J Syst Sci Complex (2019) 32: 985–996

Recursive Identification of Quantized Linear Systems*

XIAO Jianming · SONG Qijiang

DOI: 10.1007/s11424-019-8207-z
Received: 16 July 2018 / Revised: 23 November 2018

Abstract This paper studies the identification of linear systems with quantized output observations. Recursive estimates for the linear system and the quantization thresholds are derived by the stochastic approximation algorithms with expanding truncations (SAAWET). Under suitable conditions, it is shown that the estimates converge to the true values almost surely.

Keywords α-mixng, ARMA, quantized sensor, stochastic approximation, strongly consistent.

1 Introduction

Systems with binary or quantized observations have been studied a lot in the recent years for their extensive applications in practical fields, e.g., the networked control systems, biological networks, and automotive systems[1]. Due to the limited system output information, identification of quantized systems is difficult.

To date, many different types of quantized systems have been explored. In [1–8] the FIR/IIR/ARMA systems with known thresholds of binary sensors or quantized sensors are considered. In [9, 10] the FIR/ARMA systems with designed adaptive quantized sensor are dealt with. And the FIR/ARMA systems with unknown threshold of binary sensor are studied in [11–13].

In this paper we consider the identification of the ARMA systems followed by a general quantized sensor with unknown thresholds. Under reasonable conditions, the strongly consistent estimates for the parameters of the linear system and the thresholds are obtained by the SAAWET (see [14] or Appendix). While in order to guarantee the identification algorithms strong consistency, the thresholds are required to be known in [3, 5, 7], and suitable adaptive quantizers need to be designed in [9, 10].

As in [11], iid Gaussian inputs are applied to identify the underlying system, the almost surely convergence rates are also derived. Compared to the systems with binary sensor, the

XIAO Jianming
School of Mathematics, Renmin University of China, Beijing 100872, *China.* Email: jmxiao@126.com.
SONG Qijiang (Corresponding author)
School of Mathematics, Renmin University of China, Beijing 100872, *China.* Email: sqj@ruc.edu.cn.
*This research was supported by the National Natural Science Foundation of China under Grant No. 11571186.

thresholds of the general quantizer are more difficult to be estimated. With the help of the SAAWET, the thresholds could be separated and identified gradually.

The rest of the paper is organized as follows. In Section 2, the quantized system is formulated and the recursive identification algorithms are presented. In Section 3, the strong consistency of the estimates is proved. A simulation example is given in Section 4, and some concluding remarks are provided in Section 5.

2 System and Identification Algorithms

2.1 System and Estimates for the ARMA System

As shown in Figure 1, the system is given as follows

$$C(q^{-1})v_{k+1} = D(q^{-1})u_k, \tag{1}$$
$$y_k = v_k + \eta_k, \tag{2}$$
$$z_k = \begin{cases} a_1, & y_k \geq S_1, \\ a_2, & S_2 \leq y_k < S_1, \\ 0, & y_k < S_2, \end{cases} \tag{3}$$

where q^{-1} is the backward-shift operator: $q^{-1}u_k = u_{k-1}$, u_k and z_k are the system input and output, respectively, (3) represents the quantized sensor with the thresholds $S_2 < S_1$ and the different output values $a_1, a_2, 0$. The signals v_k and y_k are not directly observed. η_k is the additive noise, which is an ARMA process:

$$F(q^{-1})\eta_k = G(q^{-1})\varepsilon_k, \tag{4}$$
$$F(q^{-1}) = 1 + f_1 q^{-1} + \cdots + f_l q^{-l}, \tag{5}$$
$$G(q^{-1}) = 1 + g_1 q^{-1} + \cdots + g_m q^{-m} \tag{6}$$

with unknown orders $l, m \geq 1$ and with unknown coefficients $\{f_i, g_j\}$. Our problem is to recursively estimate $\{c_1, \cdots, c_p, d_1, \cdots, d_r\}$ in the following polynomials of the linear part

$$C(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_p q^{-p}, \tag{7}$$
$$D(q^{-1}) = 1 + d_1 q^{-1} + \cdots + d_r q^{-r} \tag{8}$$

with known orders $p, r \geq 1$ and the thresholds S_1, S_2 of the quantized sensor on the basis of the observed signals $\{u_k, z_k\}$.

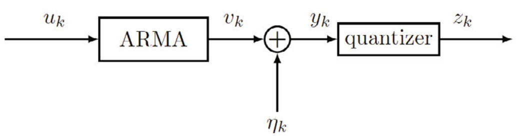

Figure 1 ARMA systems with quantized outputs and noises

Remark 2.1 The structure of the quantized sensor (3) is known. Recursive estimates and convergence analysis can be obtained similarly for the general quantized sensors, i.e.,

$$z_k = \begin{cases} a_1, & y_k \geq S_1, \\ a_2, & S_2 \leq y_k < S_1, \\ a_3, & S_3 \leq y_k < S_2, \\ \vdots & \\ a_n, & S_n \leq y_k < S_{n-1}, \\ a_{n+1}, & y_k < S_n \end{cases}$$

with $S_n < S_{n-1} < \cdots < S_2 < S_1$ and different values $a_1, a_2, \cdots, a_{n+1}, \forall n \geq 1$. For the case $a_{n+1} \neq 0$, $\{z_k - a_{n+1}\}$ could be regarded as the new output observations.

