J Syst Sci Complex (2020) 33: 100–121

Almost Automorphic Solutions for Quaternion-Valued Hopfield Neural Networks with Mixed Time-Varying Delays and Leakage Delays*

LI Yongkun · MENG Xiaofang

DOI: 10.1007/s11424-019-8051-1
Received: 26 February 2018 / Revised: 9 June 2018

Abstract This paper considers a class of quaternion-valued Hopfield neural networks with mixed time-varying delays and leakage delays. By utilizing the exponential dichotomy of linear differential equations, Banach's fixed point theorem and differential inequality techniques, the authors obtain some sufficient conditions to ensure the existence and global exponential stability of almost automorphic solutions for this class of quaternion-valued neural networks. The results are completely new. Finally, the authors give an example to illustrate the feasibility of the results.

Keywords Almost automorphic solutions, exponential stability, Hopfield neural networks, leakage delays, quaternion-valued functions.

1 Introduction

As we know, because the dynamics of neural networks plays a very important role in the design, realization and application of neural networks, and Hopfield neural networks as a kind of recurrent neural networks can be applied to the field of artificial intelligence and computer science related fields, so many scholars have devoted themselves with great interests to the study of various kinds of dynamics for Hopfield neural networks[1–11]. In addition, in reality, time delays are unavoidable, so various types of delays have been incorporated into neural networks during the past few decades[12–18], among them the mixed time delay is more practical. Moreover, since the delay in the leakage term is difficult to handle, few scholars take the leakage delay into consideration[8, 19–23]. However, it has a great impact on the dynamical behavior of the neural network. Therefore, it is significant and necessary to incorporate leakage delays into Hopfield neural networks.

LI Yongkun (Corresponding author) · MENG Xiaofang
Department of Mathematics, Yunnan University, Kunming 650091*, China.*
Email: yklie@ynu.edu.cn; xfmeng@126.com.
*This research was supported by the National Natural Sciences Foundation of People's Republic of China under Grants Nos. 11861072 and 11361072.
◇ *This paper was recommended for publication by Editor SUN Jian.*

On the one hand, the periodicity and the almost periodicity are very important dynamical behaviours of neural networks that have been intensively investigated by many authors[10, 24–28]. The almost automorphy, which was first introduced by Bochner[29], is more general than the periodicity and the almost periodicity and plays a very important role in better understanding the almost periodicity. Over the years, in the study of differential equations, almost automorphic solutions have aroused the interest of many scholars[30–32]. However, up to now, very few papers have been published on the almost automorphy of neural networks (see [9, 33–35]).

On the other hand, it is well known that the quaternion as an expansion of real numbers and complex numbers is composed of real numbers and three imaginary units i, j, k, which obey the Hamiltonian rules: $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, $i^2 = j^2 = k^2 = ijk = -1$ and the quaternion has been introduced into the neural network field and quaternion-valued neural networks (QVNNs) have been proposed for quite a long time. In recent years, the applications of quaternion-valued neural networks (QVNNs) have been widely investigated. One practical application of QVNNs is the 3D geometrical affine transformation, especially spatial rotation, which can be represented by QVNNs efficiently and compactly[36, 37]. Other practical applications of QVNNs are image impression, color night vision[38], and so on. However, since quaternion multiplication does not meet the commutative law, the method of dealing with real-valued neural networks and complex-valued neural networks can not deal with QVNNs directly, so the analysis for the equation of state becomes difficult. As far as we know, there only few papers have been published on the dynamics of QVNNs[8–11, 39–45]. But there has been no paper published on the almost automorphy for QVNNs with with mixed time-varying delays and leakage delays.

Motivated by the above discussion, in this paper, we propose the following quaternion-valued Hopfiled neural network with mixed time-varying delays and leakage delays:

$$\begin{aligned} z_p'(t) = &-a_p(t)z_p(t-\delta_p(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_q(t)) + \sum_{q=1}^{n} c_{pq}(t)g_q(z_q(t-\tau_{pq}(t))) \\ &+ \sum_{q=1}^{n} d_{pq}(t)\int_{t-\sigma_{pq}(t)}^{t} e_q(z_q(s))ds + u_p(t), \end{aligned} \tag{1}$$

where $p \in \{1, 2, \cdots, n\} := \Delta$, $z_p(t) = z_p^R(t) + iz_p^I(t) + jz_p^J(t) + kz_p^K(t) \in \mathbb{Q}$ is the state of the pth neuron at time t; $a_p(t) > 0$ is the self-feedback connection weight; $b_{pq}(t)$, $c_{pq}(t)$ and $d_{pq}(t) \in \mathbb{Q}$ are the connection weights and the delay connection weights from neuron q to neuron p, respectively; $u_p(t)$ is an external input on the pth unit at time t; $\delta_p(t)$, $\tau_{pq}(t)$ and $\sigma_{pq}(t)$ are the leakage delays and transmission delays, respectively.

The initial values are given by

$$z_p(s) = \phi_p(s), \quad s \in [-\eta, 0], \quad p \in \Delta,$$

where $\psi_p = \psi_p^R + i\psi_p^I + j\psi_p^J + k\psi_p^K, \eta = \max\{\delta, \tau, \sigma\}$, $\delta = \max_{1\le p\le n}\{\overline{\delta}_p\}$, $\tau = \max_{1\le p,q\le n}\{\overline{\tau}_{pq}\}$, $\sigma = \max_{1\le p,q\le n}\{\overline{\sigma}_{pq}\}$, $\psi_p^l \in C([-\eta, 0], \mathbb{R})$, $l \in \{R, I, J, K\} := \Lambda$.

Remark 1.1 Quaternion-valued system (1) includes real-valued systems and complex-valued systems as its special cases. In fact, in System (1),

(i) if all the coefficients and delays $a_p, b_{pq}, c_{pq}, d_{pq}, \delta_p, \tau_{pq}, \sigma_{pq}, p, q \in \Delta$ are functions from $\mathbb{R}$ to $\mathbb{R}$, and all the activation functions $f_q, g_q, e_q, q \in \Delta$ are functions from $\mathbb{R}$ to $\mathbb{R}$, then the state $z_p(t) \equiv z_p^R(t) \in \mathbb{R}$, in this case, System (1) is a real-valued system;

(ii) if the coefficients b_{pq}, c_{pq}, d_{pq} are functions from $\mathbb{R}$ to $\mathbb{C}$, and all the activation functions f_q, g_q, e_q are functions from $\mathbb{C}$ to $\mathbb{C}$, then the state $z_p(t) \equiv z_p^R(t) + iz_p^I(t) \in \mathbb{C}$, in this case, system (1) is a complex-valued system.

Our main aim in this paper is to study the existence and global exponential stability of almost automorphic solutions of (1). To the best of our knowledge, this is the first paper to study the existence and global exponential stability of almost automorphic solutions of (1) and even when System (1) is degenerated to complex-valued system, our result remains new. Our method of this paper can be used to study the almost automorphy for other types of QVNNs.

This paper is organized as follows. In Section 2, we introduce some definitions, make some preparations for later sections. In Section 3, by utilizing the Banach's fixed point theorem and differential inequality techniques, we present some sufficient conditions for the existence and global exponential stability of almost automorphic solutions of (1). In Section 4, we give an example to demonstrate the feasibility of our results. Finally, we draw a conclusion in Section 5.

2 Preliminaries

In this section, we shall recall some fundamental definitions and lemmas which are used in what follows.

Definition 2.1 (see [46]) A continuous function $f : \mathbb{R} \to \mathbb{R}^n$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} f(t + s_n) = g(t)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Denote by $AA(\mathbb{R}, \mathbb{R}^n)$ the collection of all almost automorphic functions.

Lemma 2.2 (see [46]) *Let $f, g \in AA(\mathbb{R}, \mathbb{R}^n)$. Then we have the following*

(i) *$f + g \in AA(\mathbb{R}, \mathbb{R}^n)$;*

(ii) *$\lambda f \in AA(\mathbb{R}, \mathbb{R}^n)$ for any scalar $\lambda \in \mathbb{R}$;*

(iii) *$f_\alpha \in AA(\mathbb{R}, \mathbb{R}^n)$ where $f : \mathbb{R} \to \mathbb{R}^n$ is defined by $f_\alpha(\cdot) = f(\cdot + \alpha)$;*

(iv) *let $f \in AA(\mathbb{R}, \mathbb{R}^n)$; then the range $\mathbb{R}_f = \{f(t), t \in \mathbb{R}\}$ is relatively compact in $\mathbb{R}^n$, thus f is bounded in norm;*

(v) *if $\varphi : \mathbb{R}^n \to X$ is a continuous function, then the composite function $\varphi \circ f : \mathbb{R}^n \to X$ is almost automorphic.*

(vi) $(AA(\mathbb{R},\mathbb{R}^n),\|\cdot\|_\infty)$ *is a Banach space.*

Definition 2.3 (see [46]) A function $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is said to be almost automorphic in $t\in\mathbb{R}$ for each $x\in\mathbb{R}^n$ if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} f(t+s_n,x)=g(t,x)$$

is well defined for each $t\in\mathbb{R}$, $x\in\mathbb{R}^n$ and

$$\lim_{n\to\infty} g(t-s_n,x)=f(t,x)$$

for each $t\in\mathbb{R}$, $x\in\mathbb{R}^n$. The collection of such functions will be denoted by $AA(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$.

Lemma 2.4 (see [46]) *Let $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be an almost automorphic function in $t\in\mathbb{R}$ for each $x\in\mathbb{R}^n$ and assume that f satisfies a Lipschitz condition in x uniformly in $t\in\mathbb{R}$. Let $\varphi:\mathbb{R}\to\mathbb{R}^n$ be an almost automorphic function. Then the function*

$$\phi: t\longmapsto \phi(t)=f(t,\varphi(t))$$

is almost automorphic.

Definition 2.5 A quaternion-valued function $z=z^R+iz^I+jz^J+kz^K\in C(\mathbb{R},\mathbb{Q})$ is called an almost automorphic function if z^R, z^I, z^J and z^K are almost automorphic functions.

Definition 2.6 (see [34]) System

$$x'(t)=A(t)x(t) \tag{2}$$

is said to admit an exponential dichotomy if there exist a projection P and positive constants α,β so that the fundamental solution matrix $X(t)$ satisfies

$$\begin{aligned}&|X(t)PX^{-1}(s)|\le\beta\mathrm{e}^{-\alpha(t-s)},\quad t\ge s,\\&|X(t)(I-P)X^{-1}(s)|\le\beta\mathrm{e}^{-\alpha(s-t)},\quad t\le s.\end{aligned}$$

Consider the following almost automorphic system

$$x'(t)=A(t)x(t)+f(t), \tag{3}$$

where $A(t)$ is an almost automorphic matrix function, $f(t)$ is an almost automorphic vector function.

Lemma 2.7 (see [34]) *If the linear system* (2) *admits an exponential dichotomy, then System* (3) *has a unique almost automorphic solution that can be expressed as*

$$x(t)=\int_{-\infty}^{t}X(t)PX^{-1}(s)f(s)ds-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(s)f(s)ds,$$

where $X(t)$ is the fundamental solution matrix of (2).

