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A Proposal for the Automatic Computation of Envelopes of Families of Plane Curves*

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Abstract The idea of envelope of a family of plane curves is an elementary notion in differential geometry. As such, its implementation in dynamic geometry environments is quite universal (Cabri, The Geometer's Sketchpad, Cinderella, GeoGebra, ...). Nevertheless, most of these programs return, when computing certain envelopes, both some spurious solutions and the curves that truly fit in the intuitive definition of envelope. The precise distinction between spurious and genuine parts has not been made before: This paper proposes such distinction in an algorithmic way, ready for its implementation in interactive geometry systems, allowing a finer classification of the different parts resulting from the current, advanced approach to envelope computation and, thus, yielding a more precise output, free from extraneous components.

Keywords Automated deduction in geometry, dynamic geometry, envelopes, parametric polynomial systems, symbolic computation.

1 Introduction

Recently, different issues on the automated computation of loci and envelopes have been subject of study within a research line concerning automatic reasoning tools in geometry. In particular, fostering cooperation between algebraic geometry and dynamic geometry, we proposed in [1] a taxonomy, and its algorithmic counterpart, for the automatic computation of plane algebraic loci. Customizing such taxonomy for the envelope case was first discussed in [2]. In this context, a note about the performance of well known dynamic geometry systems when computing envelopes was developed in [3], where a new symbolic approach for envelope

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computation, implemented in GeoGebra 5.0, was described. This approach, based on directly applying the locus taxonomy to the case of envelopes, was only partially successful, and so noted there[3,p.18].

The goal of this short note is to report an improvement - in fact, a complete solution - to the above mentioned situation. We introduce - through the notion of *acceptable special component*, see Subsection 2.3 — a further refinement in the taxonomy in order to deal with the undesirable and uncontrolled results of the currently implemented algorithm. Next Section shortly recalls the extension of the locus taxonomy to the envelope case, and describes a new algorithmic test to accept or discard relevant parts of the resulting curve. A pair of examples of its performance are extensively discussed, and the involved symbolic computations are shown for the sake of illustration.

2 A Complete Taxonomy for Envelope Computation

Given a family of curves $\{C_t : \mathcal{F}(x, y, t) = 0\}$ parametrized by $t \in \mathbb{R}$ we follow [4] for defining the envelope of the family:

$$\left\{(x, y) \in \mathbb{R}^2 \mid \exists\ t \text{ s.t. } \mathcal{F}(x, y, t) = 0 \text{ and } \frac{\partial \mathcal{F}(x, y, t)}{\partial t} = 0\right\}.$$

It is usual in dynamic geometry to consider the dependence of the family on a moving point (t_1, t_2), rather than on a scalar magnitude t. Bearing this observation in mind, we can consider that the given family is $\{C_P : \mathcal{F}(x, y, t_1, t_2) = 0\}$, with $P(t_1, t_2)$ constrained to lie on the curve $\{g(t_1, t_2) = 0\}$, and its envelope is the set of points (x, y) coming from the elimination of t_1, t_2 in the system

$$\begin{aligned}
&\mathcal{F}(x, y, t_1, t_2) = 0,\\
&g(t_1, t_2) = 0,\\
&\frac{\partial \mathcal{F}(x, y, t_1, t_2)}{\partial t_1}\frac{\partial g(t_1, t_2)}{\partial t_2} - \frac{\partial \mathcal{F}(x, y, t_1, t_2)}{\partial t_2}\frac{\partial g(t_1, t_2)}{\partial t_1} = 0.
\end{aligned}$$

