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Availability Analysis of a Repairable Series-Parallel System with Redundant Dependency*

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Abstract This paper investigates the steady-state availability of a repairable series-parallel system with redundant dependency. The different types of components and repairmen are taken into account, the failure rate of the operating component varies as the number of other failed components and the repair rate of the failed component is constant in each parallel redundant subsystem. To quantify the redundant dependency, a modified failure dependence function is introduced to determine the failure rate of the components in each subsystem. Markov theory and matrix analysis method are used to get the steady-state probability vector of each subsystem and the steady-state availability of the entire system. A numerical example is presented to illustrate the obtained results and to analyze the effect of redundant dependency class on the system availability.

Keywords Availability, probability, redundant dependency, repairable system, series-parallel.

1 Introduction

System availability is an important subject in repairable systems. Maintaining a high or required level of availability is often an essential requisite[1]. Repairable series-parallel systems consisting of some parallel redundant subsystems in series are frequently used in many engineering systems, e.g., power systems, manufacturing production systems, and industrial systems.

In the studies of system optimization design, the availability of the system is a major concern for the system designers. In general, choosing high reliable components or placing redundant

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components in the system is a common way to improve availability in systems design. System designers have to consider the tradeoff between system performance and the cost while using high reliable components or redundancy to improve availability. Redundancy has been recognized as a main option to improve reliability or availability since the option of high reliable components is usually expensive and beyond the scope of the system designers whose main duty is selecting standard off-the shelf products from catalogs[2]. The choice of redundancy depends on the systems configuration[3]. The systems with different redundancy strategies have different reliability or availability analysis methods. Many authors investigated the reliability or availability assessment and optimization design problems, both as homogeneous and heterogeneous redundancy allocation problems, and suggested different reliability analysis methods and optimization design algorithms. Reliability analysis has shown that the availability of homogeneous redundant parallel systems consisting of identical components is extremely affected by common cause failure. In order to reduce the effect of common cause failure and introduce flexibility and diversification into redundant parallel system design, the heterogeneous redundancy design consisting of the mixture of non-identical components is used to improve the availability of the whole system. Furthermore, in real world, the non-identical components with the same functionality might be selected in the market. Compared with homogeneous systems, availability analysis and optimization design of heterogeneous system are much more complex. In term of reliability or availability analysis and optimization design of heterogeneous system, Kim, et al.[4] investigated practical stochastic models for designing and analyzing the time-dependent reliability of nonrepairable systems with heterogeneous components. Sharma, et al.[5] handled reliability evaluation and optimal design problems for heterogeneous multi-state series-parallel systems. Chaaban, et al.[6] studied cost optimization and high available heterogeneous series-parallel redundant system design by using the universal generating function method and genetic algorithm. Recently, Mo, et al.[7] proposed a multi-valued decision diagram (MDD)-based performability analysis approach for multi-state series-parallel systems with heterogeneous components. Chowdhury and Kundu[8] studied stochastic comparisons of the lifetimes for two parallel systems having heterogeneous log-Lindley distributed components.

Failure dependencies exist in the real systems widely. Ignoring the failure dependencies and analyzing the availability of the systems under independent assumption may lead to large deviation. Different types of failure dependencies are presented by Fricks and Trivedi[9] and Pecht[10], where the following three types of the failure dependencies: Common-mode failures, multi-mode failures and other failure dependencies are described. The reliability and the availability of series-parallel systems with common-mode failures or multi-mode failures have been investigated by many researchers, see Levitin[11], Ramirez-Marquez and Coit[12], Li, et al.[13], Boddu and Xing[14], Yamashiro, et al.[15], Pham[16], Levitin[17]. There are many other failure dependencies such as standby dependencies[9], history dependent states, see Cui, et al.[18], Wang and Cui[19], sequence-dependent failures, see Xing, et al.[20], load-dependent failures, see Barros, et al.[21], Blokus[22]. For the series-parallel systems with other failure dependencies, Levitin and Amari[23] investigated the optimal loading of elements for a series-parallel system with the dependence of elements failure rates on their load. Nourelfath and Yalaoui[24] studied a production series-parallel multi-state system with binary-state components and the dependence of machines failure rates on their load, and considered the load versus failure rate relationship while optimizing planning of the production system. Blokus[25] investigated the reliability function of a series-parallel system with dependent components under the assumption that the load is distributed equally among all un-failed components of each parallel subsystem.

The redundant dependency is a special type of failure dependencies, which considers the interactions in the failure process of a system[26]. Yu, et al.[27] pointed out that the failure dependency of a system is called redundant dependency if any component can be viewed as a redundancy of another component. In [27], a redundant system of n identical components with redundant dependency was investigated and a dependency function was introduced to quantify the redundant dependency. Li[26] dealt with the reliability analysis of a k-out-of-n: G system with redundant dependency and repairmen having multiple vacations. Wang, et al.[28] studied a multi-state Markov repairable system with redundant dependency and presented a two-dimensional vector to describe accurately the performance of the system. Hu, et al.[29] analyzed the steady-state availability of a repairable series-parallel system with redundant dependency and finite repair teams, and developed an optimal design problem for the system. Recently, Yu, et al.[30] investigated availability optimization problem for a repairable system composed of redundant and dependent components. Wang, et al.[31] proposed a new load sharing parallel system model with failure dependency under the assumptions that the repair time distributions of components are arbitrary and life times are exponential distributions whose failure rates vary with the number of operating components.