Lemma 2.8 (see [34]) *Let c_p be an almost automorphic function on $\mathbb{R}$ and*

$$M[c_p] = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} c_p(s)ds > 0, \quad p \in \Delta.$$

Then the linear system

$$x'(t) = \operatorname{diag}(-c_1(t), -c_2(t), \cdots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

In order to avoid the non-commutativity of the quaternion multiplication, in the following, we will first decompose system (1) into real-valued system. To this end, we need the following assumption:

(H_1) Let $z_p = z_p^R + iz_p^I + jz_p^J + kz_p^K$, $z_p^R, z_p^I, z_p^J, z_p^K \in \mathbb{R}$, then the activation functions $f_q(z_q)$, $g_q(z_q)$ and $e_q(z_q)$ of (1) can be expressed as

$$\begin{aligned} f_q(z_q) &= f_q^R\big(z_q^R, z_q^I, z_q^J, z_q^K\big) + if_q^I\big(z_q^R, z_q^I, z_q^J, z_q^K\big) \\ &\quad + jf_q^J\big(z_q^R, z_q^I, z_q^J, z_q^K\big) + kf_q^K\big(z_q^R, z_q^I, z_q^J, z_q^K\big), \\ g_q(z_q) &= g_q^R\big(z_q^R, z_q^I, z_q^J, z_q^K\big) + ig_q^I\big(z_q^R, z_q^I, z_q^J, z_q^K\big) \\ &\quad + jg_q^J\big(z_q^R, z_q^I, z_q^J, z_q^K\big) + kg_q^K\big(z_q^R, z_q^I, z_q^J, z_q^K\big), \\ e_q(z_q) &= e_q^R\big(z_q^R, z_q^I, z_q^J, z_q^K\big) + ie_q^I\big(z_q^R, z_q^I, z_q^J, z_q^K\big) \\ &\quad + je_q^J\big(z_q^R, z_q^I, z_q^J, z_q^K\big) + ke_q^K\big(z_q^R, z_q^I, z_q^J, z_q^K\big), \end{aligned}$$

where $f_q^l, g_q^l, e_q^l : \mathbb{R}^4 \to \mathbb{R}$, $q \in \Delta$, $l \in \mathit{\Lambda}$.

Under Assumption (H_1), according to Hamilton rules, System (1) can be transformed into the following system:

$$\begin{aligned} \big(z_p^R\big)'(t) &= -a_p(t)z_p^R(t-\delta_p(t)) + \sum_{q=1}^n \Big(b_{pq}^R(t)\widetilde{f}_q^R[z_q(t)] - b_{pq}^I(t)\widetilde{f}_q^I[z_q(t)] \\ &\quad - b_{pq}^J(t)\widetilde{f}_q^J[z_q(t)] - b_{pq}^K(t)\widetilde{f}_q^K[z_q(t)]\Big) + \sum_{q=1}^n \Big(c_{pq}^R(t)\widetilde{g}_q^R[z_q(t-\tau_{pq}(t))] \\ &\quad - c_{pq}^I(t)\widetilde{g}_q^I[z_q(t-\tau_{pq}(t))] - c_{pq}^J(t)\widetilde{g}_q^J[z_q(t-\tau_{pq}(t))] \\ &\quad - c_{pq}^K(t)\widetilde{g}_q^K[z_q(t-\tau_{pq}(t))]\Big) + \sum_{q=1}^n \Big(d_{pq}^R(t)\int_{t-\sigma_{pq}(t)}^t \widetilde{e}_q^R[z_q(s)]ds \\ &\quad - d_{pq}^I(t)\int_{t-\sigma_{pq}(t)}^t \widetilde{e}_q^I[z_q(s)]ds - d_{pq}^J(t)\int_{t-\sigma_{pq}(t)}^t \widetilde{e}_q^J[z_q(s)]ds \\ &\quad - d_{pq}^K(t)\int_{t-\sigma_{pq}(t)}^t \widetilde{e}_q^K[z_q(s)]ds\Big) + u_p^R(t), \end{aligned} \tag{4}$$

$$
\begin{aligned}
\left(z_p^I\right)'(t) = & -a_p(t)z_p^I(t-\delta_p(t)) + \sum_{q=1}^{n}\Big(b_{pq}^R(t)\widetilde{f}_q^I[z_q(t)] + b_{pq}^I(t)\widetilde{f}_q^R[z_q(t)] \\
& + b_{pq}^J(t)\widetilde{f}_q^K[z_q(t)] - b_{pq}^K(t)\widetilde{f}_q^J[z_q(t)]\Big) + \sum_{q=1}^{n}\Big(c_{pq}^R(t)\widetilde{g}_q^I[z_q(t-\tau_{pq}(t))] \\
& + c_{pq}^I(t)\widetilde{g}_q^R[z_q(t-\tau_{pq}(t))] + c_{pq}^J(t)\widetilde{g}_q^K[z_q(t-\tau_{pq}(t))] \\
& - c_{pq}^K(t)\widetilde{g}_q^J[z_q(t-\tau_{pq}(t))]\Big) + \sum_{q=1}^{n}\Big(d_{pq}^R(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^I[z_q(s)]ds \\
& + d_{pq}^I(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^R[z_q(s)]ds + d_{pq}^J(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^K[z_q(s)]ds \\
& - d_{pq}^K(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^J[z_q(s)]ds\Big) + u_p^I(t), \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\left(z_p^J\right)'(t) = & -a_p(t)z_p^J(t-\delta_p(t)) + \sum_{q=1}^{n}\Big(b_{pq}^R(t)\widetilde{f}_q^J[z_q(t)] + b_{pq}^J(t)\widetilde{f}_q^R[z_q(t)] \\
& - b_{pq}^I(t)\widetilde{f}_q^K[z_q(t)] + b_{pq}^K(t)\widetilde{f}_q^I[z_q(t)]\Big) + \sum_{q=1}^{n}\Big(c_{pq}^R(t)\widetilde{g}_q^J[z_q(t-\tau_{pq}(t))] \\
& + c_{pq}^J(t)\widetilde{g}_q^R[z_q(t-\tau_{pq}(t))] - c_{pq}^I(t)\widetilde{g}_q^K[z_q(t-\tau_{pq}(t))] \\
& + c_{pq}^K(t)\widetilde{g}_q^I[z_q(t-\tau_{pq}(t))]\Big) + \sum_{q=1}^{n}\Big(d_{pq}^R(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^J[z_q(s)]ds \\
& + d_{pq}^J(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^R[z_q(s)]ds - d_{pq}^I(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^K[z_q(s)]ds \\
& + d_{pq}^K(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^I[z_q(s)]ds\Big) + u_p^J(t), \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\left(z_p^K\right)'(t) = & -a_p(t)z_p^K(t-\delta_p(t)) + \sum_{q=1}^{n}\Big(b_{pq}^R(t)\widetilde{f}_q^K[z_q(t)] + b_{pq}^K(t)\widetilde{f}_q^R[z_q(t)] \\
& + b_{pq}^I(t)\widetilde{f}_q^J[z_q(t)] - b_{pq}^J(t)\widetilde{f}_q^I[z_q(t)]\Big) + \sum_{q=1}^{n}\Big(c_{pq}^R(t)\widetilde{g}_q^K[z_q(t-\tau_{pq}(t))] \\
& + c_{pq}^K(t)\widetilde{g}_q^R[z_q(t-\tau_{pq}(t))] + c_{pq}^I(t)\widetilde{g}_q^J[z_q(t-\tau_{pq}(t))] \\
& - c_{pq}^J(t)\widetilde{g}_q^I[z_q(t-\tau_{pq}(t))]\Big) + \sum_{q=1}^{n}\Big(d_{pq}^R(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^K[z_q(s)]ds \\
& + d_{pq}^K(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^R[z_q(s)]ds + d_{pq}^I(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^J[z_q(s)]ds \\
& - d_{pq}^J(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{e}_q^I[z_q(s)]ds\Big) + u_p^K(t), \qquad (7)
\end{aligned}
$$

where $\widetilde{f}_q^l[z_q]=f_q^l\big(z_q^R,z_q^I,z_q^J,z_q^K\big)$, $\widetilde{e}_q^l[z_q]=e_q^l\big(z_q^R,z_q^I,z_q^J,z_q^K\big),\widetilde{g}_q^l[z_q]=g_q^l\big(z_q^R,z_q^I,z_q^J,z_q^K\big)$, $p,q\in\Delta,l\in\Lambda$, and

$$\begin{aligned}
b_{pq}(t)&=b_{pq}^R(t)+ib_{pq}^I(t)+jb_{pq}^J(t)+kb_{pq}^K(t),\\
c_{pq}(t)&=c_{pq}^R(t)+ic_{pq}^I(t)+jc_{pq}^J(t)+kc_{pq}^K(t),\\
d_{pq}(t)&=d_{pq}^R(t)+id_{pq}^I(t)+jd_{pq}^J(t)+kd_{pq}^K(t),\\
u_p(t)&=u_p^R(t)+iu_p^I(t)+ju_p^J(t)+ku_p^K(t).
\end{aligned}$$

According to (4)–(7), one can obtain that

$$\begin{aligned}
Z_p'(t)=&-a_p(t)Z_p(t-\delta_p(t))+\sum_{q=1}^n B_{pq}(t)\widetilde{F}_q[z_p(t)]+\sum_{q=1}^n C_{pq}(t)\widetilde{G}_q[z_p(t-\tau_{pq}(t))]\\
&+\sum_{q=1}^n D_{pq}(t)\int_{t-\sigma_{pq}(t)}^t \widetilde{E}_q[z_p(s)]ds+U_p(t),\quad p\in\Delta,
\end{aligned}\tag{8}$$

where

$$B_{pq}(t)=\begin{pmatrix} b_{pq}^R(t) & -b_{pq}^I(t) & -b_{pq}^J(t) & -b_{pq}^K(t)\\ b_{pq}^I(t) & b_{pq}^R(t) & -b_{pq}^K(t) & b_{pq}^J(t)\\ b_{pq}^J(t) & b_{pq}^K(t) & b_{pq}^R(t) & -b_{pq}^I(t)\\ b_{pq}^K(t) & -b_{pq}^J(t) & b_{pq}^I(t) & b_{pq}^R(t)\end{pmatrix},$$

$$C_{pq}(t)=\begin{pmatrix} c_{pq}^R(t) & -c_{pq}^I(t) & -c_{pq}^J(t) & -c_{pq}^K(t)\\ c_{pq}^I(t) & c_{pq}^R(t) & -c_{pq}^K(t) & c_{pq}^J(t)\\ c_{pq}^J(t) & c_{pq}^K(t) & c_{pq}^R(t) & -c_{pq}^I(t)\\ c_{pq}^K(t) & -c_{pq}^J(t) & c_{pq}^I(t) & c_{pq}^R(t)\end{pmatrix},$$