Other variants for the formulation of the envelopes computation problem include the case in which the dependence on the parameters is masked through other objects in the geometric construction. That is, if the family is multi-parametric, $\mathcal{F}(t_1, t_2, \cdots, t_n, x, y)$, with $n > 2$, where the parameters are bounded by $n - 1$ constraints, $\{g_1(t_1, t_2, \cdots, t_n) = 0, \cdots, g_{n-1}(t_1, t_2, \cdots, t_n) = 0\}$, the above conditions are replaced by $\mathcal{F} = g_1 = \cdots = g_{n-1} = 0$ and

$$\begin{vmatrix} \partial \mathcal{F}/\partial t_1 & \cdots & \partial \mathcal{F}/\partial t_n \\ \partial g_1/\partial t_1 & \cdots & \partial g_1/\partial t_n \\ \partial g_{n-1}/\partial t_1 & \cdots & \partial g_{n-1}/\partial t_n \end{vmatrix} = 0.$$

What is relevant here is to remark that, in general, computing the envelope of a family of curves is an elimination problem, and, thus, it is obviously quite similar to the more general issue of loci determination. As done for the loci case, we restrict the base field to $\mathbb{Q}$ and are aware that solutions will be found in $\mathbb{C}$. It is well known that generic elimination for computing envelopes could include extra branches due to degeneration and extra parts due to Zariski closures. In this respect we shortly recall the proposal of a taxonomy for envelopes, fully described in [2].

2.1 Classifying Envelopes

In our approach, an envelope problem in dynamic geometry is translated into a system of parametric polynomial equations $F \subseteq \mathbb{Q}[\boldsymbol{u}, \boldsymbol{t}]$ where $\boldsymbol{u} = (x, y)$ are the variables representing the coordinates of the envelope points and $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)$ are the parameters. The solution variety is

$$\boldsymbol{V}(F) = \{(\boldsymbol{u}, \boldsymbol{t}) \subset \mathbb{C}^{2+n} : \forall f \in F, f(\boldsymbol{u}, \boldsymbol{t}) = 0\},$$

and we denote by π_1 and π_2 the projections onto the variable and parameter spaces, respectively:

$$\begin{array}{rcl} \pi_1 : \mathbb{C}^{2+n} & \longrightarrow & \mathbb{C}^2 \\ (\boldsymbol{u}, \boldsymbol{t}) & \mapsto & \boldsymbol{u} \end{array} \qquad \begin{array}{rcl} \pi_2 : \mathbb{C}^{2+n} & \longrightarrow & \mathbb{C}^n \\ (\boldsymbol{u}, \boldsymbol{t}) & \mapsto & \boldsymbol{t} \end{array}$$

By definition, the envelope associated to the parametric polynomial system $F(\boldsymbol{u}, \boldsymbol{t})$ is the set $E = \pi_1(\boldsymbol{V}(F)) \subset \mathbb{C}^2$. The set E is split into two disjoint parts, concerning the normality of its elements. In fact, *normal points* are the points $\boldsymbol{u} \in \mathbb{C}^2$ of the envelope for which $\dim(\pi_2(\boldsymbol{V}(F) \cap \pi_1^{-1}(\boldsymbol{u}))) = 0$. The points $\boldsymbol{u}$ of the envelope for which $\dim(\pi_2(\boldsymbol{V}(F) \cap \pi_1^{-1}(\boldsymbol{u}))) > 0$ are called *non-normal*. The set of all normal points is called the *normal envelope* and the set of all non-normal points is called the *non-normal envelope*. Both parts of the envelope are constructible sets, see [1, Proposition 3.5]. A further subdivision of the envelope can be introduced by considering the components associated to the canonical representation of the normal and non-normal envelope as constructible sets.

A locally closed set L is a difference of algebraic varieties $L = \boldsymbol{V}(E) \setminus \boldsymbol{V}(N)$. For a locally closed set, a canonical P-representation expressed in terms of prime ideals can be obtained[5]:

$$PREP(L) = \{(\mathfrak{p}_i, \{\mathfrak{p}_{ij} : 1 \leq j \leq r_i\}) : 1 \leq j \leq r\}$$

so that

$$L = \bigcup_{i=1}^{r} \left(\boldsymbol{V}(\mathfrak{p}_i) \setminus \left(\bigcup_{j=1}^{r_i} \boldsymbol{V}(\mathfrak{p}_{ij}) \right) \right).$$

Each element $\boldsymbol{V}(\mathfrak{p}_i) \setminus \left(\bigcup_{j=1}^{r_i} \boldsymbol{V}(\mathfrak{p}_{ij}) \right)$ is called a *component* of L.

