpha t}$ and $\psi(t) = (1 + t)^\alpha$ for any $\alpha > 0$ satisfy the above four conditions, thus can be seen as ψ-type functions.

Definition 2.2 (see [28, 29]) The drive-response systems (1) and (2) are said to be general decay synchronized if there exists a constant $\varepsilon > 0$ and a ψ-type function ψ such that for any solutions $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ of System (1) and $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ of System (2), one has

$$\limsup_{t\to+\infty} \frac{\log \|y(t) - x(t)\|}{\log \psi(t)} \leq \varepsilon,$$

where $\varepsilon > 0$ can be seen the convergence rate as synchronization error approaches zero.

$\mathbf{H}_3$ For function $\psi(t)$ given in Definition 2.1, there exist a function $\varrho(t) \in C(R, R^+)$ and a constant δ such that for any $t \geq 0$

$$\widetilde{\psi}(t) \leq 1, \qquad \sup_{t\in[0,+\infty)} \int_0^t \psi^\delta(s)\varrho(s)ds < +\infty. \tag{4}$$

Now, we present a useful lemma. This lemma is essential to our later study.

Lemma 2.1 (see [28, 29]) *Under Assumption* H_3*, assume that the synchronization error* $e(t) = y(t) - x(t)$ *of driver-response systems* (1) *and* (2) *satisfy the differential equation* $\dot{e}(t) = g(t, e_t)$*, where* $e_t = e(t + s)$ *for* $s \in [-\tau, 0]$*, function* $g(t, e_t)$ *is locally bounded. If there exist a differentiable functional* $V(t, e_t) : R^+ \times C \to R^+$*, and positive constants* λ_1, λ_2 *such that for any* $(t, e_t) \in R^+ \times C$

$$(\lambda_1 \|e(t)\|)^2 \leq V(t, e_t), \qquad \left.\frac{dV(t, e_t)}{dt}\right|_{(3)} \leq -\delta V(t, e_t) + \lambda_2 \varrho(t), \tag{5}$$

where $x(t)$ *and* $y(t)$ *are solutions of the systems* (1) *and* (2) *respectively,* $\delta > 0$ *and* $\varrho(t)$ *are defined in* H_3*. Then the driver-response systems* (1) *and* (2) *are general decay synchronized in the sense of Definition* 2.2*, and the convergence rate is* $\delta/2$*.*

3 Main Results

In this section, we will obtain some sufficient conditions to insure the GDS of the systems (1) and (2). First, under Assumption H_3 designing the controller $u_i(t)$ of response system (2) as follows:

$$u_i(t) = -\alpha_i \mathrm{sign}(e_i(t)) - \frac{\beta_i \|e(t)\| e_i(t)}{e_i(t) + \varrho(t)}, \quad i \in \mathscr{I}, \tag{6}$$

where β_i and α_i for $i \in \mathscr{I}$ are control gains satisfying

$$\begin{aligned} &E_i \triangleq c_i + \beta_i - \sum_{j=1}^n \left(A_{ji} + \frac{B_{ji}}{(1-\zeta_i)} + \tau_i B_{ji} + 2\sigma_i D_{ji} + \frac{1}{2}\sigma_i^2 D_{ji} \right) > 0, \\ &\alpha_i - \sum_{j=1}^n \left(N_i |a_{ji}| + \tau_i M_i |b_{ji}| + \sigma_i O_i |d_{ji}| \right) > 0, \end{aligned} \tag{7}$$

where $A_{ij} = |a_{ij}| L_j$, $B_{ij} = |b_{ij}| H_j$ and $D_{ij} = |d_{ij}| K_j$.

Then based on the nonlinear controller (6), the following theorem can be obtained.

Theorem 3.1 *Suppose* H_1–H_3 *hold, then the response network* (2) *can be general decay synchronized with the drive network* (1) *under the nonlinear controller* (6) *if, the control gains* β_i *and* α_i *satisfy the inequality* (7).