The redundant dependency is an important factor in many repairable redundant systems, so the influence of the special kind of failure dependencies on the availability of a repairable redundant system can not be neglected. Moreover, most of the works on availability analysis for repairable series-parallel system models with failure dependencies are confined to parallel subsystems with identical components and repairmen. This motives us to develop the repairable redundant parallel subsystem with redundant dependency and heterogeneous components. In this paper, a repairable series-parallel system with redundant dependency is considered, and each parallel redundant subsystem has two different types of components and repairmen. As we know, availability is not only an important index to measure repairable system reliability, but also the premise of the system optimization design. The focus of this work is to present the analysis method of steady-state probability for each subsystem and the steady-state availability of the entire system. A modified failure dependence function is introduced to determine the failure rate of the components in each subsystem. Markov theory and matrix analysis method are used to get the steady-state probability vector of each subsystem and the steady-state availability of the entire system.

The rest of the paper is organized as follows. In Section 2, we give the description of the system model. The steady-state probability distribution and redundant dependency for each parallel redundant subsystem are analyzed in Section 3. Section 4 gives the steady-state availability of the system. An illustrative example is presented and some numerical results are discussed in Section 5. Finally, conclusions and future work are given in Section 6.

2 Problem Description

The repairable system we consider here consists of n parallel redundant subsystems connected in series. Each parallel subsystem works if and only if at least one of its components work, and the entire system works if and only if all subsystems work. The subsystem i $(i = 1, 2, \cdots, n)$ has two different types of components (say type 1_i component and type 2_i component) and two corresponding types of repairmen (say type 1_i repairman and type 2_i repairman). The subsystem i consists of m_{1i} type 1_i components, m_{2i} type 2_i components, r_{1i} type 1_i repairmen and r_{2i} type 2_i repairmen. Each type of component is assumed to have its own dedicated repairman. Here, the failed type 1_i component is repaired by the type 1_i repairman, the failed type 2_i component is repaired by the type 2_i repairman.

Furthermore, other assumptions are given as follows:

1) The entire system and the component of each subsystem have two states: Perfect functioning and complete failure.

2) In each parallel redundant subsystem, the redundant dependency is considered. The failure rate of the operating component increases with the number of other failed components and the repair rate of the failed components is constant.

3) Whenever an operating component fails, it is immediately sent to a repair facility with two different types of repairmen where it is repaired in the order of their breakdowns.

4) Each repairman can repair only one failed component at one time. The failed component must wait for repair in the queue if the all repairmen are busy until a corresponding type of repairman is available.

5) Once a component is repaired, it is as good as new and goes into operating state.

6) The different subsystems are independent of each other.

3 Analysis of the Subsystem with Redundant Dependency

We first analyze the steady-state probability of the general parallel redundant subsystem with different components and repairmen.

3.1 Probabilistic Analysis of the Parallel Redundant Subsystem

The parallel redundant subsystem i consists of m_{1i} type 1_i components, m_{2i} type 2_i components, r_{1i} type 1_i repairmen and r_{2i} type 2_i repairmen. Let $K(t)$ and $J(t)$ denote the numbers of failed type 1_i components and failed type 2_i components in the subsystem i (waiting and in repair) at time t, respectively. We can describe the states of the subsystem i by the pairs $\{(K(t), J(t)), t \geq 0\}$. Let $P^i_{k,j}(t) = P\{K(t) = k, J(t) = j : k = 0, 1, \cdots, m_{1i}, j = 0, 1, \cdots, m_{2i}\}$ be the probability of exactly k failed type 1_i components and j failed type 2_i components in the subsystem i at time t. Assume steady-state conditions, and let $P^i_{k,j} = \lim_{t\to\infty} P^i_{k,j}(t)$, where $P^i_{k,j}$ denotes the steady-state probability that there are k failed type 1_i components and j failed type 2_i components in the subsystem i. The transition rate from k failed type 1_i components to $k + 1$ failed type 1_i components when there are j failed type 2_i components is presented as $\lambda^{1i}_{(k,k+1);j}$, and $\mu^{1i}_{(k,k-1)}$ denotes the transition rate from k failed type 1_i components to $k - 1$

failed type 1_i components; the transition rate from j failed type 2_i components to $j+1$ failed type 2_i components when there are k failed type 1_i components is presented as $\lambda^{2i}_{k;(j,j+1)}$, and $\mu^{2i}_{(j,j-1)}$ denotes the transition rate from j failed type 2_i components to $j-1$ failed type 2_i components.

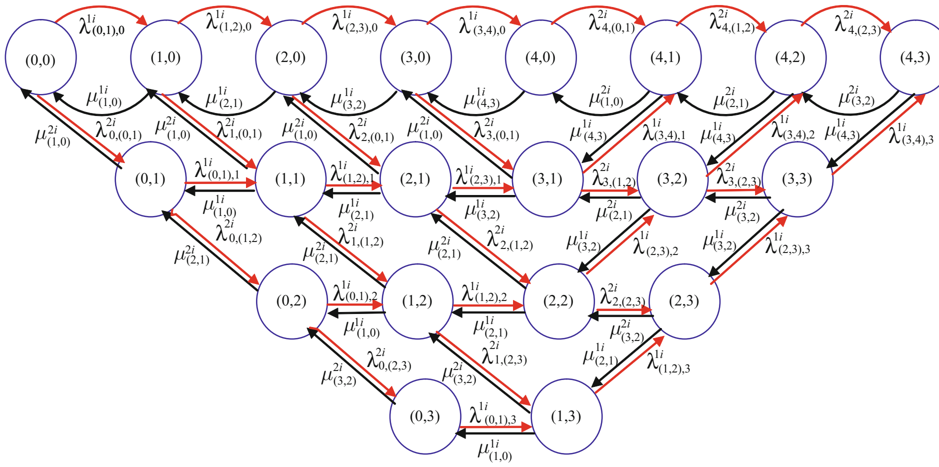

Figure 1 State transition diagram of the subsystem i with two types of components and repairmen (for $m_{1i}=4, r_{1i}=3, m_{2i}=3$ and $r_{2i}=2$)