A component C_s of the normal envelope is *special* if $\dim(C_s) > 0$ and $\dim(\pi_2(\boldsymbol{V}(F) \cap \pi_1^{-1}(C_s))) = 0$. The remaining components of the normal envelope are *normal*.

The components C_d of the non-normal envelope of dimension greater than 0 are considered *degenerate components*, whereas the zero-dimensional components are *accumulation points* of the envelope.

Let us recall that all of these objects can be algorithmically computed and have been already implemented in different programs, see [1].

Although in the case of loci we proposed in [1] accepting normal components and accumulation points as parts of the sought loci, this proposal, for envelopes, is not well adapted to our intuition, since there are constructions where it seems special components should be clearly included, and other cases where they should not, as we will show in the next examples.

2.2 Two Examples Dealing with Special Components

Consider, as a first case, the construction depicted in Figure 1. Given the circle centered at the origin and passing through $B(1,0)$, we want to find the envelope of perpendicular lines to lines BC when C moves along the circle. The involved polynomials are

$$F = t_1^2 + t_2^2 - 1, \quad t_2 y + (t_1 - 1)(x+1), \quad t_1 y - t_2 (x-1),$$

and the solution of the parametric system returns two components: $(-1,0)$, as an accumulation point (thus being part of the non-normal envelope), and the constructible set $\boldsymbol{V}(y) - \boldsymbol{V}(x+1, y)$. Note that the points of this component are, by definition, normal envelope points, and the whole component is thus labeled as special. But, of course, it should not to be considered as part of the envelope, as it arises from a degeneracy in the family when B and C coincide.

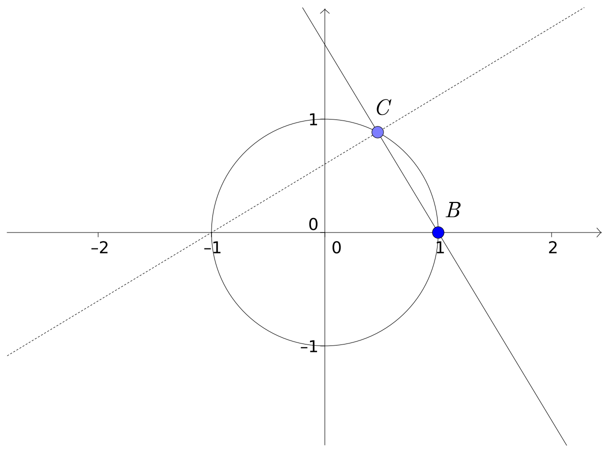

Figure 1 The envelope is just the accumulation point $(-1,0)$

As a contradictory case, we study the envelope of horizontal lines passing through a point in the unit circle centered at the origin. The construction is described as

$$F = t_1^2 + t_2^2 - 1, \quad y - t_2, \quad 2t_1,$$

and the elimination returns two components $y = \pm 1$. Both are classified as special since their dimension is 1 and they derive from the parametric points $(0, \pm 1)$. But, instead of dropping them as suggested by the taxonomy, these lines are the sought envelope (Figure 2), following its standard definition[4].