Proof Firstly, we construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V_1(t) \;=\; & \sum_{i=1}^{n}|e_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\frac{B_{ij}}{(1-\zeta_j)}|e_j(s)|ds \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n}2D_{ij}\int_{-\sigma_j(t)}^{0}\int_{t+\theta}^{t}|e_j(s)|dsd\theta. \end{aligned} \tag{8}$$

Calculating the derivative of $V_1(t)$ along the system (3), we get

$$\begin{aligned} \dot{V}_1(t) = & \sum_{i=1}^{n}\mathrm{sign}(e_i(t))\bigg\{ - c_ie_i(t) + \sum_{j=1}^{n}a_{ij}\widetilde{f}_j(e_j(t)) + \sum_{j=1}^{n}b_{ij}\widetilde{g}_j\left(e_j(t-\tau_j(t))\right) \\ & + \sum_{j=1}^{n}d_{ij}\int_{t-\sigma_j(t)}^{t}\widetilde{h}_j\left(e_j(s)\right)ds - \alpha_i\mathrm{sign}(e_i(t)) - \frac{\beta_i\|e(t)\|e_i(t)}{e_i(t)+\varrho(t)}\bigg\} \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n}B_{ij}\bigg(\frac{1}{(1-\zeta_j)}|e_j(t)| - \frac{(1-\dot{\tau}_j(t))}{(1-\zeta_j)}|e_j(t-\tau_{ij}(t))|\bigg) \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n}2D_{ij}\bigg[\int_{-\sigma_j(t)}^{0}\big(|e_j(t)| - |e_j(t+\theta)|\big)d\theta + \dot{\sigma}_j(t)\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds\bigg] \\ \le & \sum_{i=1}^{n}\bigg\{ - c_i|e_i(t)| + \sum_{j=1}^{n}|a_{ij}||\widetilde{f}_j(e_j(t))| + \sum_{j=1}^{n}|b_{ij}||\widetilde{g}_j\left(e_j(t-\tau_j(t))\right)| \\ & + \sum_{j=1}^{n}|d_{ij}|\int_{t-\sigma_j(t)}^{t}|\widetilde{h}_j(e_j(s))|ds - \alpha_i - \frac{\beta_i\|e(t)\||e_i(t)|}{|e_i(t)|+\varrho(t)} \\ & + \sum_{j=1}^{n}\frac{B_{ij}}{(1-\zeta_j)}|e_j(t)| - \sum_{j=1}^{n}B_{ij}|e_j(t-\tau_j(t))| + \sum_{j=1}^{n}2D_{ij}\sigma_j(t)|e_j(t)| \\ & - \sum_{j=1}^{n}2D_{ij}\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds + \sum_{j=1}^{n}2D_{ij}\dot{\sigma}_j(t)\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds\bigg\}. \end{aligned} \tag{9}$$

Now, using H_1, we have

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||\widetilde{f}_j(e_j(t))| \le & \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|(L_j|e_j(t)|+N_j) \\ = & \sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}|e_j(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}N_j|a_{ij}|. \end{aligned} \tag{10}$$

Similarly, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}||\widetilde{g}_j(e_j(t-\tau_j(t)))| &\le \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|(H_j|e_j(t-\tau_j(t))|+M_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}B_{ij}|e_j(t-\tau_j(t))|+\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|M_j \qquad (11)\end{aligned}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}|\int_{t-\sigma_j(t)}^{t}|\widetilde{h}_j(e_j(s))|ds \le \sum_{i=1}^{n}\sum_{j=1}^{n}D_{ij}\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds+\sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}|\sigma_j O_j. \qquad (12)$$