Markov state transition diagram is helpful in analyzing the steady-state probability distribution of the subsystem i. In the subsystem i, the Markov model is used to analyze the subsystem state transition process. The state transition diagram of the parallel redundant subsystem i with two types of components and repairmen (for $m_{1i}=4$, $r_{1i}=3$, $m_{2i}=3$ and $r_{2i}=2$) is shown in Figure 1. Referring to Figure 1, we have the following steady-state probability equations:

$$\left(\lambda^{1i}_{(0,1);0}+\lambda^{2i}_{0;(0,1)}\right)P^i_{0,0}=\mu^{1i}_{(1,0)}P^i_{1,0}+\mu^{2i}_{(1,0)}P^i_{0,1},\quad k=0, j=0, \tag{1}$$

$$\left(\lambda^{2i}_{0;(j,j+1)}+\lambda^{1i}_{(0,1);j}+\mu^{2i}_{(j,j-1)}\right)P^i_{0,j}=\lambda^{2i}_{0;(j-1,j)}P^i_{0,j-1}+\mu^{1i}_{(1,0)}P^i_{1,j}$$
$$+\mu^{2i}_{(j+1,j)}P^i_{0,j+1},\quad k=0,\quad j=1,2,\cdots,m_{2i}-1, \tag{2}$$

$$\left(\lambda^{1i}_{(0,1);m_{2i}}+\mu^{2i}_{(m_{2i},m_{2i}-1)}\right)P^i_{0,m_{2i}}=\lambda^{2i}_{0;(m_{2i}-1,m_{2i})}P^i_{0,m_{2i}-1}+\mu^{1i}_{(1,0)}P^i_{1,m_{2i}},$$
$$k=0,\quad j=m_{2i}, \tag{3}$$

$$\left(\lambda^{1i}_{(k,k+1);0}+\lambda^{2i}_{k;(0,1)}+\mu^{1i}_{(k,k-1)}\right)P^i_{k,0}=\lambda^{1i}_{(k-1,k);0}P^i_{k-1,0}+\mu^{2i}_{(1,0)}P^i_{k,1}$$
$$+\mu^{1i}_{(k+1,k)}P^i_{k+1,0},\quad k=1,2,\cdots,m_{1i}-1,\ j=0, \tag{4}$$

$$\left(\lambda^{2i}_{m_{1i};(0,1)}+\mu^{1i}_{(m_{1i},m_{1i}-1)}\right)P^i_{m_{1i},0}=\lambda^{1i}_{(m_{1i}-1,m_{1i});0}P^i_{m_{1i}-1,0}+\mu^{2i}_{(1,0)}P^i_{m_{1i},1},$$
$$k=m_{1i},\quad j=0, \tag{5}$$

$$\begin{aligned}
&\left(\lambda^{1i}_{(k,k+1);j} + \lambda^{2i}_{k;(j,j+1)} + \mu^{1i}_{(k,k-1)} + \mu^{2i}_{(j,j-1)}\right) P^i_{k,j} \\
&= \lambda^{1i}_{(k-1,k);j} P^i_{k-1,j} + \lambda^{2i}_{k;(j-1,j)} P^i_{k,j-1} + \mu^{1i}_{(k+1,k)} P^i_{k+1,j} + \mu^{2i}_{(j+1,j)} P^i_{k,j+1}, \\
&k = 1, 2, \cdots, m_{1i} - 1, \quad j = 1, 2, \cdots, m_{2i} - 1, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
&\left(\lambda^{1i}_{(k,k+1);m_{2i}} + \mu^{1i}_{(k,k-1)} + \mu^{2i}_{(m_{2i},m_{2i}-1)}\right) P^i_{k,m_{2i}} \\
&= \lambda^{1i}_{(k-1,k);m_{2i}} P^i_{k-1,m_{2i}} + \lambda^{2i}_{k;(m_{2i}-1,m_{2i})} P^i_{k,m_{2i}-1} + \mu^{1i}_{(k+1,k)} P^i_{k+1,m_{2i}}, \\
&k = 1, 2, \cdots, m_{1i} - 1, \quad j = m_{2i}, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
&\left(\lambda^{2i}_{m_{1i};(j,j+1)} + \mu^{2i}_{(j,j-1)} + \mu^{1i}_{(m_{1i},m_{1i}-1)}\right) P^i_{m_{1i},j} \\
&= \lambda^{2i}_{m_{1i};(j-1,j)} P^i_{m_{1i},j-1} + \lambda^{1i}_{(m_{1i}-1,m_{1i});j} P^i_{m_{1i}-1,j} + \mu^{2i}_{(j+1,j)} P^i_{m_{1i},j+1}, \\
&k = m_{1i}, \quad j = 1, 2, \cdots, m_{2i} - 1, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
&\left(\mu^{1i}_{(m_{1i},m_{1i}-1)} + \mu^{2i}_{(m_{2i},m_{2i}-1)}\right) P^i_{m_{1i},m_{2i}} \\
&= \lambda^{1i}_{(m_{1i}-1,m_{1i});m_{2i}} P^i_{m_{1i}-1,m_{2i}} + \lambda^{2i}_{m_{1i};(m_{2i}-1,m_{2i})} P^i_{m_{1i},m_{2i}-1}, \quad k = m_{1i}, \quad j = m_{2i}, \qquad (9)
\end{aligned}$$

and the following normalizing equation

$$\sum_{k=0}^{m_{1i}} \sum_{j=0}^{m_{2i}} P^i_{k,j} = 1. \qquad (10)$$

Based on Equations (1)–(9), the transition rate matrix Q^i of the subsystem i is obtained as the following block-tridiagonal form

$$Q^i = \begin{pmatrix} A_0 & B_0 & & & & \\ C_1 & A_1 & B_1 & & & \\ & C_2 & A_2 & B_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & C_{m_{1i}-1} & A_{m_{1i}-1} & B_{m_{1i}-1} \\ & & & & C_{m_{1i}} & A_{m_{1i}} \end{pmatrix}. \qquad (11)$$