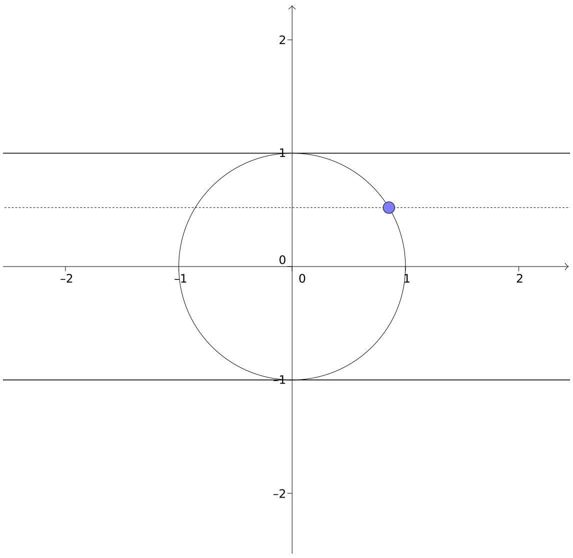

Figure 2 The envelope of horizontal lines through points in the unit circle

2.3 Distinguishing Between Special Components

As the above examples illustrate, special envelope components should be accepted as part of the envelope, but only provided that they do not come from degeneration. Thus, we propose an algorithmic test to detect the non existence of a curve of the family.

Recall that given an envelope construction described by a set of polynomials F (where the Jacobi polynomial(s), related to the partial derivatives of the involved equations, are included), a component C_s of the normal envelope is *special* if $\dim(C_s) > 0$ and $\dim(\pi_2(\boldsymbol{V}(F) \cap \ \pi_1^{-1}(C_s))) = 0$. Since the dimension of the corresponding set of parametric points is 0, our idea consists on detecting the eventual degeneration of the family curve by studying the dimension of the polynomial construction for such values of the parameters. Thus, we declare a special component C_s as *acceptable* if

$$\forall (x, y) \in C_s, \quad \exists \boldsymbol{t} \in \pi_2(\pi_1^{-1}(C_s)) \quad \text{s.t. } \dim(\pi_2^{-1}(\boldsymbol{t}) \cap \boldsymbol{V}(F^*)) \leq 1,$$

where F^* is the polynomial set without the Jacobi polynomials. It is easy to deduce that, as shown in the examples below, there is an algorithmic way to decide about the "acceptability" of a component.

The idea behind such condition is to study, for each point in the special component, the existence of the corresponding curve of the family. To this end, we look for at least a value of the parameters such that the dimension of the curve is at most 1. Then, our new protocol, namely:

- to apply the taxonomy in [1] to the case of envelopes, except in the case of special components,
- considering in this case only those that are acceptable, as defined above,

states that an acceptable special component of an envelope is part of the *true* envelope (i.e., one that meets the standard definition and our protocol requirements). Otherwise, it is not.

In what follows, we review the above examples under the new light of the acceptability of a special component. We include the Maple computations for the sake of clarity.

The construction described in Figure 1 involves the computation of the family of straight lines that are perpendicular to lines BC, where $B(1,0)$ is a fixed point on the unit circle centered at the origin, and C moves along it. Thus, the family of straight lines is implicitly defined by

$$f = (y - t_2)t_2 - (x - t_1)(1 - t_1),$$

where the parameter relation is

$$g = t_1^2 + t_2^2 - 1,$$

and the Jacobi polynomial is

$$h = \frac{\partial f}{\partial t_1}\frac{\partial g}{\partial t_2} - \frac{\partial f}{\partial t_2}\frac{\partial g}{\partial t_1} = t_2 + t_2 x - t_1 y.$$

In Maple, we define the set of polynomials and, projecting on x, y, we get the envelope candidates:

```
> with(PolynomialIdeals):F:=<f,g,h>:EliminationIdeal(F,{x,y});

                              <y^2>
```

Note that the double line $y = 0$ is the Zariski closure of the projection while the envelope, as defined in Section 2, is the constructible projection. But, in this case, both sets coincide since for each value of x with $y = 0$, there exist t_1, t_2 values satisfying $F = 0$:

```
> solve(subs(y=0,{t2+t2*x-t1*y, t1^2+t2^2-1, (y-t2)*t2-(x-t1)*(1-
  t1)}),{t1,t2,x});

            {t1 = 1,t2 = 0,x = x}, {t1 = -1,t2 = 0,x = -1},
            {t1 = RootOf(_Z^2+t2^2-1,t2=t2,x=-1}
```