Using H_2 and from (7), (9)–(12), we have

$$\begin{aligned}\dot{V}_1(t) \le& \sum_{i=1}^{n}\Bigg\{-c_i|e_i(t)|+\sum_{j=1}^{n}A_{ij}|e_j(t)|+\sum_{j=1}^{n}N_j|a_{ij}|+\sum_{j=1}^{n}B_{ij}|e_j(t-\tau_j(t))|\\ &+\sum_{j=1}^{n}|b_{ij}|M_j+\sum_{j=1}^{n}D_{ij}\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds+\sum_{j=1}^{n}|d_{ij}|\sigma_j O_j-\alpha_i-\frac{\beta_i\|e(t)\||e_i(t)|}{\|e(t)\|+\varrho(t)}\\ &+\sum_{j=1}^{n}\frac{B_{ij}}{(1-\zeta_j)}|e_j(t)|-\sum_{j=1}^{n}B_{ij}|e_j(t-\tau_{ij}(t))|+\sum_{j=1}^{n}2D_{ij}\sigma_j(t)|e_j(t)|\\ &-\sum_{j=1}^{n}2D_{ij}\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds+\sum_{j=1}^{n}D_{ij}\int_{t-\sigma_j(t)}^{t}|e_j(s)|ds\Bigg\}\\ =&\sum_{i=1}^{n}\Bigg\{-\Bigg[c_i+\beta_i-\sum_{j=1}^{n}\left(A_{ji}+\frac{B_{ji}}{(1-\zeta_i)}+2\sigma_i D_{ji}\right)\Bigg]|e_j(t)|-\Bigg[\alpha_i-\sum_{j=1}^{n}\big(N_i|a_{ji}|\\ &+M_i|b_{ji}|+\sigma_i O_i|d_{ji}|\Bigg]+\beta_i|e_i(t)|-\frac{\beta_i\|e(t)\||e_i(t)|}{\|e(t)\|+\varrho(t)}\Bigg\}\\ \le&\sum_{i=1}^{n}-\Bigg[c_i+\beta_i-\sum_{j=1}^{n}\big(A_{ji}+\frac{B_{ji}}{(1-\zeta_i)}+2\sigma_i D_{ji}\big)\Bigg]|e_j(t)|+\sum_{i=1}^{n}\frac{\beta_i\varrho(t)|e_i(t)|}{\|e(t)\|+\varrho(t)}. \qquad (13)\end{aligned}$$

Next, we construct the following Lyapunov-Krasovskii functional:

$$V_2(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}B_{ij}\int_{-\tau_j}^{0}\int_{t+s}^{t}|e_j(\varepsilon)|d\varepsilon ds+\sum_{i=1}^{n}\sum_{j=1}^{n}D_{ij}\int_{-\sigma_j}^{0}\int_{s}^{0}\int_{t+\theta}^{t}|e_j(\varepsilon)|d\varepsilon d\theta ds.$$

Calculating the derivative of $V_2(t)$, we get

$$\begin{aligned}\dot{V}_2(t)=&\sum_{i=1}^{n}\sum_{j=1}^{n}\left[B_{ij}\big(\tau_j|e_j(t)|-\int_{t-\tau_j}^{t}|e_j(s)|ds\big)+D_{ij}\int_{-\sigma_j}^{0}\int_{s}^{0}\big(|e_j(t)|-|e_j(t+\theta)|\big)d\theta ds\right]\\ \le&\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\tau_j B_{ij}+\frac{1}{2}\sigma_j^2 D_{ij}\right)|e_j(t)|-A, \qquad (14)\end{aligned}$$

where
$$A=\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg[B_{ij}\int_{t+s}^{t}|e_j(s)|ds+D_{ij}\int_{-\sigma_j}^{0}\int_{t+s}^{t}|e_j(\omega)|d\omega ds\bigg].$$
Finally, we construct the following Lyapunov-Krasovskii functional:
$$V(t)=V_1(t)+V_2(t).$$
Then, there exists a scalar $\chi>1$ such that
$$\sum_{i=1}^{n}|e_i(t)|\le V(t)\le\chi\sum_{i=1}^{n}|e_i(t)|+\frac{\chi}{E}A, \tag{15}$$
where $E=\min_{i\in\mathscr{I}}\{E_i\}$.