The matrix Q^i is a square matrix of order $(m_{1i} + 1) \cdot (m_{2i} + 1)$, and $(A_0 + B_0)\, e = (C_{m_{1i}} + A_{m_{1i}})\, e = (C_k + A_k + B_k)\, e = 0$, $k = 1, 2, \cdots, m_{1i} - 1$, e is an $m_{2i} + 1$ dimensions column vector with each component equal to one. Each block of the matrix Q^i is given in the following

$$A_0 = \left(\begin{matrix} \lambda^{1i}_{(0,1);0} + \lambda^{2i}_{0;(0,1)} & -\lambda^{2i}_{0;(0,1)} & 0 & \cdots \\ -\mu^{2i}_{(1,0)} & \lambda^{1i}_{(0,1);1} + \lambda^{2i}_{0;(1,2)} + \mu^{2i}_{(1,0)} & -\lambda^{2i}_{0;(1,2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{matrix} \right.$$

$$
\left.\begin{matrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
-\mu^{2i}_{(m_{2i}-1,m_{2i}-2)} & \lambda^{1i}_{(0,1);m_{2i}-1} + \lambda^{2i}_{0;(m_{2i}-1,m_{2i})} + \mu^{2i}_{(m_{2i}-1,m_{2i}-2)} & -\lambda^{2i}_{0;(m_{2i}-1,m_{2i})} \\
0 & -\mu^{2i}_{(m_{2i},m_{2i}-1)} & \lambda^{1i}_{(0,1);m_{2i}} + \mu^{2i}_{(m_{2i},m_{2i}-1)}
\end{matrix}\right),
$$

$$
A_k = \left(\begin{matrix}
\lambda^{1i}_{(k,k+1);0} + \lambda^{2i}_{k;(0,1)} + \mu^{1i}_{(k,k-1)} & -\lambda^{2i}_{k;(0,1)} & 0 & \cdots \\
-\mu^{2i}_{(1,0)} & \lambda^{1i}_{(k,k+1);1} + \lambda^{2i}_{k;(1,2)} + \mu^{1i}_{(k,k-1)} + \mu^{2i}_{(1,0)} & -\lambda^{2i}_{k;(1,2)} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
-\mu^{2i}_{(m_{2i}-1,m_{2i}-2)} & \lambda^{1i}_{(k,k+1);m_{2i}-1} + \lambda^{2i}_{k;(m_{2i}-1,m_{2i})} + \mu^{1i}_{(k,k-1)} + \mu^{2i}_{(m_{2i}-1,m_{2i}-2)} & -\lambda^{2i}_{k;(m_{2i}-1,m_{2i})} \\
0 & -\mu^{2i}_{(m_{2i},m_{2i}-1)} & \lambda^{1i}_{(k,k+1);m_{2i}} + \mu^{1i}_{(k,k-1)} + \mu^{2i}_{(m_{2i},m_{2i}-1)}
\end{matrix}\right),
$$

$$k = 1, 2, \cdots, m_{1i} - 1,$$

$$
A_{m_{1i}} = \left(\begin{matrix}
\lambda^{2i}_{m_{1i};(0,1)} + \mu^{1i}_{(m_{1i},m_{1i}-1)} & -\lambda^{2i}_{m_{1i};(0,1)} & 0 & \cdots \\
-\mu^{2i}_{(1,0)} & \lambda^{2i}_{m_{1i};(1,2)} + \mu^{1i}_{(m_{1i},m_{1i}-1)} + \mu^{2i}_{(1,0)} & -\lambda^{2i}_{m_{1i};(1,2)} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
-\mu^{2i}_{(m_{2i}-1,m_{2i}-2)} & \lambda^{2i}_{m_{1i};(m_{2i}-1,m_{2i})} + \mu^{1i}_{(m_{1i},m_{1i}-1)} + \mu^{2i}_{(m_{2i}-1,m_{2i}-2)} & -\lambda^{2i}_{m_{1i};(m_{2i}-1,m_{2i})} \\
0 & -\mu^{2i}_{(m_{2i},m_{2i}-1)} & \mu^{1i}_{(m_{1i},m_{1i}-1)} + \mu^{2i}_{(m_{2i},m_{2i}-1)}
\end{matrix}\right),
$$

$$
B_k = \begin{pmatrix}
-\lambda^{1i}_{(k,k+1);0} & & & & & \\
& -\lambda^{1i}_{(k,k+1);1} & & & & \\
& & -\lambda^{1i}_{(k,k+1);2} & & & \\
& & & \ddots & & \\
& & & & -\lambda^{1i}_{(k,k+1);m_{2i}-1} & \\
& & & & & -\lambda^{1i}_{(k,k+1);m_{2i}}
\end{pmatrix},
$$

$$k = 0, 1, \cdots, m_{1i} - 1,$$

$$C_k = \begin{pmatrix} -\mu^{1i}_{(k,k-1)} & & & & & \\ & -\mu^{1i}_{(k,k-1)} & & & & \\ & & -\mu^{1i}_{(k,k-1)} & & & \\ & & & \ddots & & \\ & & & & -\mu^{1i}_{(k,k-1)} & \\ & & & & & -\mu^{1i}_{(k,k-1)} \end{pmatrix},$$

$$k = 1, 2, \cdots, m_{1i},$$

where A_k, B_k and C_k are square matrices of order $m_{2i} + 1$.