$$D_{pq}(t)=\begin{pmatrix} d_{pq}^R(t) & -d_{pq}^I(t) & -d_{pq}^J(t) & -d_{pq}^K(t)\\ d_{pq}^I(t) & d_{pq}^R(t) & -d_{pq}^K(t) & d_{pq}^J(t)\\ d_{pq}^J(t) & d_{pq}^K(t) & d_{pq}^R(t) & -d_{pq}^I(t)\\ d_{pq}^K(t) & -d_{pq}^J(t) & d_{pq}^I(t) & d_{pq}^R(t)\end{pmatrix},$$

$$Z_p(t)=\begin{pmatrix} z_p^R(t)\\ z_p^I(t)\\ z_p^J(t)\\ z_p^K(t)\end{pmatrix},\quad U_p(t)=\begin{pmatrix} u_p^R(t)\\ u_p^I(t)\\ u_p^J(t)\\ u_p^K(t)\end{pmatrix},\quad \widetilde{F}_q[z_p(t)]=\begin{pmatrix} \widetilde{f}_q^R[z_p(t)]\\ \widetilde{f}_q^I[z_p(t)]\\ \widetilde{f}_q^J[z_p(t)]\\ \widetilde{f}_q^K[z_p(t)]\end{pmatrix},$$

$$\widetilde{G}_q[z_p(t-\tau_{pq}(t))]=\begin{pmatrix} \widetilde{g}_q^R[z_p(t-\tau_{pq}(t))]\\ \widetilde{g}_q^I[z_p(t-\tau_{pq}(t))]\\ \widetilde{g}_q^J[z_p(t-\tau_{pq}(t))]\\ \widetilde{g}_q^K[z_p(t-\tau_{pq}(t))]\end{pmatrix},\quad \widetilde{E}_q[z_p(s)]=\begin{pmatrix} \widetilde{e}_q^R[z_p(s)]\\ \widetilde{e}_q^I[z_p(s)]\\ \widetilde{e}_q^J[z_p(s)]\\ \widetilde{e}_q^K[z_p(s)]\end{pmatrix}.$$

The initial condition associated with (8) is of the form

$$Z_p(s) = \Psi_p(s), \quad s \in [-\eta, 0],$$

where $\Psi_p(s) = (\psi_p^R(s), \psi_p^I(s), \psi_p^J(s), \psi_p^K(s))^{\mathrm{T}}, \psi_p^l \in C([-\eta, 0], \mathbb{R})$, $p \in \Delta$, $l \in \Lambda$.

According to Remark 1.1, we have

Remark 2.9 Under Assumption (H_1), corresponding to case (i) of Remark 1.1, quaternion-valued system (1) reduces to (4) and corresponding to case (ii) of Remark 1.1, quaternion-valued system (1) reduces to (4)–(5).

Remark 2.10 If $Z(t) = (z_1^R(t), z_1^I(t), z_1^J(t), z_1^K(t), \cdots, z_n^R(t), z_n^I(t), z_n^J(t), z_n^K(t))^{\mathrm{T}}$ is an almost automorphic solution of System (8), then $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))^{\mathrm{T}}$, where $z_p(t) = z_p^R(t) + iz_p^I(t) + jz_p^J(t) + kz_p^K(t), p \in \Delta$ must be an almost automorphic solution of (1). Thus, the problem of finding an almost automorphic solution for (1) reduces to finding one for the system of (8). For considering the stability of solution of (1), we just need to consider the stability of solution of System (8).

3 Main Results

In this section, we will study the existence and global exponential stability of almost automorphic solutions of System (8).

Let $\mathbb{B} = \{\phi = (\phi_1^R, \phi_1^I, \phi_1^J, \phi_1^K, \phi_2^R, \phi_2^I, \phi_2^J, \phi_2^K, \cdots, \phi_n^R, \phi_n^I, \phi_n^J, \phi_n^K)^{\mathrm{T}} := (\phi_1, \phi_2, \cdots, \phi_n)^{\mathrm{T}} \in C^1(\mathbb{R}, \mathbb{R}^{4n}) \mid \phi, \phi' \in AA(\mathbb{R}, \mathbb{R}^{4n})\}$ with the norm $\|\phi\|_{\mathbb{B}} = \max\{\sup_{t\in\mathbb{R}} \|\phi(t)\|, \sup_{t\in\mathbb{R}} \|\phi'(t)\|\}$, where $\|\phi(t)\| = \max_{p\in\Delta, l\in\Lambda}\{|\phi_p^l(t)|\}$, $\|\phi'(t)\| = \max_{p\in\Delta, l\in\Lambda}\{|(\phi_p^l)'(t)|\}$, then $\mathbb{B}$ is a Banach space. For the convenience, we will introduce the notation: $\overline{f} = \sup_{t\in\mathbb{R}} |f(t)|$, $\underline{f} = \inf_{t\in\mathbb{R}} |f(t)|$, where f is a bounded continuous function.

Throughout the rest of the paper, we assume that the following conditions hold:

(H_2) There exist positive constants α_q^l, β_q^l and γ_q^l such that

$$\begin{aligned}
\big|f_q^l\big(z_q^R, z_q^I, z_q^J, z_q^K\big) - f_q^l\big(y_q^R, y_q^I, y_q^J, y_q^K\big)\big| &\le \alpha_q^R|z_q^R - y_q^R| + \alpha_q^I|z_q^I - y_q^I| \\
&\quad + \alpha_q^J|z_q^J - y_q^J| + \alpha_q^K|z_q^K - y_q^K|, \\
\big|g_q^l\big(z_q^R, z_q^I, z_q^J, z_q^K\big) - g_q^l\big(y_q^R, y_q^I, y_q^J, y_q^K\big)\big| &\le \beta_q^R|z_q^R - y_q^R| + \beta_q^I|z_q^I - y_q^I| \\
&\quad + \beta_q^J|z_q^J - y_q^J| + \beta_q^K|z_q^K - y_q^K|, \\
\big|e_q^l\big(z_q^R, z_q^I, z_q^J, z_q^K\big) - e_q^l\big(y_q^R, y_q^I, y_q^J, y_q^K\big)\big| &\le \gamma_q^R|z_q^R - y_q^R| + \gamma_q^I|z_q^I - y_q^I| \\
&\quad + \gamma_q^J|z_q^J - y_q^J| + \gamma_q^K|z_q^K - y_q^K|,
\end{aligned}$$

and $f_q^l(0,0,0,0) = g_q^l(0,0,0,0) = e_q^l(0,0,0,0) = 0$, where $q \in \Delta, l \in \Lambda$.

(H_3) Function $a_p \in C(\mathbb{R}, \mathbb{R}^+)$ with $M[a_p] > 0$ is almost automorphic, $\delta_p, \tau_{pq}, \sigma_{pq} \in C(\mathbb{R}, \mathbb{R}^+)$, $U_p \in C(\mathbb{R}, \mathbb{R}^{4\times 1})$ and $B_{pq}, C_{pq}, D_{pq} \in C(\mathbb{R}, \mathbb{R}^{4\times 4})$ are almost automorphic, where $p, q \in \Delta$.

(H_4) There exists a constant κ such that

$$\max_{p\in\Delta}\left\{\max_{l\in\Lambda}\left\{\frac{\Gamma_p\kappa + \overline{u}_p^l}{\underline{a}_p}, \left(1 + \frac{\overline{a}_p}{\underline{a}_p}\right)\left(\Gamma_p\kappa + \overline{u}_p^l\right)\right\}\right\} \le \kappa$$

and

$$\max_{p\in\Delta}\left\{\frac{\Gamma_p}{\underline{a}_p},\ \left(1+\frac{\overline{a}_p}{\underline{a}_p}\right)\Gamma_p\right\}:=r<1,$$

where

$$\begin{aligned}
\Gamma_p &= \overline{a}_p\overline{\delta}_p + B_p + C_p + D_p, \quad p\in\Delta,\\
B_p &= \sum_{q=1}^{n}\left(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\right)\left(\alpha_q^R+\alpha_q^I+\alpha_q^J+\alpha_q^K\right), \quad p\in\Delta,\\
C_p &= \sum_{q=1}^{n}\left(\overline{c}_{pq}^R+\overline{c}_{pq}^I+\overline{c}_{pq}^J+\overline{c}_{pq}^K\right)\left(\beta_q^R+\beta_q^I+\beta_q^J+\beta_q^K\right), \quad p\in\Delta,\\
D_p &= \sum_{q=1}^{n}\overline{\sigma}_{pq}\left(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\right)\left(\gamma_q^R+\gamma_q^I+\gamma_q^J+\gamma_q^K\right), \quad p\in\Delta.
\end{aligned}$$

Theorem 3.1 *Let* (H_1)–(H_4) *hold. Then System* (8) *has a unique almost automorphic solution in the region* $\mathbb{B}^*=\{\phi\in\mathbb{B}\mid \|\phi\|_{\mathbb{B}}\le\kappa\}$.

Proof For any $\phi\in\mathbb{B}^*$, $p\in\Delta$, we consider the following almost automorphic system:

$$\begin{aligned}
Z_p'(t) &= -a_p(t)Z_p(t)+a_p(t)\int_{t-\delta_p(t)}^{t}\phi_p'(s)ds+\sum_{q=1}^{n}B_{pq}(t)\widetilde{F}_q[\phi_q(t)]\\
&\quad+\sum_{q=1}^{n}C_{pq}(t)\widetilde{G}_q[\phi_q(t-\tau_{pq}(t))]+\sum_{q=1}^{n}D_{pq}(t)\int_{t-\sigma_{pq}(t)}^{t}\widetilde{E}_q[\phi_q(s)]ds+U_p(t). \qquad (9)
\end{aligned}$$

It follows from Lemma 2.8 that the linear system

$$Z_p'(t)=-a_p(t)Z_p(t), \quad p\in\Delta$$

admits an exponential dichotomy on $\mathbb{R}$. Thus, by Lemma 2.7, we obtain that System (9) has exactly one almost automorphic solution that can be expressed as follows

$$\begin{aligned}
Z_p^{\phi}(t) &= \int_{-\infty}^{t}\mathrm{e}^{-\int_s^t a_p(u)du}\bigg[a_p(s)\int_{s-\delta_p(s)}^{s}\phi_p'(u)du+\sum_{q=1}^{n}B_{pq}(s)\widetilde{F}_q[\phi_q(s)]\\
&\quad+\sum_{q=1}^{n}C_{pq}(s)\widetilde{G}_q[\phi_q(s-\tau_{pq}(s))]+\sum_{q=1}^{n}D_{pq}(s)\int_{s-\sigma_{pq}(s)}^{s}\widetilde{E}_q[\phi_q(u)]du+U_p(s)\bigg], \quad p\in\Delta.
\end{aligned}$$

From Lemma 2.2, Lemma 2.4 and Definition 2.5, we can obtain $(Z_1^{\phi},Z_2^{\phi},\cdots,Z_n^{\phi})^{\mathrm{T}}\in\mathbb{B}$. Now, we define a mapping $\Phi:\mathbb{B}^*\to\mathbb{B}$ by setting

$$\Phi\phi=((\Phi\phi)_1,(\Phi\phi)_2,\cdots,(\Phi\phi)_n)^{\mathrm{T}}=(Z_1^{\phi},Z_2^{\phi},\cdots,Z_n^{\phi})^{\mathrm{T}}, \quad \forall\,\phi\in\mathbb{B}^*, \quad p\in\Delta.$$