The above computation is also useful to study the normality of the envelope points. Taking a point $(x, 0)$, we lift it up to $F = 0$, and after projecting this lifting on t_1, t_2, we check the finite character of this projection. So, all points $(x, 0)$ are normal points except $(-1, 0)$, that is a non-normal point. The envelope has two components: The accumulation point $\{(-1, 0)\}$, a non-normal one, and the constructible set $\boldsymbol{V}(y) - \boldsymbol{V}(x + 1, y)$, that is a normal component. Furthermore, it is special since for all point $(x, 0)$ with $x \neq -1$ the solutions of t_1, t_2 in F are just $(1, 0)$.

The last step consists of studying its acceptability. To this end, choosing $t_1 = 1, t_2 = 0$, we lift to the family of curves without the Jacobi polynomial, and we check the dimension:

```
> HilbertDimension(<t1^2+t2^2-1, (y-t2)*t2-(x-t1)*(1-t1), t1-1,
t2>,{x,y,t1,t2});

                            2
```

concluding that this component is not acceptable as part of the sought envelope.

The second example illustrates the opposite case, one where the special components are part of the true envelope. Recall that in this case the family of curves are the horizontal lines through a point moving along the unit circle centered at the origin (Figure 2). Thus, the family is implicitly defined by $f = y - t_2$, and the parametric point is constrained by $g = t_1^2 + t_2^2 - 1$. The Jacobi polynomial is $h = 2t_1$.

The envelope candidates are the lines $y = \pm 1$:

```
> restart:with(PolynomialIdeals):F:=<f,g,h>:
  EliminationIdeal(F,{x,y});

                            <-1+y^2>
```

As in the previous problem, the elimination variety, $y^2 = 1$, is the Zariski closure of the projection instead the constructible projection. The discussion of such issue is solved simultaneously with the study about normality of points. Taking a point $(x, 1)$ (resp. $(x, -1)$), the lifting to $F = 0$ is projected on t_1, t_2. Both cases return a finite number of solutions, thus being all points normal:

```
> solve(subs(y=1,{t1^2+t2^2-1, t1, y-t2}),{t1,t2,x});

                            {t1 = 0, t2 = 1, x = x}

> solve(subs(y=-1,{t1^2+t2^2-1, t1, y-t2}),{t1,t2,x});

                            {t1 = 0, x = x, t2 = -1}
```

Thus, $y = \pm 1$ are components of the normal envelope, and they are also special since their solutions for t_1, t_2 in F are $(0, 1)$ (resp. $(0, -1)$). Finally, we check if $y = 1$ (resp. $y = -1$) is acceptable, taking $t_1 = 0, t_2 = 1$ (resp. $t_1 = 0, t_2 = -1$) and lifting it up to the polynomial set F without the Jacobi polynomial:

```
> HilbertDimension(<t1^2+t2^2-1, y-t2,t1,t2-1>,{x,y,t1,t2});

                            1

> HilbertDimension(<t1^2+t2^2-1, y-t2,t1,t2+1>,{x,y,t1,t2});

                            1
```

concluding that both components are acceptable, and the envelope is the expected one.

3 Conclusion

In this note we have motivated, proposed and exemplified a new algorithmic test, allowing a finer classification of the different parts of the curve resulting from the standard envelope computation and, thus, yielding a more precise output, free from extraneous components.

We think the proposal is well fitted into the intuitive idea of envelope, and deserves its implementation in interactive geometry systems, in order to be automatically tested and analyzed over suitable benchmark collections.

Acknowledgements With the occasion of Professor Wen-tsun Wu's Centennial Birthday, we would like to honor, through this humble contribution, the decisive role of Professor Wen-tsun Wu's work in the development of symbolic computation methods for the automatization of geometric reasoning.

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