Let P^i denote steady-state probability vector corresponding to the matrix Q^i, and it is partitioned as $P^i = \left(P^i_0, P^i_1, \cdots, P^i_{m_{1i}}\right)$, where $P^i_k = (P^i_{k,0}, P^i_{k,1}, \cdots, P^i_{k,m_{2i}})$, $k = 0, 1, \cdots, m_{1i}$. According to the transition rate matrix Q^i and Equations (1)–(9), the steady-state probability equations of the subsystem i can be written in matrix form as $P^iQ^i = 0$, that is,

$$\left(P^i_0, P^i_1, \cdots, P^i_{m_{1i}}\right) \begin{pmatrix} A_0 & B_0 & & & & \\ C_1 & A_1 & B_1 & & & \\ & C_2 & A_2 & B_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & C_{m_{1i-1}} & A_{m_{1i-1}} & B_{m_{1i-1}} \\ & & & & C_{m_{1i}} & A_{m_{1i}} \end{pmatrix} = 0, \tag{12}$$

then we have

$$P^i_0 A_0 + P^i_1 C_1 = 0, \tag{13}$$

$$P^i_{k-1} B_{k-1} + P^i_k A_k + P^i_{k+1} C_{k+1} = 0, \quad k = 1, 2, \cdots, m_{1i} - 1, \tag{14}$$

$$P^i_{m_{1i}-1} B_{m_{1i}-1} + P^i_{m_{1i}} A_{m_{1i}} = 0. \tag{15}$$

The normalizing equation can be written as

$$\sum_{k=0}^{m_{1i}} P^i_k e = 1, \tag{16}$$

where e represents an $m_{2i} + 1$ dimensions column vector with all components equal to one.

Based on the method in PÉRez-OcÓN and Montoro-Cazorla[32,33] and Ke and Wang[34], let $D_1, D_2, \cdots, D_{m_{1i}-1}, D_{m_{1i}}$ be matrices satisfying the following conditions

$$B_{k-1} + D_k(A_k + D_{k+1}C_{k+1}) = 0, \quad k = 1, 2, \cdots, m_{1i-1}, \tag{17}$$

$$B_{m_{1i-1}} + D_{m_{1i}} A_{m_{1i}} = 0. \tag{18}$$

Assume that the matrices $A_{m_{1i}}$ and $A_k + D_{k+1}C_{k+1}$, $k = 1, 2, \cdots, m_{1i} - 1$, are non-singular. According to (15), we have

$$P^i_{m_{1i}} = P^i_{m_{1i}-1} D_{m_{1i}}, \tag{19}$$

where $D_{m_{1i}} = -B_{m_{1i}-1}A_{m_{1i}}^{-1}$. According to (14) ($k = m_{1i} - 1$) and (19), we have

$$P_{m_{1i}-1}^i = P_{m_{1i}-2}^i D_{m_{1i}-1}, \tag{20}$$

where $D_{m_{1i}-1} = -B_{m_{1i}-2}(A_{m_{1i}-1} + D_{m_{1i}}C_{m_{1i}})$.

Similarly, we can obtain

$$P_k^i = P_{k-1}^i D_k, \quad k = 1, 2, \cdots, m_{1i} - 1, \tag{21}$$

where $D_k = -B_{k-1}\left(A_k + D_{k+1}C_{k+1}\right)^{-1}$, $k = 1, 2, \cdots, m_{1i} - 1$.

After substitutions, (13)–(15) can be expressed as

$$\begin{cases} P_0^i A_0 + P_1^i C_1 = 0, \\ P_k^i = P_{k-1}^i D_k, \quad k = 1, 2, \cdots, m_{1i}, \end{cases} \tag{22}$$

where $D_k = -B_{k-1}\left(A_k + D_{k+1}C_{k+1}\right)^{-1}$, $k = 1, 2, \cdots, m_{1i} - 1$, $D_{m_{1i}} = -B_{m_{1i}-1}A_{m_{1i}}^{-1}$. Therefore, we have

$$P_k^i = P_0^i D_1 D_2 \cdots D_k = P_0^i \prod_{k_1=1}^{k} D_{k_1}, \quad k = 1, 2, \cdots, m_{1i}, \tag{23}$$

and P_0^i can be determined by

$$\begin{cases} P_0^i\left(A_0 + D_1 C_1\right) = 0, \\ \sum\limits_{k=0}^{m_{1i}} P_k^i e = P_0^i \left(I + \sum\limits_{k=1}^{m_{1i}} \prod\limits_{k_1=1}^{k} D_{k_1}\right) e = 1, \end{cases} \tag{24}$$

where I is an identity matrix of order $m_{2i} + 1$, and e is an $m_{2i} + 1$ dimensions column vector with each component equal to one.

Based on the analysis of the above, the steps to calculate the steady-state probability vector of the parallel redundant subsystem i is given as

1) Calculate the matrices $D_{m_{1i}} = -B_{m_{1i}-1}A_{m_{1i}}^{-1}$ and $D_k = -B_{k-1}(A_k + D_{k+1}C_{k+1})^{-1}$, $k = m_{1i} - 1, m_{1i} - 2, \cdots, 1$.