Assumptions imposed on the system (1)–(8) are as follows:

A1 $C(q^{-1})$ and $D(q^{-1})$ are coprime, $c_p \neq 0, d_r \neq 0$, and $C(q^{-1})$ is stable (i.e., all roots of $q^p C(q^{-1}) = 0$ are inside the open unit disk).

A2 $F(q^{-1})$ and $G(q^{-1})$ are coprime, and $F(q^{-1})$ is stable.

A3 $\{\varepsilon_k\}$ is a sequence of zero mean iid Gaussian random variables with unknown variance σ_ε^2. $\{u_k\}$ and $\{\varepsilon_k\}$ are mutually independent, and $u_k = 0, \varepsilon_k = 0$ for $k < 0$.

A4 The input sequence $\{u_k\}$ is a sequence of iid random variables and $u_k \sim \mathcal{N}(0, 1)$.

By A1 we have

$$v_{k+1} = C^{-1}(q^{-1})D(q^{-1})u_k = \sum_{i=0}^{k} h_i u_{k-i}, \tag{9}$$

where $\{h_i\}$ are impulse responses with $h_0 = 1$, and $|h_i|$ is of the same order as $\mathrm{e}^{-\mu i}$, i.e., $|h_i| = O(\mathrm{e}^{-\mu i}),\ \mu > 0, \forall i \geq 1$. By A4 and (9), it is clear that $v_k \sim \mathcal{N}(0, \sigma_{v,k}^2)$, where $\sigma_{v,k}^2 = \sum_{i=0}^{k-1} h_i^2$ and

$$\mathbb{E}u_k v_{k+i+1} = h_i, \quad i = 0, 1, \cdots, \quad k \geq 0, \tag{10}$$

in which $\mathbb{E}$ denotes the expectation operator. By (4), A2 and A3 it follows that $\{u_k\}$ and $\{\eta_k\}$ are mutually independent, which together with (2) and (10) implies

$$\mathbb{E}u_k y_{k+i+1} = h_i, \quad i = 0, 1, \cdots, \quad k \geq 0. \tag{11}$$

Let $\sigma_{y,k}^2 \triangleq \mathbb{E}y_k^2$ and $\sigma_{\eta,k}^2 \triangleq \mathbb{E}\eta_k^2$. It is clear that $y_k \sim \mathcal{N}(0, \sigma_{y,k}^2)$ with $\sigma_{y,k}^2 = \sigma_{v,k}^2 + \sigma_{\eta,k}^2$. By the correlation analysis as in [11], it follows that

$$\mathbb{E}[u_k | y_{k+i+1}] = \frac{h_i}{\sigma_{y,k+i+1}^2} y_{k+i+1}, \text{ a.s.}, \tag{12}$$

$\forall i \geq 0, k \geq 0$. By (3) and (12) we have

$$\begin{aligned}\mathbb{E}u_k z_{k+i+1} &= \mathbb{E}[z_{k+i+1}\mathbb{E}[u_k|y_{k+i+1}]]\\ &= \frac{h_i}{\sigma^2_{y,k+i+1}}\mathbb{E}z_{k+i+1}y_{k+i+1}\\ &= \frac{h_i}{\sigma_{y,k+i+1}}\left((a_1-a_2)\int_{\frac{S_1}{\sigma_{y,k+i+1}}}^{\infty} x\varphi_0(x)dx + a_2\int_{\frac{S_2}{\sigma_{y,k+i+1}}}^{\infty} x\varphi_0(x)dx\right)\\ &\xrightarrow[k\to\infty]{} \frac{h_i}{\sqrt{2\pi}\sigma_y}\left((a_1-a_2)\exp\left\{-\frac{S_1^2}{2\sigma_y^2}\right\} + a_2\exp\left\{-\frac{S_2^2}{2\sigma_y^2}\right\}\right), \quad \forall i \geq 0, \qquad (13)\end{aligned}$$

where $\varphi_0(x) = \frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}$, $\sigma_y^2 \triangleq \sigma_v^2 + \sigma_\eta^2$ with $\sigma_v^2 \triangleq \lim_{k\to\infty}\sigma^2_{v,k} = \sum_{i=0}^{\infty} h_i^2$ and $\sigma_\eta^2 \triangleq \lim_{k\to\infty}\sigma^2_{\eta,k}$.