First, we show that for any $\phi\in\mathbb{B}^*$, $\Phi\phi\in\mathbb{B}^*$. In fact,

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}|(\Phi\phi)_p^R(t)|\\
&=\sup_{t\in\mathbb{R}}\bigg|\int_{-\infty}^{t}\mathrm{e}^{-\int_s^t a_p(u)du}\bigg[a_p(s)\int_{s-\delta_p(s)}^{s}(\phi_p^R)'(u)du+\sum_{q=1}^{n}\Big(b_{pq}^R(s)\widetilde{f}_q^R[\phi_q(s)]\\
&\quad -b_{pq}^I(s)\widetilde{f}_q^I[\phi_q(s)]-b_{pq}^J(s)\widetilde{f}_q^J[\phi_q(s)]-b_{pq}^K(s)\widetilde{f}_q^K[\phi_q(s)]\Big)\\
&\quad +\sum_{q=1}^{n}\Big(c_{pq}^R(s)\widetilde{g}_q^R[\phi_q(s-\tau_{pq}(s))]-c_{pq}^I(s)\widetilde{g}_q^I[\phi_q(s-\tau_{pq}(s))]\\
&\quad -c_{pq}^J(s)\widetilde{g}_q^J[\phi_q(s-\tau_{pq}(s))]-c_{pq}^K(s)\widetilde{g}_q^K[\phi_q(s-\tau_{pq}(s))]\Big)\\
&\quad +\sum_{q=1}^{n}\bigg(d_{pq}^R(s)\int_{s-\sigma_{pq}(s)}^{s}\widetilde{e}_q^R[\phi_q(u)]ds-d_{pq}^I(s)\int_{s-\sigma_{pq}(s)}^{s}\widetilde{e}_q^I[\phi_q(u)]ds\\
&\quad -d_{pq}^J(s)\int_{s-\sigma_{pq}(s)}^{s}\widetilde{e}_q^J[\phi_q(u)]ds-d_{pq}^K(s)\int_{s-\sigma_{pq}(s)}^{s}\widetilde{e}_q^K[\phi_q(u)]ds\bigg)+u_p^R(s)\bigg]ds\bigg|\\
&\le\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\mathrm{e}^{-\int_s^t a_p(u)du}\bigg[\overline{a}_p\overline{\delta}_p\|\phi\|_{\mathbb{B}}+\sum_{q=1}^{n}\Big(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\Big)\Big(\alpha_q^R+\alpha_q^I\\
&\quad +\alpha_q^J+\alpha_q^K\Big)\|\phi\|_{\mathbb{B}}+\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R+\overline{c}_{pq}^I+\overline{c}_{pq}^J+\overline{c}_{pq}^K\Big)\Big(\beta_q^R+\beta_q^I+\beta_q^J+\beta_q^K\Big)\|\phi\|_{\mathbb{B}}\\
&\quad +\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\Big)\Big(\gamma_q^R+\gamma_q^I+\gamma_q^J+\gamma_q^K\Big)\|\phi\|_{\mathbb{B}}+\overline{u}_p^R\bigg]ds\\
&\le\frac{1}{\underline{a}_p}\bigg(\bigg[\overline{a}_p\overline{\delta}_p+\sum_{q=1}^{n}\Big(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\Big)\Big(\alpha_q^R+\alpha_q^I+\alpha_q^J+\alpha_q^K\Big)\\
&\quad +\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R+\overline{c}_{pq}^I+\overline{c}_{pq}^J+\overline{c}_{pq}^K\Big)\Big(\beta_q^R+\beta_q^I+\beta_q^J+\beta_q^K\Big)\\
&\quad +\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\Big)\Big(\gamma_q^R+\gamma_q^I+\gamma_q^J+\gamma_q^K\Big)\bigg]\kappa+\overline{u}_p^R\bigg)\\
&=\frac{1}{\underline{a}_p}\Big(\big(\overline{a}_p\overline{\delta}_p+B_p+C_p+D_p\big)\kappa+\overline{u}_p^R\Big)=\frac{\Gamma_p\kappa+\overline{u}_p^R}{\underline{a}_p},\quad p\in\Delta.
\end{aligned}\tag{10}$$

In a similar way, we have

$$\sup_{t\in\mathbb{R}}|(\Phi\phi)_p^l(t)|\le\frac{\Gamma_p\kappa+\overline{u}_p^l}{\underline{a}_p},\quad p\in\Delta,\ l=I,J,K.\tag{11}$$

On the other hand, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\big|\big((\Phi\phi)_p^R\big)'(t)\big|\\
&\le\sup_{t\in\mathbb{R}}\bigg(\overline{a}_p\overline{\delta}_p\big|(\phi_p^R)'(t)\big|+\sum_{q=1}^{n}\Big(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\Big)\Big(\alpha_q^R\big|\phi_q^R(t)\big|
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_q^I\big|\phi_q^I(t)\big| + \alpha_q^J\big|\phi_q^J(t)\big| + \alpha_q^K\big|\phi_q^K(t)\big|\Big) + \sum_{q=1}^{n}\Big(\overline{c}_{pq}^R + \overline{c}_{pq}^I + \overline{c}_{pq}^J + \overline{c}_{pq}^K\Big)\\
&\times\Big(\beta_q^R\big|\phi_q^R(t-\tau_{pq}(t))\big| + \beta_q^I\big|\phi_q^I(t-\tau_{pq}(t))\big| + \beta_q^J\big|\phi_q^J(t-\tau_{pq}(t))\big|\\
&+\beta_q^K\big|\phi_q^K(t-\tau_{pq}(t))\big|\Big) + \sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R + \overline{d}_{pq}^I + \overline{d}_{pq}^J + \overline{d}_{pq}^K\Big) + \Big(\gamma_q^R\big|\phi_q^R(t)\big|\\
&+\gamma_q^I\big|\phi_q^I(t)\big| + \gamma_q^J\big|\phi_q^J(t)\big| + \gamma_q^K\big|\phi_q^K(t)\big|\Big) + \overline{u}_p^R\Big)\\
&+\overline{a}_p\int_{-\infty}^{t}\mathrm{e}^{-\int_s^t a_p(u)du}\Big[\overline{a}_p\overline{\delta}_p\big|(\phi_p^R)'(t)\big| + \sum_{q=1}^{n}\Big(\overline{b}_{pq}^R + \overline{b}_{pq}^I + \overline{b}_{pq}^J + \overline{b}_{pq}^K\Big)\\
&\times\Big(\alpha_q^R\big|\phi_q^R(s)\big| + \alpha_q^I\big|\phi_q^I(s)\big| + \alpha_q^J\big|\phi_q^J(s)\big| + \alpha_q^K\big|\phi_q^K(s)\big|\Big)\\
&+\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R + \overline{c}_{pq}^I + \overline{c}_{pq}^J + \overline{c}_{pq}^K\Big)\Big(\beta_q^R\big|\phi_q^R(s-\tau_{pq}(s))\big| + \beta_q^I\big|\phi_q^I(s-\tau_{pq}(s))\big|\\
&+\beta_q^J\big|\phi_q^J(s-\tau_{pq}(s))\big| + \beta_q^K\big|\phi_q^K(s-\tau_{pq}(s))\big|\Big) + \sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R + \overline{d}_{pq}^I + \overline{d}_{pq}^J\\
&+\overline{d}_{pq}^K\Big)\Big(\gamma_q^R\big|\phi_q^R(s)\big| + \gamma_q^I\big|\phi_q^I(s)\big| + \gamma_q^J\big|\phi_q^J(s)\big| + \gamma_q^K\big|\phi_q^K(s)\big|\Big) + \overline{u}_p^R\Big]ds\\
\le{}& \overline{a}_p\overline{\delta}_p\kappa + \sum_{q=1}^{n}\Big(\overline{b}_{pq}^R + \overline{b}_{pq}^I + \overline{b}_{pq}^J + \overline{b}_{pq}^K\Big)\Big(\alpha_q^R + \alpha_q^I + \alpha_q^J + \alpha_q^K\Big)\kappa\\
&+\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R + \overline{c}_{pq}^I + \overline{c}_{pq}^J + \overline{c}_{pq}^K\Big)\Big(\beta_q^R + \beta_q^I + \beta_q^J + \beta_q^K\Big)\kappa\\
&+\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R + \overline{d}_{pq}^I + \overline{d}_{pq}^J + \overline{d}_{pq}^K\Big)\Big(\gamma_q^R + \gamma_q^I + \gamma_q^J + \gamma_q^K\Big)\kappa + \overline{u}_p^R\\
&+\frac{\overline{a}_p}{\underline{a}_p}\Big[\overline{a}_p\overline{\delta}_p\kappa + \sum_{q=1}^{n}\Big(\overline{b}_{pq}^R + \overline{b}_{pq}^I + \overline{b}_{pq}^J + \overline{b}_{pq}^K\Big)\Big(\alpha_q^R + \alpha_q^I + \alpha_q^J\\
&+\alpha_q^K\Big)\kappa + \sum_{q=1}^{n}\Big(\overline{c}_{pq}^R + \overline{c}_{pq}^I + \overline{c}_{pq}^J + \overline{c}_{pq}^K\Big)\Big(\beta_q^R + \beta_q^I + \beta_q^J + \beta_q^K\Big)\kappa\\
&+\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R + \overline{d}_{pq}^I + \overline{d}_{pq}^J + \overline{d}_{pq}^K\Big)\Big(\gamma_q^R + \gamma_q^I + \gamma_q^J + \gamma_q^K\Big)\kappa + \overline{u}_p^R\Big]\\
={}& \left(1 + \frac{\overline{a}_p}{\underline{a}_p}\right)\Big(\big(\overline{a}_p\overline{\delta}_p + B_p + C_p + D_p\big)\kappa + \overline{u}_p^R\Big)\\
={}& \left(1 + \frac{\overline{a}_p}{\underline{a}_p}\right)\big(\varGamma_p\kappa + \overline{u}_p^R\big), \quad p\in\Delta.
\end{aligned}
\tag{12}
$$

In a similar way, we have

$$
\sup_{t\in\mathbb{R}}\big|\big((\varPhi\phi)_p^l\big)'(t)\big| \le \left(1 + \frac{\overline{a}_p}{\underline{a}_p}\right)\big(\varGamma_p\kappa + \overline{u}_p^l\big), \quad p\in\Delta,\ l = I, J, K. \tag{13}
$$

It follows from (10)–(13) and (H_3) that

$$\|\Phi\phi\|_{\mathbb{B}} \leq \kappa,$$

which implies that $\Phi\phi \in \mathbb{B}^*$. Next, we show that $\Phi : \mathbb{B}^* \to \mathbb{B}^*$ is a contraction operator. In fact, for any $\phi, \psi \in \mathbb{B}^*$, we can get