2) Calculate P_0^i according to $P_0^i\left(A_0 + D_1 C_1\right) = 0$ and

$$\sum_{k=0}^{m_{1i}} P_k^i e = P_0^i \left(I + \sum_{k=1}^{m_{1i}} \prod_{k_1=1}^{k} D_{k_1}\right) e = 1.$$

3) Calculate P_k^i according to $P_k^i = P_0^i \prod_{k_1=1}^{k} D_{k_1}$, $k = 1, 2, \cdots, m_{1i}$.

A computer program can be developed to obtain $P_k^i = \left(P_{k,0}^i, P_{k,1}^i, \cdots, P_{k,m_{2i}}^i\right)$ and $P_{k,j}^i$, $k = 0, 1, \cdots, m_{1i}$, $j = 0, 1, \cdots, m_{2i}$.

3.2 Redundant Dependency Analysis

In the real world, a wide variety of dependencies exist among the failure behavior of some engineering systems. The types of failure dependencies among systems are different and depend on the functional and constructional configuration of the systems. Yu, et al.[27] showed that the failure dependency of a system is called redundant dependency if any of the components can be viewed as a redundancy of one another component. In this paper, the redundant dependency is considered for each parallel redundant subsystem.

Based on the dependence function in [27] and the parallel redundant subsystem i with different components, we introduce a modified dependence function $g((m_{1i}-k),(m_{2i}-j))$ to quantify the redundant dependency, where k and j denote the numbers of the failed types 1_i and 2_i components, respectively, $k=0,1,\cdots,m_{1i}$, $j=0,1,\cdots,m_{2i}$. It is assumed that the failure rate of components in the subsystem i depends on the dependence function $g((m_{1i}-k),(m_{2i}-j))$ and their inherent failure rate (failure rate at failure independency). The failure rates of the two types of components in the subsystem i can be expressed as

$$\frac{\lambda_{1i}}{g((m_{1i}-k),(m_{2i}-j))}, \quad k=0,1,\cdots,m_{1i}, \tag{25}$$

$$\frac{\lambda_{2i}}{g((m_{1i}-k),(m_{2i}-j))}, \quad j=0,1,\cdots,m_{2i}, \tag{26}$$

where $k+j\leq m_{1i}+m_{2i}-2$, λ_{1i} is the inherent failure rate of the type 1_i component, λ_{2i} is the inherent failure rate of the type 2_i component, and $g(1,0)=g(0,1)\equiv 1$. For the subsystem i with redundant dependency, we can obtain the failure transition rates as

$$\lambda^{1i}_{(k,k+1);j}=\frac{(m_{1i}-k)\lambda_{1i}}{g((m_{1i}-k),(m_{2i}-j))}, \quad k=0,1,\cdots,m_{1i}-1, \quad j=0,1,\cdots,m_{2i}, \tag{27}$$

$$\lambda^{2i}_{k;(j,j+1)}=\frac{(m_{2i}-j)\lambda_{2i}}{g((m_{1i}-k),(m_{2i}-j))}, \quad j=0,1,\cdots,m_{2i}-1, \quad k=0,1,\cdots,m_{1i}. \tag{28}$$

In this study, we assume that the repair of components is independent. Let μ_{1i} and μ_{2i} be the repair rates of the types 1_i and 2_i components, respectively. We can obtain the repair transition rate as

$$\mu^{1i}_{(k,k-1)}=\begin{cases} k\mu_{1i} & k\leq r_{1i}, \\ r_{1i}\mu_{1i} & k>r_{1i}, \end{cases} \quad k=1,2,\cdots,m_{1i}, \tag{29}$$

$$\mu^{2i}_{(j,j-1)}=\begin{cases} j\mu_{2i} & j\leq r_{2i}, \\ r_{2i}\mu_{2i} & j>r_{2i}, \end{cases} \quad j=1,2,\cdots,m_{2i}. \tag{30}$$

In the parallel redundant subsystem i, the failure rate of the components with redundant dependency is less than that of the components with failure independency, i.e., $g((m_{1i}-k),(m_{2i}-j)) \geq 1$. The dependence function $g((m_{1i}-k),(m_{2i}-j))$ is used to describe the strength of the redundant dependency. Obviously, if k and j are fixed, the bigger the value of $g((m_{1i}-k),(m_{2i}-j))$, the stronger the redundant dependency is. According to (25) and (26), if $g((m_{1i}-k),(m_{2i}-j))$

$\equiv 1$, the failure rates of the types 1_i and 2_i components are λ_{1i} and λ_{2i}, respectively. It shows no dependency in the subsystem i. When $g\left((m_{1i}-k),(m_{2i}-j)\right)>1$, the redundant dependency can be classified into weak dependence, moderate dependence and strong dependence. As an example, the dependence function $g\left((m_{1i}-k),(m_{2i}-j)\right)$ is uniformed as

$$g\left((m_{1i}-k),(m_{2i}-j)\right)=\left((m_{1i}-k)+(m_{2i}-j)\right)^{\alpha_i}, \qquad \alpha_i \geq 0, \tag{31}$$

where the redundant dependency is called weak dependence, moderate dependence and strong dependence when $0<\alpha_i<1$, $1\leq\alpha_i\leq 2$ and $\alpha_i>2$, respectively. And the failure transition rates of the components in the subsystem i with redundant dependency can be written as

$$\lambda^{1i}_{(k,k+1);j}=\frac{(m_{1i}-k)\lambda_{1i}}{\left((m_{1i}-k)+(m_{2i}-j)\right)^{\alpha_i}}, \quad k=0,1,\cdots,m_{1i}-1, \quad j=0,1,\cdots,m_{2i}, \tag{32}$$

$$\lambda^{2i}_{k;(j,j+1)}=\frac{(m_{2i}-j)\lambda_{2i}}{\left((m_{1i}-k)+(m_{2i}-j)\right)^{\alpha_i}}, \quad j=0,1,\cdots,m_{2i}-1, \quad k=0,1,\cdots,m_{1i}. \tag{33}$$