Let $\rho \triangleq \frac{1}{\sqrt{2\pi}\sigma_y}\Big((a_1-a_2)\exp\{-\frac{S_1^2}{2\sigma_y^2}\} + a_2\exp\{-\frac{S_2^2}{2\sigma_y^2}\}\Big)$. By (13) we obtain

$$\mathbb{E}u_k z_{k+i+1} \xrightarrow[k\to\infty]{} \rho h_i, \quad \forall i \geq 0. \qquad (14)$$

The following additional assumption needs to be imposed on the system:

A5 $\rho = \frac{1}{\sqrt{2\pi}\sigma_y}\Big((a_1-a_2)\exp\{-\frac{S_1^2}{2\sigma_y^2}\} + a_2\exp\{-\frac{S_2^2}{2\sigma_y^2}\}\Big) \neq 0$.

As in [11], by setting $h_i = 0$ for $i < 0$, we derive the Yule-Walker equations:

$$H\begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_p\end{pmatrix} = \begin{pmatrix} -h_{r+1}\\ -h_{r+2}\\ \vdots\\ -h_{r+p}\end{pmatrix}, \qquad (15)$$

$$h_i = -\sum_{j=1}^{p} c_j h_{i-j} + d_i, \quad i = 1, 2, \cdots, r, \qquad (16)$$

where H is the Hankel matrix defined as follows

$$H \triangleq \begin{pmatrix} h_r & h_{r-1} & \cdots & h_{r+1-p}\\ h_{r+1} & h_r & \cdots & h_{r+2-p}\\ \vdots & \vdots & \ddots & \vdots\\ h_{r+p-1} & h_{r+p-2} & \cdots & h_r\end{pmatrix}.$$

By A1 the Hankel matrix H is nonsingular[15]. Thus, for estimating $\{c_i,\ d_i\}$, it suffices to estimate $\{h_i,\ i = 1, 2, \cdots, p+r\}$.

Let $h_{i,k}$ be the estimate for h_i at time k, and let $h_{i,k} = 0$ for $i < 0$. Define

$$H_k \triangleq \begin{pmatrix} h_{r,k} & h_{r-1,k} & \cdots & h_{r+1-p,k}\\ h_{r+1,k} & h_{r,k} & \cdots & h_{r+2-p,k}\\ \vdots & \vdots & \ddots & \vdots\\ h_{r+p-1,k} & h_{r+p-2,k} & \cdots & h_{r,k}\end{pmatrix}. \qquad (17)$$

Then the estimates for $\{c_i,\, d_i\}$ are given as follows

$$[c_{1,k}\ \cdots\ c_{p,k}]^{\mathrm{T}} = \begin{cases} -H_k^{-1}[h_{r+1,k}\ \cdots\ h_{r+p,k}]^{\mathrm{T}}, & \text{if } \det(H_k)\neq 0, \\ [\underbrace{0\ \cdots\ 0}_{p}]^{\mathrm{T}}, & \text{otherwise}, \end{cases} \tag{18}$$

$$d_{i,k} = h_{i,k} + \sum_{l=1}^{p} c_{l,k}h_{i-l,k}, \quad i = 1,2,\cdots,r. \tag{19}$$

Since $h_0 = 1$ and $\rho \neq 0$, for estimating $\{h_i, i = 1,2,\cdots,p+r\}$, it suffices to estimate $\{\rho h_i, i = 0,1,\cdots,p+r\}$. Let $I_{[\cdot]}$ be the indicator function of the set $[\cdot]$ and $\{M_k\}$ be a sequence nondecreasing positive numbers diverging to infinity. Then by the SAAWET, $\{\theta_{i,k}\}$ are recursively defined as follows

$$\theta_{i,k+1} = \Big[\theta_{i,k} - \frac{1}{k+1}(\theta_{i,k} - u_k z_{k+i+1})\Big]\cdot I_{[|\theta_{i,k}-\frac{1}{k+1}(\theta_{i,k}-u_k z_{k+i+1})|\le M_{\delta_{i,k}}]}, \tag{20}$$

$$\delta_{i,k} = \sum_{j=1}^{k-1} I_{[|\theta_{i,j}-\frac{1}{j+1}(\theta_{i,j}-u_j z_{j+i+1})|>M_{\delta_{i,j}}]} \tag{21}$$

with $\delta_{i,0} = 0$ and an arbitrary initial value $\theta_{i,0}$. And the estimates for h_i, $i \geq 1$, at time k are given by

$$h_{i,k} \triangleq \begin{cases} \theta_{i,k}/\theta_{0,k}, & \text{if } \theta_{0,k}\neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{22}$$