Calculating the derivative of $V(t)$ and from (13)–(14), we get
$$\begin{aligned}\dot V(t)\le&\sum_{i=1}^{n}-\bigg[c_i+\beta_i-\sum_{j=1}^{n}\bigg(A_{ji}+\frac{B_{ji}}{(1-\zeta_i)}+\tau_iB_{ji}+2\sigma_iD_{ji}+\frac{1}{2}\sigma_i^2D_{ji}\bigg)\bigg]|e_j(t)|\\&+\sum_{i=1}^{n}\frac{\beta_i\varrho(t)\|e_i(t)\|}{\|e(t)\|+\varrho(t)}-A\\\le&\sum_{i=1}^{n}-E_i|e_i(s)|+\max_{i\in\mathscr{I}}\{\beta_i\}\frac{\|e(t)\|\varrho(t)}{\|e(t)\|+\varrho(t)}-A.\end{aligned} \tag{16}$$

Let $\beta=\max_{i\in\mathscr{I}}\{\beta_i\}>0$ and using the inequality $0\le ab/(a+b)\le a$ for any $a>0,\ b>0$, we have
$$\dot V(t)\le\sum_{i=1}^{n}-E_i|e_i(t)|+\beta\varrho(t)-A. \tag{17}$$
Now taking a small enough δ such that $\delta\chi<E$, then from the inequalities (15) and (17), we get
$$\begin{aligned}\frac{d}{dt}V(t)+\delta V(t)\le&\sum_{i=1}^{n}-E_i|e_i(t)|+\beta\varrho(t)-A+\delta\bigg(\chi\sum_{i=1}^{n}|e_i(t)|+\frac{\chi}{E}A\bigg)\\\le&(\delta\chi-E)\sum_{i=1}^{n}|e_i(t)|+\bigg(\frac{\delta\chi}{E}-1\bigg)A+\beta\varrho(t)\\\le&\beta\varrho(t),\end{aligned}$$
which means that
$$\dot V(t)+\delta V(t)\le\beta\varrho(t). \tag{18}$$
Then, from Lemma 2, the drive-response systems (1) and (2) achieve GDS under the adaptive nonlinear controller (6). The convergence rate of $e(t)$ approaching zero is $\delta/2$. The proof is completed. ▌

Remark 3.2 (see [29]) The function ψ is used as the decay function, so ψ-type stability is also said to be stability with general decay rate. When $\psi(t)=\mathrm{e}^{\alpha t}$ and $\psi(t)=(1+t)^{\alpha}$ for any $\alpha>0$, ψ-type stability may be specialized as exponential synchronization and polynomial synchronization.

If, in H_1 we assume that the activation functions $f_j(u), g_j(u), h_j(u)$ are globally Lipschitz, i.e., the constants $N_j = M_j = O_j = 0$, then H_1 turns to

$\overline{\mathbf{H}}_1$: $f_j(u), g_j(u), h_j(u)$ are globally Lipschitz continuous, i.e., there exist constants L_j, H_j, $K_j > 0$, such that

$$|f_j(v_1)-f_j(v_2)| \leq L_j|v_1-v_2|, \quad |g_j(v_1)-g_j(v_2)| \leq H_j|v_1-v_2|, \quad |h_j(v_1)-h_j(v_2)| \leq K_j|v_1-v_2|,$$

where $v_1, v_2 \in R$.

In addition, the controller (6) in the system (2) becomes

$$u_i(t) = -\frac{\beta_i\|e(t)\|e_i(t)}{e_i(t)+\varrho(t)}, \quad i \in \mathscr{I}. \tag{19}$$

Then from Theorem 3.1, we have the following corollary.