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} |(\Phi\phi - \Phi\psi)_p^R(t)| \\
&\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_p(u)du} \Big[\overline{a}_p \overline{\delta}_p \big|(\phi_p^R)'(s) - (\psi_p^R)'(s)\big| + \sum_{q=1}^{n} \Big(\overline{b}_{pq}^R + \overline{b}_{pq}^I + \overline{b}_{pq}^J + \overline{b}_{pq}^K \Big) \\
&\quad \times \Big(\alpha_q^R \big|\phi_q^R(s) - \psi_q^R(s)\big| + \alpha_q^I \big|\phi_q^I(s) - \psi_q^I(s)\big| + \alpha_q^J \big|\phi_q^J(s) - \psi_q^J(s)\big| + \alpha_q^K \big|\phi_q^K(s) \\
&\quad - \psi_q^K(s)\big| \Big) + \sum_{q=1}^{n} \Big(\overline{c}_{pq}^R + \overline{c}_{pq}^I + \overline{c}_{pq}^J + \overline{c}_{pq}^K \Big) \Big(\beta_q^R \big|\phi_q^R(s - \tau_{pq}(s)) - \psi_q^R(s - \tau_{pq}(s))\big| \\
&\quad + \beta_q^I \big|\phi_q^I(s - \tau_{pq}(s)) - \psi_q^I(s - \tau_{pq}(s))\big| + \beta_q^J \big|\phi_q^J(s - \tau_{pq}(s)) - \psi_q^J(s - \tau_{pq}(s))\big| \\
&\quad + \beta_q^K \big|\phi_q^K(s - \tau_{pq}(s)) - \psi_q^K(s - \tau_{pq}(s))\big| \Big) + \sum_{q=1}^{n} \overline{\sigma}_{pq} \Big(\overline{d}_{pq}^R + \overline{d}_{pq}^I + \overline{d}_{pq}^J + \overline{d}_{pq}^K \Big) \\
&\quad \times \Big(\gamma_q^R \big|\phi_q^R(s) - \psi_q^R(s)\big| + \gamma_q^I \big|\phi_q^I(s) - \psi_q^I(s)\big| + \gamma_q^J \big|\phi_q^J(s) - \psi_q^J(s)\big| \\
&\quad + \gamma_q^K \big|\phi_q^K(s) - \psi_q^K(s)\big| \Big) \Big] ds \\
&\leq \frac{1}{\underline{a}_p} \Big[\overline{a}_p \overline{\delta}_p + \sum_{q=1}^{n} \Big(\overline{b}_{pq}^R + \overline{b}_{pq}^I + \overline{b}_{pq}^J + \overline{b}_{pq}^K \Big) \Big(\alpha_q^R + \alpha_q^I + \alpha_q^J + \alpha_q^K \Big) \\
&\quad + \sum_{q=1}^{n} \Big(\overline{c}_{pq}^R + \overline{c}_{pq}^I + \overline{c}_{pq}^J + \overline{c}_{pq}^K \Big) \Big(\beta_q^R + \beta_q^I + \beta_q^J + \beta_q^K \Big) \\
&\quad + \sum_{q=1}^{n} \overline{\sigma}_{pq} \Big(\overline{d}_{pq}^R + \overline{d}_{pq}^I + \overline{d}_{pq}^J + \overline{d}_{pq}^K \Big) \Big(\gamma_q^R + \gamma_q^I + \gamma_q^J + \gamma_q^K \Big) \Big] \|\phi - \psi\|_{\mathbb{B}} \\
&= \frac{\Gamma_p}{\underline{a}_p} \|\phi - \psi\|_{\mathbb{B}}, \quad p \in \Delta. \qquad (14)
\end{aligned}$$

In a similar way, we have

$$\sup_{t\in\mathbb{R}} |(\Phi\phi - \Phi\psi)_p^l(t)| \leq \frac{\Gamma_p}{\underline{a}_p} \|\phi - \psi\|_{\mathbb{B}}, \quad p \in \Delta,\ l = I, J, K. \qquad (15)$$

On the other hand, we can obtain

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \big| \big((\Phi\phi - \Phi\psi)_p^R\big)'(t) \big| &\leq \Big(1 + \frac{\overline{a}_p}{\underline{a}_p} \Big) \Big(\overline{a}_p \overline{\delta}_p + B_p + C_p + D_p \Big) \|\phi - \psi\|_{\mathbb{B}} \\
&= \Big(1 + \frac{\overline{a}_p}{\underline{a}_p} \Big) \Gamma_p \|\phi - \psi\|_{\mathbb{B}}, \quad p \in \Delta. \qquad (16)
\end{aligned}$$

In a similar way, we have

$$\sup_{t\in\mathbb{R}} \big| \big((\Phi\phi - \Phi\psi)_p^l\big)'(t) \big| \leq \Big(1 + \frac{\overline{a}_p}{\underline{a}_p} \Big) \Gamma_p \|\phi - \psi\|_{\mathbb{B}}, \quad p \in \Delta,\ l = I, J, K. \qquad (17)$$

By (14)–(17), we have

$$\|\Phi(\phi)-\Phi(\psi)\|_{\mathbb{B}}\leq r\|\phi-\psi\|_{\mathbb{B}}.$$

In view of (H_4), we see that Φ is a contraction mapping from $\mathbb{B}^*$ to $\mathbb{B}^*$. Therefore, Φ has a unique fixed point in $\mathbb{B}^*$, that is, (8) has a unique almost automorphic solution in $\mathbb{B}^*$. The proof is complete. ∎

Theorem 3.2 *Assume that* (H_1)–(H_4) *hold, then System* (8) *has a unique almost automorphic solution that is globally exponentially stable.*

Proof From Theorem 3.1, we see that System (8) has an almost automorphic solution $Z^*(t)=(Z_1^*(t),Z_2^*(t),\cdots,Z_n^*(t))^{\mathrm{T}}$ with initial value $\psi^*(t)=(\psi_1^*(t),\psi_2^*(t),\cdots,\psi_n^*(t))^{\mathrm{T}}$. Suppose that $Z(t)=(Z_1(t),Z_2(t),\cdots,Z_n(t))^{\mathrm{T}}$ is an arbitrary solution of System (8) with initial value $\psi(t)=(\psi_1(t),\psi_2(t),\cdots,\psi_n(t))^{\mathrm{T}}$. Set $Y(t)=Z(t)-Z^*(t)$, then, according to (8), we have

$$\begin{aligned}Y_p'(t)&=-a_p(t)Y_p(t-\delta_p(t))+\sum_{q=1}^{n}B_{pq}(t)\big(\widetilde{F}_q[z_q(t)]-\widetilde{F}_q[z_q^*(t)]\big)\\&\quad+\sum_{q=1}^{n}C_{pq}(t)\big(\widetilde{G}_q[z_q(t-\tau_{pq}(t))]-\widetilde{G}_q(z_q^*[(t-\tau_{pq}(t))]\big)\\&\quad+\sum_{q=1}^{n}D_{pq}(t)\int_{t-\sigma_{pq}(t)}^{t}\big(\widetilde{E}_q[z_q(s)]-\widetilde{E}_q[z_q^*(s)]\big)ds,\quad p\in\Delta.\end{aligned}\tag{18}$$

For $p\in\Delta$, let Π_p and Θ_p be defined as follows:

$$\begin{aligned}\Pi_p(\zeta)&=\underline{a}_p-\zeta-\big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\zeta\overline{\tau}_{pq}}+D_p\big),\\\Theta_p(\zeta)&=\underline{a}_p-\zeta-\big(\overline{a}_p+\underline{a}_p\big)\big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\zeta\overline{\tau}_{pq}}+D_p\big).\end{aligned}$$

By (H_4), we have

$$\begin{aligned}\Pi_p(0)&=\underline{a}_p-\big(\overline{a}_p\overline{\delta}_p+B_p+C_p+D_p\big)>0,\quad p\in\Delta,\\\Theta_p(0)&=\underline{a}_p-\big(\overline{a}_p+\underline{a}_p\big)\big(\overline{a}_p\overline{\delta}_p+B_p+C_p+D_p\big)>0,\quad p\in\Delta.\end{aligned}$$

Since Π_p and Θ_p are continuous on $[0,+\infty)$ and $\Pi_p(\zeta),\Theta_p(\zeta)\to-\infty$, as $\zeta\to+\infty$, there exist $\xi_p,\xi_p^*>0$ such that $\Pi_p(\xi_p)=\Theta_p(\xi_p^*)=0$ and $\Pi_p(\zeta)>0$ for $\zeta\in(0,\xi_p)$, $\Theta_p(\zeta)>0$ for $\zeta\in(0,\xi_p^*)$, $p\in\Delta$. Take $\vartheta=\min_{p\in\Delta}\{\xi_p,\xi_p^*\}$, we have $\Pi_p(\vartheta)\geq0,\Theta_p(\vartheta)\geq0$. So, we can choose a positive constant $0<\lambda<\min\{\vartheta,\min_{p\in\Delta}\{\underline{a}_p\}\}$ such that

$$\Pi_p(\lambda)>0,\quad\Theta_p(\lambda)>0,\quad p\in\Delta,$$

which imply that for $p\in\Delta$,

$$\frac{1}{\underline{a}_p-\lambda}\Big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\lambda\overline{\tau}_{pq}}+D_p\Big)<1$$

and

$$\Big(1+\frac{\overline{a}_p}{\underline{a}_p-\lambda}\Big)\Big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\lambda\overline{\tau}_{pq}}+D_p\Big)<1.$$

Let $M = \max_{p\in\Delta}\left\{\frac{\underline{a}_p}{\Gamma_p}\right\}$, then by ($H_4$) we have $M > 1$. Thus,

$$\frac{1}{M} - \min_{p\in\Delta}\left\{\frac{1}{\underline{a}_p - \lambda}\Big(\overline{a}_p\overline{\delta}_p + B_p + C_p e^{\lambda\overline{\tau}_{pq}} + D_p\Big)\right\} < 0.$$

Let

$$\|Y(t)\| = \max_{p\in\Delta}\left\{\max_{l\in\Lambda}\left\{|z_p^l(t) - z_p^{*l}(t)|, |(z_p^l)'(t) - (z_p^{*l})'(t)|\right\}\right\},$$
$$\|\varphi\|_0 = \max_{p\in\Delta}\left\{\max_{l\in\Lambda}\left\{\sup_{s\in[-\eta,0]}\big|\psi_p^l(s) - \psi_p^{*l}(s)\big|, \sup_{s\in[-\eta,0]}\big|(\psi_p^l)'(s) - (\psi_p^{*l})'(s)\big|\right\}\right\}.$$

Hence, for any $\varepsilon > 0$, it is obvious that

$$\|Y(0)\| < \|\varphi\|_0 + \varepsilon \tag{19}$$

and

$$\|Y(t)\| < (\|\varphi\|_0 + \varepsilon)e^{-\lambda t} < M(\|\varphi\|_0 + \varepsilon)e^{-\lambda t}, \quad \forall\, t \in [-\eta, 0]. \tag{20}$$

We claim that

$$\|Y(t)\| < M(\|\varphi\|_0 + \varepsilon)e^{-\lambda t}, \quad \forall\, t > 0. \tag{21}$$