2.2 Estimation of the Thresholds

Since y_k is Gaussian and $y_k \sim N(0,\sigma_{y,k}^2)$, by (3) we have

$$\begin{aligned} \mathbb{E}I_{[z_k=a_1]} &= P\{y_k \geq S_1\} = \int_{\frac{S_1}{\sigma_{y,k}}}^{\infty} \varphi_0(x)dx \\ &\xrightarrow[k\to\infty]{} \int_{\frac{S_1}{\sigma_y}}^{\infty} \varphi_0(x)dx = 1-\phi_0(S_1/\sigma_y), \end{aligned} \tag{23}$$

$$\begin{aligned} \mathbb{E}I_{[z_k=a_2]} &= P\{S_2 \leq y_k < S_1\} \\ &= P\{y_k \geq S_2\} - P\{y_k \geq S_1\} \\ &\xrightarrow[k\to\infty]{} \big(1-\phi_0(S_2/\sigma_y)\big) - \big(1-\phi_0(S_1/\sigma_y)\big) \end{aligned} \tag{24}$$

with $\phi_0(t) \triangleq \int_{-\infty}^{t}\varphi_0(x)dx$.

Let $\gamma^{(i)} \triangleq 1-\phi_0(S_i/\sigma_y), i = 1,2$. Then by the SAAWET, the estimates for $\gamma^{(1)},\gamma^{(2)}$ are given as follows:

$$\gamma_{k+1}^{(1)} = \Big[\gamma_k^{(1)} - \frac{1}{k+1}(\gamma_k^{(1)} - I_{[z_k=a_1]})\Big]\cdot I_{[|\gamma_k^{(1)}-\frac{1}{k+1}(\gamma_k^{(1)}-I_{[z_k=a_1]})|\le M_{\delta_k^{(1)}}]}, \tag{25}$$

$$\delta_k^{(1)} = \sum_{j=1}^{k-1} I_{[|\gamma_j^{(1)}-\frac{1}{j+1}(\gamma_j^{(1)}-I_{[z_j=a_1]})|>M_{\delta_j^{(1)}}]} \tag{26}$$

with $\delta_0^{(1)} = 0$ and an arbitrary initial value $\gamma_0^{(1)}$, and

$$\gamma_{k+1}^{(2)} = \left[\gamma_k^{(2)} - \frac{1}{k+1}(\gamma_k^{(2)} - \gamma_k^{(1)} - I_{[z_k=a_2]})\right] \cdot I_{[|\gamma_k^{(2)} - \frac{1}{k+1}(\gamma_k^{(2)} - \gamma_k^{(1)} - I_{[z_k=a_2]})| \le M_{\delta_k^{(2)}}]}, \tag{27}$$

$$\delta_k^{(2)} = \sum_{j=1}^{k-1} I_{[|\gamma_j^{(2)} - \frac{1}{j+1}(\gamma_j^{(2)} - \gamma_j^{(1)} - I_{[z_j=a_2]})| > M_{\delta_j^{(2)}}]} \tag{28}$$

with $\delta_0^{(2)} = 0$ and an arbitrary initial value $\gamma_0^{(2)}$.

By noticing that S_i/σ_y is the single root of the equation $1 - \phi_0(x) - \gamma^{(i)} = 0, i = 1, 2$, the SAAWET can be applied to estimate S_i/σ_y as follows:

$$\Theta_{k+1}^{(i)} = \left[\Theta_k^{(i)} + \frac{1}{k+1}(1 - \phi_0(\Theta_k^{(i)}) - \gamma_k^{(i)})\right] \cdot I_{[|\Theta_k^{(i)} + \frac{1}{k+1}(1-\phi_0(\Theta_k^{(i)}) - \gamma_k^{(i)})| \le M_{\Delta_k^{(i)}}]}, \tag{29}$$

$$\Delta_k^{(i)} = \sum_{j=1}^{k-1} I_{[|\Theta_j^{(i)} + \frac{1}{j+1}(1-\phi_0(\Theta_j^{(i)}) - \gamma_j^{(i)})| > M_{\Delta_j^{(i)}}]} \tag{30}$$

with $\Delta_0^{(i)} = 0$ and an arbitrary initial value $\Theta_0^{(i)}, i = 1, 2$.

By A5 and the estimations of ρ and S_i/σ_y, the estimate for S_i at time k is given as follows

$$s_k^{(i)} \triangleq \begin{cases} \Theta_k^{(i)} \dfrac{(a_1-a_2)\exp\left\{-\frac{1}{2}\left(\Theta_k^{(1)}\right)^2\right\} + a_2 \exp\left\{-\frac{1}{2}\left(\Theta_k^{(2)}\right)^2\right\}}{\sqrt{2\pi}\theta_{0,k}}, & \text{if } \theta_{0,k} \ne 0, \\ 0, & \text{if } \theta_{0,k} = 0, \end{cases} \tag{31}$$

$i = 1, 2$.