Corollary 3.3 *Suppose* $\overline{\mathrm{H}}_1, \mathrm{H}_2, \mathrm{H}_3$ *hold, then the response network* (2) *can be general decay synchronized with the drive network* (1) *under the nonlinear controller* (19) *if, the control gains* β_i *satisfy the following inequality:*

$$c_i + \beta_i - \sum_{j=1}^{n}\left(A_{ji} + \frac{B_{ji}}{(1-\zeta_i)} + \tau_i B_{ji} + 2\sigma_i D_{ji} + \frac{1}{2}\sigma_i^2 D_{ji}\right) > 0.$$

In System (1), if $b_{ij} = 0$, then the system (1) is reduced to the following form n-dimensional RNNs with distributed time delays

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}\int_{t-\sigma_j(t)}^{t} h_j(x_j(s))ds + I_i. \tag{20}$$

Accordingly, the response system (2) is degenerated to

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} d_{ij}\int_{t-\sigma_j(t)}^{t} h_j(y_j(s))ds + I_i + u_i(t). \tag{21}$$

Accordingly, the assumptions H_1 and $\overline{\mathrm{H}}_1$ turn to

$\mathbf{H}_1^*$ For each $j \in \mathcal{I}$, the activation functions $f_j(u)$ and $h_j(u)$ are continuous and there exist constants $L_j, K_j, N_j, O_j > 0$, such that

$$|f_j(v_1) - f_j(v_2)| \leq L_j|v_1 - v_2| + N_j, \quad |h_j(v_1) - h_j(v_2)| \leq K_j|v_1 - v_2| + O_j, \quad v_1, v_2 \in R.$$

$\overline{\mathbf{H}}_1^*$ For each $j \in \mathcal{I}$, the activation functions $f_j(u), h_j(u)$ are globally Lipschitz continuous, i.e., there exist constants $L_j, K_j > 0$, such that

$$|f_j(v_1) - f_j(v_2)| \leq L_j|v_1 - v_2|, \quad |h_j(v_1) - h_j(v_2)| \leq K_j|v_1 - v_2|,$$

where $v_1, v_2 \in R$.

Then also from Theorem 3.1, we have the following two corollaries.

Corollary 3.4 *Suppose* $\mathrm{H}_1^*, \mathrm{H}_2$ *and* H_3 *hold, then the response network* (21) *can be general decay synchronized with the drive network* (20) *under the nonlinear controller* (6) *if, the control gains* β_i *and* α_i *satisfy the following inequalities:*

$$\begin{aligned} &c_i + \beta_i - \sum_{j=1}^{n} \left(A_{ji} + 2\sigma_i D_{ji} + \frac{1}{2}\sigma_i^2 D_{ji} \right) > 0, \\ &\alpha_i - \sum_{j=1}^{n} \left(N_i |a_{ji}| + \sigma_i O_i |d_{ji}| \right) > 0. \end{aligned} \tag{22}$$

Corollary 3.5 *Suppose* $\overline{\mathrm{H}}_1^*, \mathrm{H}_2$ *and* H_3 *hold, then the response network* (21) *can be general decay synchronized with the drive network* (20) *under the nonlinear controller* (19) *if, the control gains* β_i *satisfy the following inequality:*

$$c_i + \beta_i - \sum_{j=1}^{n} \left(A_{ji} + 2\sigma_i D_{ji} + \frac{1}{2}\sigma_i^2 D_{ji} \right) > 0. \tag{23}$$

4 Numerical Simulations

In this section, one example is given to illustrate the effectiveness of our results obtained in this paper.

Example 4.1 For $n = 2$, we consider the following chaotic recurrent neural network system with time-varying delays

$$\begin{aligned} \dot{x}_i(t) = &-c_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(x_j(t - \tau_j(t))) \\ &+ \sum_{j=1}^{2} d_{ij} \int_{t-\sigma_j(t)}^{t} h_j(x_j(s)) ds + I_i, \end{aligned} \tag{24}$$

where $f_1(u) = f_2(u) = \tanh(u)$, $g_1(u) = g_2(u) = \tanh(u) - \sin(u)$, $h_1(u) = h_2(u) = \tanh(u) - \cos(u)$. The parameters of System (24) are assumed that $c_1 = c_2 = 1$, $a_{11} = 2$, $a_{12} = -0.11$, $a_{21} = -2.5$, $a_{22} = 3.2$, $b_{11} = -1.6$, $b_{12} = -0.1$, $b_{21} = 0.18$, $b_{22} = -2.4, d_{11} = 0.2, d_{12} = 0, d_{21} = -0.2, d_{22} = 0.15$, $\tau_j(t) = \mathrm{e}^t/(1 + \mathrm{e}^t)$, $\sigma_j(t) = \mathrm{e}^t/(5 + \mathrm{e}^t)$ and $I_i = 0$ for $i = 1, 2$.