4 Steady-State Availability of the Repairable Series-Parallel System

A repairable series-parallel system can be represented by several parallel redundant subsystems linked in series. The steady-state availability of the repairable system denotes the expected long run behavior of the system. Based on the system structure considered here, the steady-state availability of the system can be determined by

$$A=\prod_{i=1}^{n}A_i, \tag{34}$$

where A_i is the steady-state availability of the parallel redundant subsystem i. By using steady-state probability vector $P^i=\left(P^i_0,P^i_1,\cdots,P^i_{m_{1i}}\right)$, $P^i_k=\left(P^i_{k,0},P^i_{k,1},\cdots,P^i_{k,m_{2i}}\right)$, $k=0,1,\cdots,m_{1i}$, the steady-state availability of the subsystem i can be obtained as

$$A_i=\sum_{k=0}^{m_{1i}-1}P^i_k e+\sum_{j=0}^{m_{2i}-1}P^i_{m_{1i},j}=1-P^i_{m_{1i},m_{2i}}. \tag{35}$$

Using (35) through (34), we have

$$A=\prod_{i=1}^{n}\left(1-P^i_{m_{1i},m_{2i}}\right). \tag{36}$$

It is assumed that the dependence function $g\left((m_{1i}-k),(m_{2i}-j)\right)$ is equal to $\left((m_{1i}-k)+(m_{2i}-j)\right)^{\alpha_i}(\alpha_i\geq 0)$ in the parallel redundant subsystem i, and then A_i is a function in terms of α_i, that is, $A_i=A_i\ (\alpha_i)$, $i=1,2,\cdots,n$. The steady-state availability of the system with redundant dependency can be written as

$$A\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=\prod_{i=1}^{n}A_i(\alpha_i), \tag{37}$$

where α_i is called dependence strength parameter.

5 Numerical Example

Some numerical results of the steady-state availability of the repairable series-parallel system for different redundant dependencies are presented in this section. We consider a repairable series-parallel system with two parallel redundant subsystems. The subsystem 1 consists of $m_{11} = 2$ type 1_1 components, $m_{21} = 1$ type 2_1 component, $r_{11} = 1$ type 1_1 repairman and $r_{21} = 1$ type 2_1 repairman. The subsystem 2 consists of $m_{12} = 3$ type 1_2 components, $m_{22} = 2$ type 2_2 components, $r_{12} = 2$ type 2_1 repairmen and $r_{22} = 1$ type 2_2 repairman. The inherent failure rates λ_{1i}, λ_{2i} and the repair rates μ_{1i}, μ_{2i}, $i = 1, 2$, are presented in Table 1. The dependence function for the subsystem i is $((m_{1i} - k) + (m_{2i} - j))^{\alpha_i}$, $\alpha_i \geq 0$, $i = 1, 2$.

Table 1 Inherent failure rate and repair rate for the subsystem i

Subsystem i	λ_{1i}	λ_{2i}	μ_{1i}	μ_{2i}
1	0.03	0.04	0.05	0.08
2	0.01	0.02	0.03	0.04

According to Section 3, the transition rate matrix Q^1 of the parallel redundant subsystem 1 is obtained as the following form

$$Q^1 = \begin{pmatrix} A_0 & B_0 & 0 \\ C_1 & A_1 & B_1 \\ 0 & C_2 & A_2 \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} 0.06/3^{\alpha_1} + 0.04/3^{\alpha_1} & -0.04/3^{\alpha_1} \\ -0.08 & 0.06/2^{\alpha_1} + 0.08 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -0.06/3^{\alpha_1} & 0 \\ 0 & -0.06/2^{\alpha_1} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -0.03/2^{\alpha_1} & 0 \\ 0 & -0.03/1^{\alpha_1} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.03/2^{\alpha_1} + 0.04/2^{\alpha_1} + 0.05 & -0.04/2^{\alpha_1} \\ -0.08 & 0.03/1^{\alpha_1} + 0.05 + 0.08 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.04/1^{\alpha_1} + 0.05 & -0.04/1^{\alpha_1} \\ -0.08 & 0.05 + 0.08 \end{pmatrix}.$$

Let $\alpha_1 = 1$, the steady-state probability vector of the subsystem 1 is obtained as

$$P_0^1 = \left(P_{0,0}^1, P_{0,1}^1\right) = (0.5415,\ 0.0903), \quad P_1^1 = \left(P_{1,0}^1, P_{1,1}^1\right) = (0.2166,\ 0.0542),$$
$$P_2^1 = \left(P_{2,0}^1, P_{2,1}^1\right) = (0.0650,\ 0.0324).$$

The steady-state availability of the subsystem 1 is obtained

$$A_1(\alpha_1) = A_1(1) = 1 - P_{2,1}^1 = 0.9676.$$

Similarly, we can obtain the steady-state availability of the subsystem 2. In Table 2, we present the values of the entire system steady-state availability $A(\alpha_1, \alpha_2)$ for different values of α_1 and α_2.