3 Consistency of Estimates

As in [11], let $a \triangleq \max\{p, r+1\}$, $b \triangleq \max\{l, m+1\}$, $c_i \triangleq 0$ if $p < i \le a$, $d_j \triangleq 0$ if $r < j \le a$, $f_k \triangleq 0$ if $l < k \le b$, and $g_n \triangleq 0$ if $m < n \le b$. Define

$$V_k \triangleq [v_k \ \cdots \ v_{k-a+1} \ u_k \ \cdots \ u_{k-a+1} \ \eta_k \ \cdots \ \eta_{k-b+1} \ \varepsilon_{k+1} \ \cdots \ \varepsilon_{k-b+2}]^{\mathrm{T}} \in \boldsymbol{R}^{2(a+b)}.$$

Lemma 3.1 (see [11]) *Assume* A1–A4 *hold. Then $\{V_k\}$ is a zero mean α-mixing process with the mixing coefficients α_k exponentially decay to zero: $\alpha_k \le d\lambda^k$ for some $d > 0$ and $\lambda \in (0, 1), \forall k \ge 1$.*

Remark 3.1 The definition of the α-mixing process please refer to [16]. It is well known that the mixing property is hereditary, i.e., the process $\{h(X_k)\}$ for any Borel measurable function $h(\cdot)$ possesses the same mixing property as $\{X_k\}$ does. By Lemma 3.1 and the hereditary property of α-mixing, the processes $\{I_{[z_k=a_i]} - \mathbb{E}I_{[z_k=a_i]}\}, i = 1, 2$, and $\{u_k z_{k+i+1} - \mathbb{E}u_k z_{k+i+1}\}, \forall i \ge 0$, are all α-mixing processes with the mixing coefficients exponentially decay to zero (under Conditions A1–A4).

Lemma 3.2 (see [15]) *Let $\{X_k\}$ be a zero mean α-mixing process with the mixing coefficients α_k exponentially decay to zero: $\alpha_k \le d\lambda^k$ for some $d > 0$ and $\lambda \in (0, 1), \forall k \ge 1$. If there exists a constant $\varepsilon > 0$ such that $\sum_{k=1}^{\infty}(\mathbb{E}|X_k|^{2+\varepsilon})^{\frac{2}{2+\varepsilon}} < \infty$, then $\sum_{k=1}^{\infty} X_k < \infty$ a.s.*

Lemma 3.3 *Assume* A1–A4 *hold. Then,* $\forall \nu \in (0, \frac{1}{2})$,

$$\sum_{k=1}^{\infty} \frac{1}{k^{1-\nu}}(I_{[z_k=a_i]} - \mathbb{E}I_{[z_k=a_i]}) < \infty \quad a.s., \ i = 1, 2, \tag{32}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{1-\nu}}(u_k z_{k+i+1} - \mathbb{E}u_k z_{k+i+1}) < \infty \quad a.s., \ \forall i \geq 0. \tag{33}$$

Proof By Remark 3.1 and Lemma 3.2, it suffices to prove

$$\sum_{k=1}^{\infty} \frac{1}{k^{2-2\nu}}(\mathbb{E}|I_{[z_k=a_i]} - \mathbb{E}I_{[z_k=a_i]}|^{2+\varepsilon})^{\frac{2}{2+\varepsilon}} < \infty, \quad i = 1, 2, \tag{34}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{2-2\nu}}(\mathbb{E}|u_k z_{k+i+1} - \mathbb{E}u_k z_{k+i+1}|^{2+\varepsilon})^{\frac{2}{2+\varepsilon}} < \infty, \quad \forall i \geq 0 \tag{35}$$

for any $\nu \in (0, \frac{1}{2})$ and some $\varepsilon > 0$, respectively.

Since $|I_{[z_k=a_i]} - \mathbb{E}I_{[z_k=a_i]}| \leq 2$, it is clear that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2-2\nu}}(\mathbb{E}|I_{[z_k=a_i]} - \mathbb{E}I_{[z_k=a_i]}|^{2+\varepsilon})^{\frac{2}{2+\varepsilon}} < \infty, \quad \forall \varepsilon > 0, \ \forall \nu \in \left(0, \frac{1}{2}\right).$$

On the other hand, by (3) we have $|u_k z_{k+i+1} - \mathbb{E}u_k z_{k+i+1}| \leq \max\{|a_1|, |a_2|\}(|u_k| + \mathbb{E}|u_k|)$. Then by the Cr-inequality and the Jesen inequality, for any $\varepsilon > 0$ we obtain

$$\mathbb{E}|u_k z_{k+i+1} - \mathbb{E}u_k z_{k+i+1}|^{2+\varepsilon} \leq (2\max\{|a_1|, |a_2|\})^{2+\varepsilon}\mathbb{E}|u_k|^{2+\varepsilon} < \infty, \tag{36}$$

which implies (35). ∎

Lemma 3.4 *Assume* A1–A4 *hold. Then,* $\forall \nu \in (0, \frac{1}{2})$,

$$|\mathbb{E}I_{[z_k=a_1]} - \gamma^{(1)}| = o\left(\frac{1}{k^{\nu}}\right), \tag{37}$$

$$|\mathbb{E}I_{[z_k=a_2]} + \gamma^{(1)} - \gamma^{(2)}| = o\left(\frac{1}{k^{\nu}}\right), \tag{38}$$

$$|\mathbb{E}u_k z_{k+i+1} - \rho h_i| = o\left(\frac{1}{k^{\nu}}\right). \tag{39}$$