The corresponding response system is described by

$$\begin{aligned} \dot{y}_i(t) = &-c_i y_i(t) + \sum_{j=1}^{2} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(y_j(t - \tau_j(t))) \\ &+ \sum_{j=1}^{2} d_{ij} \int_{t-\sigma_j(t)}^{t} h_j(y_j(s)) ds + I_i + u_i(t), \end{aligned} \tag{25}$$

where c_i, a_{ij}, $b_{ij}, d_{ij}, f_j(t), g_j(t), h_j(t)$ $\tau_j(t), \sigma_j(t)$ and I_i are the same as in System (24).

The numerical simulations of System (24) and System (25) with initial values $x_1(s) = 0.2$, $x_2(s) = 0.5$ and $y_1(s) = -1.3$, $y_2(s) = 2.1$ for $s \in [-1, 0]$ are represented in Figures 1 and 2, we can see that System (24) and System (25) have chaotic attractors.

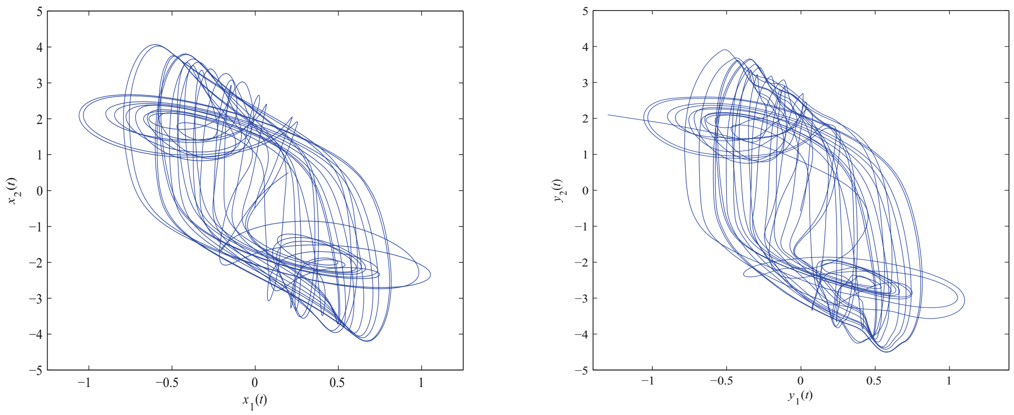

Figure 1 The chaotic behavior of delayed recurrent neural network system (24) and (25)

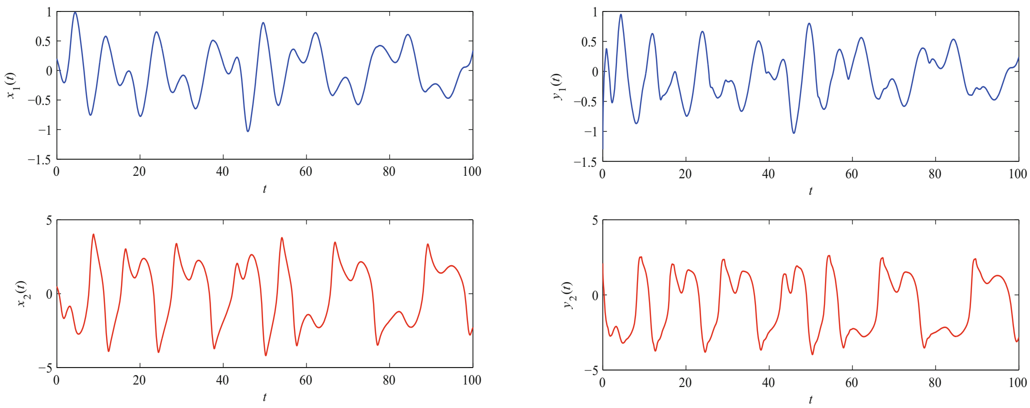

Figure 2 The chaotic behavior of delayed recurrent neural network system (24) and (25)