In the contrary case, then there must be some $p \in \{1, 2, \cdots, n\}$ and $t_1 > 0$ such that

$$\begin{cases} |Y_p(t_1)| = \|Y(t_1)\| = M(\|\varphi\|_0 + \varepsilon)e^{-\lambda t_1}, \\ \|Y(t)\| < M(\|\varphi\|_0 + \varepsilon)e^{-\lambda t}, \quad t < t_1. \end{cases} \tag{22}$$

Multiplying both sides of (18) by $e^{\int_0^t a_p(u)du}$ and integrating over $[0, t]$, we get

$$\begin{aligned} Y_p(t) = \bigg\{ & Y_p(0)e^{-\int_0^t a_p(u)du} + \int_0^t e^{-\int_s^t a_p(u)du}\bigg(a_p(s)\int_{s-\delta_p(s)}^s Y_p'(u)du \\ & + \sum_{q=1}^n B_{pq}(s)\big(\widetilde{F}_q[z_q(s)] - \widetilde{F}_q[z_q^*(s)]\big) + \sum_{q=1}^n C_{pq}(s)\big(\widetilde{G}_q[z_q(s - \tau_{pq}(s))] \\ & - \widetilde{G}_q[z_q^*(s - \tau_{pq}(s))]\big) + \sum_{q=1}^n D_{pq}(s)\int_{s-\sigma_{pq}(s)}^s \big(\widetilde{E}_q[z_q(u)] - \widetilde{E}_q[z_q^*(u)]\big)du\bigg)ds\bigg\}. \end{aligned}$$

Thus, by $M > 1$, (19), (20) and (22) imply that

$$\begin{aligned} & \big|(z - z^*)_p^R(t_1)\big| \\ \le{} & \big|z_p^R(0) - z_p^{*R}(0)\big|e^{-\int_0^{t_1} a_p(u)du} + \int_0^{t_1} e^{-\int_s^{t_1} a_p(u)du}\Big[\overline{a}_p\overline{\delta}_p\big|(z_p^R)'(s) - (z_p^{*R})'(s)\big| \\ & + \sum_{q=1}^n \Big(\overline{b}_{pq}^R + \overline{b}_{pq}^I + \overline{b}_{pq}^J + \overline{b}_{pq}^K\Big)\Big(\alpha_q^R\big|z_q^R(s) - z_q^{*R}(s)\big| + \alpha_q^I\big|z_q^I(s) - z_q^{*I}(s)\big| \end{aligned}$$

$$
\begin{aligned}
&+\alpha_q^J\big|z_q^J(s)-z_q^{*J}(s)\big|+\alpha_q^K\big|z_q^K(s)-z_q^{*K}(s)\big|\Big)+\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R+\overline{c}_{pq}^I+\overline{c}_{pq}^J+\overline{c}_{pq}^K\Big)\\
&\times\Big(\beta_q^R\big|z_q^R(s-\tau_{pq}(s))-z_q^{*R}(s-\tau_{pq}(s))\big|+\beta_q^I\big|z_q^I(s-\tau_{pq}(s))-z_q^{*I}(s-\tau_{pq}(s))\big|\\
&+\beta_q^J\big|z_q^J(s-\tau_{pq}(s))-z_q^{*J}(s-\tau_{pq}(s))\big|+\beta_q^K\big|z_q^K(s-\tau_{pq}(s))-z_q^{*K}(s-\tau_{pq}(s))\big|\Big)\\
&+\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\Big)\Big(\gamma_q^R\big|z_q^R(s)-z_q^{*R}(s)\big|+\gamma_q^I\big|z_q^I(s)-z_q^{*I}(s)\big|\\
&+\gamma_q^J\big|z_q^J(s)-z_q^{*J}(s)\big|+\gamma_q^K\big|z_q^K(s)-z_q^{*K}(s)\big|\Big)\Big]ds\\
\le{}&(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1}\mathrm{e}^{-\int_0^{t_1}(a_p(u)-\lambda)du}+\int_0^{t_1}\mathrm{e}^{-\int_s^{t_1}\left(a_p(u)-\lambda\right)du}\Big(\overline{a}_p\overline{\delta}_p\\
&+\sum_{q=1}^{n}\Big(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\Big)\Big(\alpha_q^R+\alpha_q^I+\alpha_q^J+\alpha_q^K\Big)+\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R+\overline{c}_{pq}^I\\
&+\overline{c}_{pq}^J+\overline{c}_{pq}^K\Big)\Big(\beta_q^R+\beta_q^I+\beta_q^J+\beta_q^K\Big)\mathrm{e}^{\lambda\overline{\tau}_{pq}}+\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\Big)\\
&\times\Big(\gamma_q^R+\gamma_q^I+\gamma_q^J+\gamma_q^K\Big)\Big)dsM(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1}\\
\le{}&M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1}\bigg(\frac{1}{M}-\frac{1}{\underline{a}_p-\lambda}\Big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\lambda\overline{\tau}_{pq}}+D_p\Big)\bigg)\mathrm{e}^{(\lambda-a_p(u))t_1}\\
<{}&M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1}.
\end{aligned}\tag{23}
$$

Similarly, we can get

$$
|(z-z^*)_p^l(t_1)|<M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1},\quad l=I,J,K. \tag{24}
$$

On the other hand, we have

$$
\begin{aligned}
&\big|\big((z-z^*)_p^R\big)'(t_1)\big|\\
\le{}&\overline{a}_p\big|z_p^R(0)-z_p^{*R}(0)\big|\mathrm{e}^{-\int_0^{t_1}a_p(u)du}+\overline{a}_p\overline{\delta}_p\big|(z_p^R)'(t)-(z_p^{*R})'(t)\big|\\
&+\sum_{q=1}^{n}\Big(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\Big)\Big(\alpha_q^R\big|z_q^R(t)-z_q^{*R}(t)\big|+\alpha_q^I\big|z_q^I(t)-z_q^{*I}(t)\big|\\
&+\alpha_q^J\big|z_q^J(t)-z_q^{*J}(t)\big|+\alpha_q^K\big|z_q^K(t)-z_q^{*K}(t)\big|\Big)+\sum_{q=1}^{n}\Big(\overline{c}_{pq}^R+\overline{c}_{pq}^I+\overline{c}_{pq}^J+\overline{c}_{pq}^K\Big)\\
&\times\Big(\beta_q^R\big|z_q^R(t-\tau_{pq}(t))-z_q^{*R}(t-\tau_{pq}(t))\big|+\beta_q^I\big|z_q^I(t-\tau_{pq}(t))-z_q^{*I}(t-\tau_{pq}(t))\big|\\
&+\beta_q^J\big|z_q^J(t-\tau_{pq}(t))-z_q^{*J}(t-\tau_{pq}(t))\big|+\beta_q^K\big|z_q^K(t-\tau_{pq}(t))-z_q^{*K}(t-\tau_{pq}(t))\big|\Big)\\
&+\sum_{q=1}^{n}\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\Big)\Big(\gamma_q^R\big|z_q^R(t)-z_q^{*R}(t)\big|+\gamma_q^I\big|z_q^I(t)-z_q^{*I}(t)\big|\\
&+\gamma_q^J\big|z_q^J(t)-z_q^{*J}(t)\big|+\gamma_q^K\big|z_q^K(t)-z_q^{*K}(t)\big|\Big)+\overline{a}_p\int_0^{t_1}\mathrm{e}^{-\int_s^{t_1}a_p(u)du}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big[\overline{a}_p\overline{\delta}_p\big|(z_p^R)'(s)-(z_p^{*R})'(s)\big|+\sum_{q=1}^n\Big(\overline{b}_{pq}^R+\overline{b}_{pq}^I+\overline{b}_{pq}^J+\overline{b}_{pq}^K\Big)\Big(\alpha_q^R\big|z_q^R(s)-z_q^{*R}(s)\big|\\
&+\alpha_q^I\big|z_q^I(s)-z_q^{*I}(s)\big|+\alpha_q^J\big|z_q^J(s)-z_q^{*J}(s)\big|+\alpha_q^K\big|z_q^K(s)-z_q^{*K}(s)\big|\Big)\\
&+\sum_{q=1}^n\Big(\overline{c}_{pq}^R+\overline{c}_{pq}^I+\overline{c}_{pq}^J+\overline{c}_{pq}^K\Big)\Big(\beta_q^R\big|z_q^R(s-\tau_{pq}(s))-z_q^{*R}(s-\tau_{pq}(s))\big|\\
&+\beta_q^I\big|z_q^I(s-\tau_{pq}(s))-z_q^{*I}(s-\tau_{pq}(s))\big|+\beta_q^J\big|z_q^J(s-\tau_{pq}(s))-z_q^{*J}(s-\tau_{pq}(s))\big|\\
&+\beta_q^K\big|z_q^K(s-\tau_{pq}(s))-z_q^{*K}(s-\tau_{pq}(s))\big|\Big)+\sum_{q=1}^n\overline{\sigma}_{pq}\Big(\overline{d}_{pq}^R+\overline{d}_{pq}^I+\overline{d}_{pq}^J+\overline{d}_{pq}^K\Big)\\
&\times\Big(\gamma_q^R\big|z_q^R(s)-z_q^{*R}(s)\big|+\gamma_q^I\big|z_q^I(s)-z_q^{*I}(s)\big|+\gamma_q^J\big|z_q^J(s)-z_q^{*J}(s)\big|\\
&+\gamma_q^K\big|z_q^K(s)-z_q^{*K}(s)\big|\Big)\Big]ds\\
\le{}& M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1}\bigg\{\Big(\frac{1}{M}-\frac{1}{\underline{a}_p-\lambda}\Big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\lambda\overline{\tau}_{pq}}+D_p\Big)\overline{a}_p\mathrm{e}^{(\lambda-a_p(u))t_1}\\
&+\Big(1+\frac{\overline{a}_p}{\underline{a}_p-\lambda}\Big)\Big(\overline{a}_p\overline{\delta}_p+B_p+C_p\mathrm{e}^{\lambda\overline{\tau}_{pq}}+D_p\Big)\bigg\}\\
<{}& M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1}. \qquad (25)
\end{aligned}
$$

Similarly, we have

$$
\big|\big((z-z^*)_p^l\big)'(t_1)\big|<M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1},\quad l=I,J,K. \qquad (26)
$$

It follows from (23)–(26) that

$$
\|Y(t_1)\|<M(\|\varphi\|_0+\varepsilon)\mathrm{e}^{-\lambda t_1},
$$

which contradicts the first equation of (22). Therefore, (21) holds. Letting $\varepsilon\to 0^+$ leads to

$$
\|Y(t)\|\le M\|\varphi\|_0\mathrm{e}^{-\lambda t},\quad \forall\, t>0.
$$

Hence, the almost automorphic solution of System (8) is globally exponentially stable. The proof is complete. ∎