Proof Since $\sigma_v^2 - \sigma_{v,k}^2 = \sum_{i=k}^{\infty} h_i^2$ with $|h_i| = O(\mathrm{e}^{-\mu i}), \mu > 0, i \geq 1$, for any $\nu \in (0, \frac{1}{2})$ we have

$$\sigma_v^2 - \sigma_{v,k}^2 = o\left(\frac{1}{k^{\nu}}\right). \tag{40}$$

And similarly by A3 we obtain

$$\sigma_\eta^2 - \sigma_{\eta,k}^2 = o\left(\frac{1}{k^{\nu}}\right) \tag{41}$$

for any $\nu \in (0, \frac{1}{2})$.

By noticing $\sigma_{y,k} \xrightarrow[k\to\infty]{} \sigma_y$, it follows that

$$\begin{aligned}\left|\frac{1}{\sigma_y} - \frac{1}{\sigma_{y,k}}\right| &= \frac{\sigma_y - \sigma_{y,k}}{\sigma_y \sigma_{y,k}} = \frac{\sigma_y^2 - \sigma_{y,k}^2}{\sigma_y \sigma_{y,k}(\sigma_y + \sigma_{y,k})} \\ &= \frac{(\sigma_v^2 - \sigma_{v,k}^2) + (\sigma_\eta^2 - \sigma_{\eta,k}^2)}{\sigma_y \sigma_{y,k}(\sigma_y + \sigma_{y,k})} \\ &= o\left(\frac{1}{k^\nu}\right) \end{aligned} \tag{42}$$

for any $\nu \in (0, \frac{1}{2})$.

By (23) and (24) we have

$$|\mathbb{E}I_{[z_k=a_1]} - \gamma^{(1)}| = \left|\int_{\frac{S_1}{\sigma_{y,k}}}^{\frac{S_1}{\sigma_y}} \varphi_0(x)dx\right| \le |S_1| \cdot \left|\frac{1}{\sigma_y} - \frac{1}{\sigma_{y,k}}\right| = o\left(\frac{1}{k^\nu}\right), \tag{43}$$

$$\begin{aligned}|\mathbb{E}I_{[z_k=a_2]} + \gamma^{(1)} - \gamma^{(2)}| &\le \left|\int_{\frac{S_1}{\sigma_{y,k}}}^{\frac{S_1}{\sigma_y}} \varphi_0(x)dx\right| + \left|\int_{\frac{S_2}{\sigma_{y,k}}}^{\frac{S_2}{\sigma_y}} \varphi_0(x)dx\right| \\ &\le (|S_1| + |S_2|) \cdot \left|\frac{1}{\sigma_y} - \frac{1}{\sigma_{y,k}}\right| = o\left(\frac{1}{k^\nu}\right)\end{aligned} \tag{44}$$

for any $\nu \in (0, \frac{1}{2})$.

On the other hand, by (13) it follows that

$$|\mathbb{E}u_k z_{k+i+1} - \rho h_i| \le |h_i|\big(|a_1 - a_2|F_1 + |a_2|F_2\big) \tag{45}$$

with

$$\begin{aligned}F_j &\triangleq \frac{1}{\sigma_{y,k+i+1}}\left|\int_{\frac{S_j}{\sigma_{y,k+i+1}}}^{\frac{S_j}{\sigma_y}} x\varphi_0(x)dx\right| + \left|\frac{1}{\sigma_{y,k+i+1}} - \frac{1}{\sigma_y}\right| \cdot \left|\int_{\frac{S_j}{\sigma_y}}^{\infty} x\varphi_0(x)dx\right| \\ &= O\left(\left|\frac{1}{\sigma_{y,k+i+1}} - \frac{1}{\sigma_y}\right|\right), \quad j = 1, 2.\end{aligned} \tag{46}$$

Combing (42), (45) and (46) we have (39). ∎

Noticing $\zeta_{i,k} \triangleq u_k z_{k+i+1} - \rho h_i = (u_k z_{k+i+1} - \mathbb{E}u_k z_{k+i+1}) + (\mathbb{E}u_k z_{k+i+1} - \rho h_i)$, by (33), (39) and Proposition 1 we derive the following convergence results for the linear system.