The nonlinear controller $u_i(t)$ is designed as follows

$$u_i(t) = -\alpha_i \text{sign}(e_i(t)) - \frac{\beta_i \|e(t)\| e_i(t)}{e_i(t) + \varrho(t)}, \quad i \in \mathscr{I}, \tag{26}$$

where $e_i(t) = y_i(t) - x_i(t)$ for $i = 1, 2$.

It is not difficult to estimate that $L_j = H_j = K_j = 1$, $N_j = 0.05, M_j = 0.04, O_j = 0.035$ and $\tau_j = \sigma_j = 1$. Thus, the assumptions H_1 and H_2 are satisfied. Letting $\varrho(t) = \mathrm{e}^{-0.1t}$, $\psi(t) = \mathrm{e}^t$ and choosing $\alpha_1 = 0.5, \alpha_2 = 0.6$, $\beta_1 = 7, \beta_2 = 5$. Then, the assumption H_3 and the inequality (7) of Theorem 3.1 are satisfied. Therefore, according to Theorem 3.1, the drive-response systems (24) and (25) can be achieved GDS under the controller (26). The time evolution of synchronization errors between systems (24) and (25) are demonstrated in Figure 3. The synchronization curves between the systems (24) and (25) are shown in Figure 4.

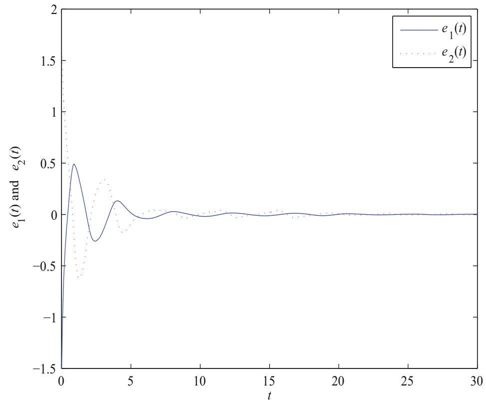

Figure 3 The evaluation of synchronization error $e_1(t)$ and $e_2(t)$

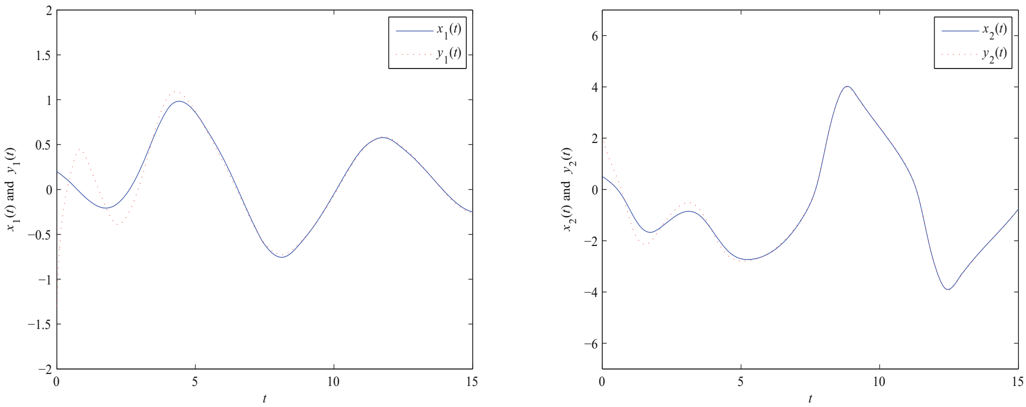

Figure 4 Synchronization curves of $x_1(t),\ y_1(t)$ and $x_2(t),\ y_2(t)$

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