4 An Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Section 3.

Example 4.1 Consider the following QVNNs with time-varying leakage delays:

$$
\begin{aligned}
z_p'(t)={}&-a_p(t)z_p(t-\delta_p(t))+\sum_{q=1}^2 b_{pq}(t)f_q(z_q(t))+\sum_{q=1}^2 c_{pq}(t)g_q(z_q(t-\tau_{pq}(t)))\\
&+\sum_{q=1}^2 d_{pq}(t)\int_{t-\sigma_{pq}(t)}^t e_q(z_q(s))ds+u_p(t), \qquad (27)
\end{aligned}
$$

where $p = 1, 2$, $z_p = z_p^R + iz_p^I + jz_p^J + kz_p^K \in \mathbb{Q}$, and the coefficients are as follows:

$$
\begin{aligned}
&a_1(t) = 1.8 + 0.2|\cos(\sqrt{3}t)|, \quad a_2(t) = 2.6 + 0.4|\sin(\sqrt{2}t)|,\\
&f_q(x_q) = \frac{1}{15}|x_q^R + x_q^K| + i\frac{1}{20}\sin^2(x_q^I + x_q^J) + j\frac{1}{40}(|x_q^J + 1| - |x_q^J - 1|) + k\frac{1}{15}\tanh x_q^K,\\
&g_q(x_q) = \frac{1}{32}\sin^2(x_q^R + x_q^J) + i\frac{1}{16}|x_q^I| + j\frac{1}{16}\sin(x_q^J + x_q^K) + k\frac{1}{32}(|x_q^K - 1| - |x_q^K - 1|),\\
&e_q(x_q) = \frac{1}{8}\sin^2 x_q^R + i\frac{1}{4}|x_q^R + x_q^I| + j\frac{1}{8}(|x_q^J - 1| - |x_q^J - 1|) + k\frac{1}{4}\sin x_q^K,\\
&b_{11}(t) = b_{12}(t) = 0.07\sin(\sqrt{2}t) - i0.09\cos(\sqrt{2}t) + j0.08\cos t - k0.05\sin t,\\
&b_{21}(t) = b_{22}(t) = 0.06\cos(\sqrt{5}t) + i0.04\cos(\sqrt{2}t) - j0.09\sin(\sqrt{3}t) - k0.08\cos(2t),\\
&c_{11}(t) = c_{12}(t) = 0.08\cos t - i0.07\sin(\sqrt{5}t) + j0.075\sin t + k0.055\sin t,\\
&c_{21}(t) = c_{22}(t) = 0.06\cos(2t) - i0.065\sin t + j0.085\sin(\sqrt{5}t) + k0.09\cos t,\\
&d_{11}(t) = d_{12}(t) = 0.18\cos(\sqrt{2}t) + i0.16\sin t - j0.24\cos(\sqrt{3}t) + k0.3\cos t,\\
&d_{21}(t) = d_{22}(t) = 0.28\cos(\sqrt{5}t) + i0.15\cos(\sqrt{2}t) - j0.2\sin(\sqrt{3}t) - k0.15\cos(2t),\\
&u_1(t) = u_2(t) = 0.04\sin(\sqrt{3}t) - i0.02\cos(\sqrt{2}t) + j0.035\sin^2 t + k0.03\sin(2t),\\
&\delta_1(t) = 0.02 + 0.005\sin t, \quad \delta_2(t) = 0.02 + 0.05\cos(\sqrt{2}t), \quad \tau_{11}(t) = \tau_{12}(t) = 1 + \sin^2 t,\\
&\tau_{21}(t) = \tau_{22}(t) = 2 + \sin t, \quad \sigma_{pq}(t) = 0.05 + 0.01\sin t, \quad p, q \in \Delta.
\end{aligned}
$$

By a simple calculation, we have

$$
\begin{aligned}
&\underline{a}_1 = 1.8, \quad \overline{a}_1 = 2, \quad \underline{a}_2 = 2.6, \quad \overline{a}_2 = 3, \quad \alpha_q^R = \alpha_q^K = \frac{1}{15},\\
&\alpha_q^I = \alpha_q^J = \frac{1}{20}, \quad \beta_q^R = \beta_q^J = \beta_q^I = \beta_q^K = \frac{1}{16}, \quad \gamma_q^R = \gamma_q^I = \gamma_q^J = \gamma_q^K = \frac{1}{4},\\
&\overline{b}_{11}^R = \overline{b}_{12}^R = 0.07, \quad \overline{b}_{11}^I = \overline{b}_{12}^I = 0.09, \quad \overline{b}_{11}^J = \overline{b}_{12}^J = 0.08, \quad \overline{b}_{11}^K = \overline{b}_{12}^K = 0.05,\\
&\overline{b}_{21}^R = \overline{b}_{22}^R = 0.06, \quad \overline{b}_{21}^I = \overline{b}_{22}^I = 0.04, \quad \overline{b}_{21}^J = \overline{b}_{22}^J = 0.09, \quad \overline{b}_{21}^K = \overline{b}_{22}^K = 0.08,\\
&\overline{c}_{11}^R = \overline{c}_{12}^R = 0.08, \quad \overline{c}_{11}^I = \overline{c}_{12}^I = 0.07, \quad \overline{c}_{11}^J = \overline{c}_{12}^J = 0.075, \quad \overline{c}_{11}^K = \overline{c}_{12}^K = 0.055,\\
&\overline{c}_{21}^R = \overline{c}_{22}^R = 0.06, \quad \overline{c}_{21}^I = \overline{c}_{22}^I = 0.065, \quad \overline{c}_{21}^J = \overline{c}_{22}^J = 0.085, \quad \overline{c}_{21}^K = \overline{c}_{22}^K = 0.09,\\
&\overline{d}_{11}^R = \overline{d}_{12}^R = 0.18, \quad \overline{d}_{11}^I = \overline{d}_{12}^I = 0.16, \quad \overline{d}_{11}^J = \overline{d}_{12}^J = 0.24, \quad \overline{d}_{11}^K = \overline{d}_{12}^K = 0.3,\\
&\overline{d}_{21}^R = \overline{d}_{22}^R = 0.28, \quad \overline{d}_{21}^I = \overline{d}_{22}^I = 0.15, \quad \overline{d}_{21}^J = \overline{d}_{22}^J = 0.2 \quad \overline{d}_{21}^K = \overline{d}_{22}^K = 0.15,\\
&\overline{u}_1^R = \overline{u}_2^R = 0.04, \quad \overline{u}_1^I = \overline{u}_2^I = 0.02, \quad \overline{u}_1^J = \overline{u}_2^J = 0.035, \quad \overline{u}_1^K = \overline{u}_2^K = 0.03,\\
&\overline{\delta}_1 = \overline{\delta}_2 = 0.025, \quad \overline{\tau}_{11} = \overline{\tau}_{12} = 2, \quad \overline{\tau}_{21} = \overline{\tau}_{22} = 3, \quad \overline{\sigma}_{pq} = 0.06, \quad p, q \in \Delta.
\end{aligned}
$$

Then, we have

$$\max_{1 \le p \le 2}\left\{\max_{l \in \Lambda}\left\{\frac{\mathit{\Gamma}_p\kappa + \overline{u}_p^l}{\underline{a}_p}, \left(1 + \frac{\overline{a}_p}{\underline{a}_p}\right)\left(\mathit{\Gamma}_p\kappa + \overline{u}_p^l\right)\right\}\right\} = 1.9978 \le \kappa = 2$$

and

$$\max_{1 \le p \le 2}\left\{\frac{\mathit{\Gamma}_p}{\underline{a}_p}, \left(1 + \frac{\overline{a}_p}{\underline{a}_p}\right)\mathit{\Gamma}_p\right\} = 0.9559 = r < 1,$$

which means that (H_4) is satisfied for $\kappa = 2$. Obviously, conditions (H_1)–(H_3) are also satisfied. Therefore, according to Theorem 3.2, (27) has a unique almost automorphic solution, which is globally exponentially stable (see Figures 1–3).

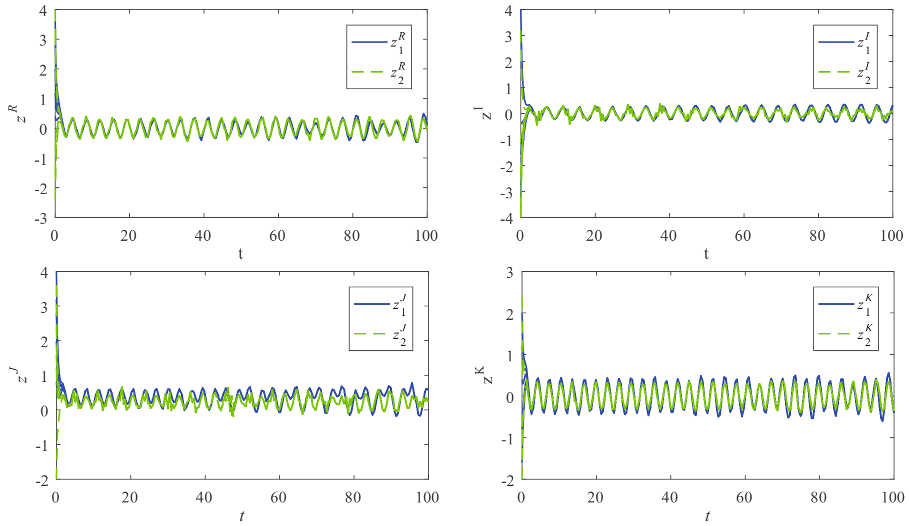

Figure 1 Transient states of four parts of the the QVNNs (27) in Example 4.1

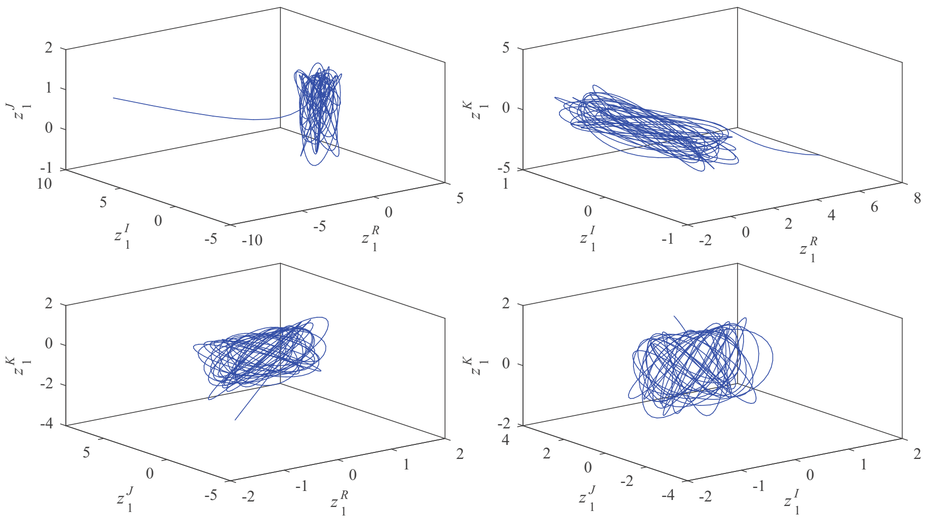

Figure 2 Curves of z_1^l ($l \in \Lambda$) in 3-dimensional space for stable case

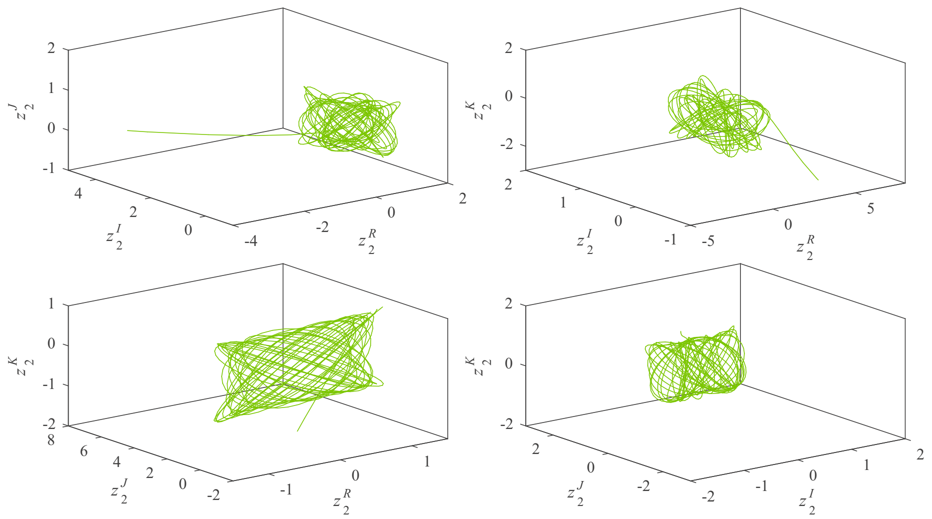

Figure 3 Curves of z_2^l $(l \in \Lambda)$ in 3-dimensional space for stable case

Remark 4.2 No existing results can directly derive that (27) has a unique almost automorphic solution, which is globally exponentially stable.