Theorem 3.1 *Assume* A1–A5 *hold. Then,* $\theta_{i,k}$*, defined by* (20) *and* (21)*, are strongly consistent with following convergence rate:*

$$|\theta_{0,k} - \rho| = o\left(\frac{1}{k^\nu}\right), \text{ and } |\theta_{i,k} - \rho h_i| = o\left(\frac{1}{k^\nu}\right) \quad a.s., \quad i = 1, 2, \cdots, p + r \tag{47}$$

for any $\nu \in (0, \frac{1}{2})$*. And consequently,* $h_{i,k}$*, defined by* (22)*, are also strongly consistent:*

$$|h_{i,k} - h_i| = o\left(\frac{1}{k^\nu}\right) \quad a.s., \quad i = 1, 2, \cdots, p + r \tag{48}$$

for any $\nu \in (0, \frac{1}{2})$.

Theorem 3.2 *Assume* A1–A5 *hold. Then, for any* $\nu \in (0, \frac{1}{2})$,

$$|c_{i,k} - c_i| = o\left(\frac{1}{k^\nu}\right), \text{ and } |d_{j,k} - d_j| = o\left(\frac{1}{k^\nu}\right) \quad a.s., \\ i = 1, 2, \cdots, p, \quad j = 1, 2, \cdots, r, \tag{49}$$

where $c_{i,k}$ *and* $d_{j,k}$ *are given by* (18) *and* (19), *respectively.*

For the thresholds of the quantizer, the convergence results are presented in the following theorems.

Theorem 3.3 *Assume* A1–A5 *hold. Then*

$$|\gamma_k^{(1)} - \gamma^{(1)}| = o\left(\frac{1}{k^\nu}\right) \quad a.s., \ \forall \nu \in \left(0, \frac{1}{2}\right), \tag{50}$$

$$|\gamma_k^{(2)} - \gamma^{(2)}| = o\left(\frac{1}{k^\nu}\right) \quad a.s., \ \forall \nu \in \left(0, \frac{1}{2}\right), \tag{51}$$

$$|\Theta_k^{(i)} - S_i/\sigma_y| = o\left(\frac{1}{k^\tau}\right) \quad a.s., \ \exists \tau \in \left(0, \frac{1}{2}\right) \tag{52}$$

with $\gamma_k^{(i)}$, $\Theta_k^{(i)}$ *being defined by* (25)–(30).

Proof Let $\zeta_k^{(1)} = I_{[z_k=a_1]} - \gamma^{(1)} = (I_{[z_k=a_1]} - \mathbb{E}I_{[z_k=a_1]}) + (\mathbb{E}I_{[z_k=a_1]} - \gamma^{(1)})$, by (32), (37) and Proposition 1 we obtain (50).

Also let $\zeta_k^{(2)} = I_{[z_k=a_2]} + \gamma_k^{(1)} - \gamma^{(2)} = (I_{[z_k=a_2]} - \mathbb{E}I_{[z_k=a_2]}) + (\mathbb{E}I_{[z_k=a_2]} + \gamma^{(1)} - \gamma^{(2)}) + (\gamma_k^{(1)} - \gamma^{(1)})$. By (32), (38), (50) and Proposition 1 we have (51).

By Theorem 3 in [11], (52) holds. ∎

Theorem 3.4 *Assume* A1–A5 *hold. Then,* $s_k^{(i)}$ *defined by* (31) *converges to* S_i *almost surely: For some* $\tau \in (0, \frac{1}{2})$,

$$|s_k^{(i)} - S_i| = o\left(\frac{1}{k^\tau}\right) \quad a.s., \tag{53}$$

$i = 1, 2$.

Proof Let

$$\varrho_k \triangleq \begin{cases} \dfrac{(a_1 - a_2)\exp\left\{-\frac{1}{2}\left(\Theta_k^{(1)}\right)^2\right\} + a_2 \exp\left\{-\frac{1}{2}\left(\Theta_k^{(2)}\right)^2\right\}}{\sqrt{2\pi}\theta_{0,k}}, & \text{if } \theta_{0,k} \neq 0, \\ 0, & \text{if } \theta_{0,k} = 0. \end{cases}$$

By (52) and proceed a similar proof as Theorem 4 in [11], we have

$$|\varrho_k - \sigma_y| = o\left(\frac{1}{k^\tau}\right) \quad \text{a.s.} \tag{54}$$

with the same constant $\tau \in (0, \frac{1}{2})$ as in (52).

By A5 and (31), it follows that

$$\begin{aligned}|s_k^{(i)} - S_i| &= |\varrho_k \Theta_k^{(i)} - S_i| \\ &= |\varrho_k \Theta_k^{(i)} - \sigma_y \Theta_k^{(i)} + \sigma_y \Theta_k^{(i)} - S_i| \\ &\le |\varrho_k - \sigma_y| \cdot |\Theta_k^{(i)}| + |\sigma_y \Theta_k^{(i)} - S_i|, \quad i = 1, 2.\end{aligned} \tag{55}$$

Thus, by combing (52), (54) and (55), we obtain (53). ∎

4 Simulation Examples

Let the quantized system be considered as follows

$$\begin{aligned}& v_{k+1} + 0.2v_k + 0.6v_{k-1} = u_k - 0.3u_{k-1} + 1.2u_{k-2}, \\ & y_k = v_k + \eta_k, \\ & z_k = \begin{cases} -0.6, & y_k \ge 0.2, \\ 0.8, & -0.5 \le y_k < 0.2, \\ 0, & y_k < -0.5, \end{cases} \\ & \eta_k - 0.7\eta_{k-1} = \varepsilon_k + 0.5\varepsilon_{k-1},\end{aligned}$$

in which the input $\{u_k\}$ and the driven noise $\{\varepsilon_k\}$ are mutually independent, and $\{u_k\}$ and $\{\varepsilon_k\}$ are both iid Gaussian: $u_k \sim \mathcal{N}(0, 1)$ and $\varepsilon_k \sim \mathcal{N}(0, 0.3^2)$. It is noticed that $c_1 = 0.2, c_2 = 0.6, d_1 = -0.3, d_2 = 1.2$, $S_1 = 0.2$, and $S_2 = -0.5$. Take $M_k = 6^k$.

It is clear that A1–A5 hold. The estimates for $\{c_1, c_2, d_1, d_2\}$ and $\{S_1, S_2\}$ are plotted at different time steps in Figure 2, in which the solid lines denote the true values of the parameters, while the dashed lines denote the estimates. It is shown that the estimates converge to the true parameters as the time steps increase.

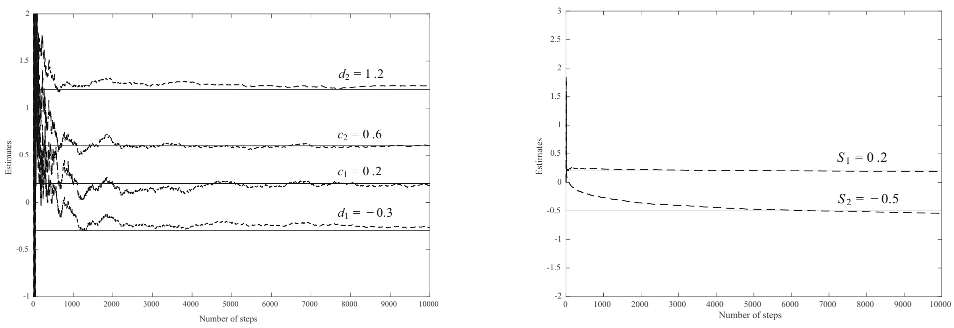

Figure 2 Estimates for the parameters of the system

5 Concluding Remarks

This paper is focused on the identification of quantized ARMA systems and with ARMA noises. By the SAAWET, the recursive estimates for the linear system and the thresholds of the quantizer are presented. Under reasonable conditions, the estimates are proved to be strongly consistent. The almost surely convergence rates are also obtained for the parameters of the ARMA system and the thresholds of the quantizer.

For further research, it is of interest to remove the Gaussian restriction of the noise, and to consider quantized Wiener systems.

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Appendix

Let $g(\cdot)$ be an $\boldsymbol{R} \longrightarrow \boldsymbol{R}$ function with only one root x^0, and let the observation at time k be $y_k = g(x_k) + \zeta_k$, where ζ_k is the observation noise, and x_k is the estimate for x^0 generated by SAAWET:

$$x_{k+1} = \left[x_k + \frac{1}{k+1} y_k\right] \cdot I_{[|x_k + \frac{1}{k+1} y_k| \le M_{\delta_k}]}, \tag{56}$$

$$\delta_k = \sum_{j=1}^{k-1} I_{[|x_j + \frac{1}{j+1} y_j| > M_{\delta_j}]}, \quad \delta_0 = 0 \tag{57}$$

with an arbitrary initial value x_0, where $\{M_k\}$ is a sequence of nondecreasing positive numbers diverging to infinity.

For convergence analysis of the SAAWET, we need the following propositions[14].

Proposition 1 *Let $g(x) = -(x - x_0)$. If the observation noise ζ_k can be decomposed into two parts $\zeta_k = \zeta_k^{(1)} + \zeta_k^{(2)}$ such that*

$$\sum_{k=1}^{\infty} \frac{1}{k^{1-\nu}} \zeta_k^{(1)} < \infty \quad a.s., \text{ and } \zeta_k^{(2)} = O\left(\frac{1}{k^{\nu}}\right) \quad a.s.$$

for some $\nu \in (0, \frac{1}{2})$, then x_k defined by (56)–(57) converges to x^0 a.s. with the following convergence rate: $|x_k - x^0| = o(\frac{1}{k^{\nu}})$ a.s.