5 Conclusion

In this paper, we have investigated QVNNs with time-varying leakage delays. By employing the Banach's fixed point theorem and differential inequality techniques, we obtain the existence and global exponential stability of almost automorphic solutions for QVNNs. An example has been given to demonstrate the effectiveness of our results. Our results of this paper is new. Furthermore, the method of this paper can be applied to study the almost automorphic problem for other types of QVNNs.

References

[1] Liao X F and Yu J B, Robust stability for interval Hopfield neural networks with time delay, *IEEE Transactions on Neural Networks*, 1998, **9**(5): 1042–1045.

[2] Van D D P and Zou X, Global attractivity in delayed Hopfield neural network models, *SIAM Journal on Applied Mathematics*, 1998, **58**(6): 1878–1890.

[3] Liao X X and Xiao D M, Global exponential stability of Hopfield neural networks with time-varying delays, *Acta Electronica Sinica*, 2000, **28**(4): 87–90.

[4] Chen T, Global exponential stability of delayed Hopfield neural networks, *Neural Networks*, 2001, **14**(8): 977–980.

[5] Dharani S, Rakkiyappan R, and Cao J D, New delay-dependent stability criteria for switched Hopfield neural networks of neutral type with additive time-varying delay components, *Neurocomputing*, 2015, **151**(2): 827–834.

[6] Duan L, Huang L H, and Guo Z Y, Stability and almost periodicity for delayed high-order Hopfield neural networks with discontinuous activations, *Nonlinear Dynamics*, 2014, **77**(4): 1469–1484.

[7] Aouiti C, Coirault P, Miaadi F, et al., Finite time boundedness of neutral high-order Hopfield neural networks with time delay in the leakage term and mixed time delays, *Neurocomputing*, 2017, **260**: 378–392.

[8] Li Y K, Meng X F, and Xiong L L, Pseudo almost periodic solutions for neutral type high-order Hopfield neural networks with mixed time-varying delays and leakage delays on time scales, *International Journal of Machine Learning and Cybernetics*, 2017, **8**(6): 1915–1927.

[9] Zhao L L, Li Y K, and Li B, Weighted pseudo-almost automorphic solutions of high-order Hopfield neural networks with neutral distributed delays, *Neural Computing and Applications*, 2018, **29**(7): 513–527.

[10] Li Y K, Qin J L, and Li B, Anti-periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays, *Neural Processing Letters*, 2019, **49**(3): 1217–1237.

[11] Li Y K, Wang H M, and Meng X F, Almost automorphic synchronization of quaternion-valued high-order Hopfield neural networks with time-varying and distributed delays, *IMA Journal of Mathematical Control and Information*, 2018, in press, https://doi.org/10.1093/imamci/dny015.

[12] Chen J, Meng S, and Sun J, Stability analysis of networked control systems with aperiodic sampling and time-varying delay, *IEEE Transactions on Cybernetics*, 2017, **47**(8): 2312–2320.

[13] Sun J and Chen J, A survey on Lyapunov-based methods for stability of linear time-delay systems, *Frontiers of Computer Science*, 2017, **11**(4): 555–567.

[14] Sun J, Han Q L, Chen J, et al., Less conservative stability criteria for linear systems with interval time-varying delays, *International Journal of Robust and Nonlinear Control*, 2015, **25**(4): 475–485.

[15] Chen G L, Xia J W, and Zhuang G L, Improved passivity analysis for neural networks with Markovian jumping parameters and interval time-varying delays, *Neurocomputing*, 2015, **155**: 253–260.

[16] Chen G L, Xia J W, and Zhuang G L, Delay-dependent stability and dissipativity analysis of generalized neural networks with Markovian jump parameters and two delay components, *Journal of the Franklin Institute*, 2016, **353**(9): 2137–2158.

[17] Xia J W, Chen G L, and Sun W, Extended dissipative analysis of generalized Markovian switching neural networks with two delay components, *Neurocomputing*, 2017, **260**: 275–283.

[18] Chen G L, Xia J W, Park, J H, et al., New delay-dependent global robust passivity analysis for stochastic neural networks with Markovian jumping parameters and interval time-varying delays, *Complexity*, 2016, **21**(6): 167–179.

[19] Liu B W, Global exponential stability for BAM neural networks with time-varying delays in the leakage terms, *Nonlinear Analysis: Real World Applications*, 2013, **14**(1): 559–566.

[20] Li R X and Cao J D, Stability analysis of reaction-diffusion uncertain memristive neural networks with time-varying delays and leakage term, *Applied Mathematics and Computation*, 2016, **278**: 54–69.

[21] Li Y K and Li Y Q, Existence and exponential stability of almost periodic solution for neutral delay BAM neural networks with time-varying delays in leakage terms, *Journal of the Franklin Institute*, 2013, **350**(9): 2808–2825.

[22] Zhang H and Shao J Y, Existence and exponential stability of almost periodic solutions for CNNs with time-varying leakage delays, *Neurocomputing*, 2013, **121**: 226–233.

[23] Li Y K and Meng X F, Existence and global exponential stability of pseudo almost periodic solutions for neutral type quaternion-valued neural networks with delays in the leakage term on time scales, *Complexity*, 2017, **2017**(10): 1–15.

[24] Chen X Y, Huang L H, and Guo Z Y, Finite time stability of periodic solution for Hopfield neural networks with discontinuous activations, *Neurocomputing*, 2013, **103**: 43–49.

[25] Yang L and Li Y K, Existence and exponential stability of periodic solution for stochastic Hopfield neural networks on time scales, *Neurocomputing*, 2015, **167**: 543–550.

[26] Aouiti C, M'hamdi M S, Cao J D, et al., Piecewise pseudo almost periodic solution for impulsive generalised high-order Hopfield neural networks with leakage delays, *Neural Processing Letters*, 2017, **45**(2): 615–648.

[27] Liu B W, Almost periodic solutions for Hopfield neural networks with continuously distributed delays, *Mathematics and Computers in Simulation*, 2007, **73**(5): 327–335.

[28] Bai C Z, Existence and stability of almost periodic solutions of hopfield neural networks with continuously distributed delays, *Nonlinear Analysis: Theory, Methods & Applications*, 2009, **71**(11): 5850–5859.

[29] Bochner S, A new approach to almost periodicity, *Proceedings of the National Academy of Sciences of the United States of America*, 1962, **48**(12): 2039–2043.

[30] Bugajewski D and N'Guérékata G M, On the topological structure of almost automorphic and asymptotically almost automorphic solutions of differential and integral equations in abstract spaces, *Nonlinear Analysis: Theory, Methods & Applications*, 2004, **59**(8): 1333–1345.

[31] Hino Y and Murakami S, Almost automorphic solutions for abstract functional differential equations, *Journal of Mathematical Analysis and Applications*, 2003, **286**(2): 741–752.

[32] Ezzinbi K and N'Guérékata G M, Almost automorphic solutions for some partial functional differential equations, *Journal of Mathematical Analysis and Applications*, 2007, **328**(1): 344–358.

[33] Chérif F, Sufficient conditions for global stability and existence of almost automorphic solution of a class of RNNs, *Differential Equations & Dynamical Systems*, 2014, **22**(2): 191–207.

[34] Li Y K and Yang L, Almost automorphic solution for neutral type high-order Hopfield neural networks with delays in leakage terms on time scales, *Applied Mathematics and Computation*, 2014, **242**: 679–693.

[35] Li Y K, Meng X F, and Zhang X M, Almost automorphic solutions for fuzzy Cohen-Grossberg neural networks with mixed time delays, *Mathematical Problems in Engineering*, 2015, **2015**: 1–14.

[36] Isokawa T, Matsui T, and Nishimura H, Quaternionic neural networks: Fundamental properties and applications, *Complex Valued Neural Networks: Utilizing High-Dimensional Parameters*, 2009, 411–439.

[37] Matsui N, Isokawa T, Kusamichi H, et al., Quaternion neural network with geometrical operators, *Journal of Intelligent & Fuzzy Systems*, 2004, **15**(3–4): 149–164.

[38] Isokawa T, Kusakabe T, Matsui N, et al., Quaternion neural network and its application, *Knowledge-Based Intelligent Information and Engineering Systems*, 2003, **2774**: 318–324.

[39] Li Y K, Li B, Yao S S, et al., The global exponential pseudo almost periodic synchronization of quaternion-valued cellular neural networks with time-varying delays, *Neurocomputing*, 2018, **303**: 75–87.

[40] Li Y K, Meng X F, and Ye Y, Almost periodic synchronization for quaternion-valued neural networks with time-varying delays, *Complexity*, 2018, **2018**: Article ID 6504590, 13 pages.

[41] Liu Y, Zhang D D, and Lu J Q, Global exponential stability for quaternion-valued recurrent neural networks with time-varying delays, *Nonlinear Dynamics*, 2017, **87**(1): 553–565.

[42] Tu Z W, Cao J D, Alsaedi A, et al., Global dissipativity analysis for delayed quaternion-valued neural networks, *Neural Networks*, 2017, **89**: 97–104.

[43] Chen X F, Li Z S, Song Q K, et al., Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties, *Neural Networks*, 2017, **91**: 55–65.

[44] Shu H Q, Song Q K, Liu Y R, et al., Global μ-stability of quaternion-valued neural networks with non-differentiable time-varying delays, *Neurocomputing*, 2017, **247**: 202–212.

[45] Liu Y, Zhang D D, Lu J Q, et al., Global μ-stability criteria for quaternion-valued neural networks with unbounded time-varying delays, *Information Sciences*, 2016, **360**: 273–288.

[46] N'Guérékata G M, *Topics in Almost Automorphy*, Springer, New York, 2005.