Table 2 Steady-state availability $A(\alpha_1, \alpha_2)$ for different values of α_1 and α_2

α_1	α_2	$A(\alpha_1, \alpha_2)$	α_1	α_2	$A(\alpha_1, \alpha_2)$
0	0	0.9135	0.5	0	0.9275
	0.5	0.9168		0.5	0.9308
	1.0	0.9176		1.0	0.9316
	1.5	0.9178		1.5	0.9318
1.0	0	0.9631	1.5	0	0.9785
	0.5	0.9666		0.5	0.9821
	1.0	0.9674		1.0	0.9829
	1.5	0.9675		1.5	0.9830

Some interesting results can be observed from Table 2. The system steady-state availability is 0.9135 when $\alpha_1 = \alpha_2 = 0$, that is, the system with independent components has the lowest availability. The system steady-state availability increases as α_1 and α_2 increase. The increase with increasing α_1 is rapid when α_2 is a fixed value, and the increase with increasing α_2 is slow when α_1 is a fixed value. This means that the effect of α_1 on the system steady-state availability is more than that of α_2. The results show that the strongest redundant dependency class obtains the highest steady-state availability for the system.

6 Conclusions and Future Work

A repairable series-parallel system model with redundant dependency and different components is studied in this paper. Each parallel redundant subsystem is composed of two different types of components and the corresponding types of repairmen. A modified failure dependence function is proposed to quantify the redundant dependency and analyze the failure rate of the components in each subsystem. By using Markov model theory and matrix analysis method, the steady-state probability vector of each subsystem and the steady-state availability of the entire system are obtained. Some numerical results of the steady-state availability of the system for different redundant dependencies are presented to illustrate the effect of redundant dependency class on the system availability. This study reveals that the redundant dependency is an essential and effective option to improve the steady-state availability of the repairable series-parallel system.

System optimization design plays a key role in repairable system design and have been effectively applied to enhance system performance in many real world problems. Therefore, in the design phase of repairable series-parallel system, system designers typically try to determine the optimal numbers of components and repair teams for each subsystem, in order to minimize the system cost while satisfying the system availability constraint or maximize the system availability while satisfying the system cost constraint. In this paper, the research focus is mainly on system availability analysis. Further work will mainly concern system optimization design for repairable series-parallel system with redundant dependency and different types of components and teams. Moreover, some effective optimization algorithms can be developed to solve the optimization model in further research.

References

[1] Wang K H and Chen Y J, Comparative analysis of availability between three systems with general repair times, reboot delay and switching failures, *Applied Mathematics and Computation*, 2009, **215**(1): 384–394.

[2] Wang Y and Li L, Heterogeneous redundancy allocation for series-parallel multi-state systems using hybrid particle swarm optimization and local search, *IEEE Transactions On Systems, Man, and Cybernetics — Part A: Systems and Humans*, 2012, **42**(2): 464–474.

[3] Zhang X, Reliability analysis of a cold standby repairable system with repairman extra work, *Journal of Systems Science and Complexity*, 2015, **28**(5): 1015–1032.

[4] Kim H and Kim P, Reliability models for a nonrepairable system with heterogeneous components having a phase-type time-to-failure distribution, *Reliability Engineering and System Safety*, 2017, **159**: 37–46.

[5] Sharma V K, Agarwal M, and Sen K, Reliability evaluation and optimal design in heterogeneous multi-state series-parallel systems, *Information Sciences*, 2011, **181**(2): 362–378.

[6] Chaaban W, Schwarz M, and Börcsök J, Cost optimization and high available heterogeneous series-parallel redundant system design using genetic algorithms, Eds. by Mastorakis N, Bulucea A, Tsekouras G, *Computational Problems in Science and Engineering*, Lecture Notes in Electrical Engineering, Springer, Cham, 2015, **343**: 283–305.

[7] Mo Y C, Liu Y, and Cui L R, Performability analysis of multi-state series-parallel systems with heterogeneous components, *Reliability Engineering and System Safety*, 2018, **171**: 48–56.

[8] Chowdhury S and Kundu A, Stochastic comparison of parallel systems with log-Lindley distributed components, *Operations Research Letters*, 2017, **45**(3): 199–205.

[9] Fricks R M and Trivedi K S, Modeling failure dependencies in reliability analysis using stochastic petri nets, *Proceeding of European Simulation Multiconference*, Turkey, Istanbul, 1997.

[10] Pecht M, *Product Reliability, Maintainability, and Supportability Handbook*, CRC Press, Florida, 2009.

[11] Levitin G, Incorporating common-cause failures into nonrepairable multistate series-parallel system analysis, *IEEE Transactions on Reliability*, 2001, **50**(4): 380–388.

[12] Ramirez-Marquez J E and Coit D W, Optimization of system reliability in the presence of common cause failures, *Reliability Engineering and System Safety*, 2007, **92**(10): 1421–1434.

[13] Li C Y, Chen X, Yi X S, et al., Heterogeneous redundancy optimization for multi-state series-parallel systems subject to common cause failures, *Reliability Engineering and System Safety*, 2010, **95**(3): 202–207.

[14] Boddu P and Xing L, Optimal design of heterogeneous series-parallel systems with common-cause failures, *International Journal of Performability Engineering*, 2011, **7**(5): 455–466.

[15] Yamashiro M, Satoh M, and Yuasa Y, Parallel-series and series-parallel redundant systems consisting of units having two kinds of failure rates, *Microelectronics Reliability*, 1992, **32**(5): 611–613.

[16] Pham H, Optimal system-profit design of series-parallel systems with multiple failure modes, *Reliability Engineering and System Safety*, 1992, **37**(2): 151–155.

[17] Levitin G, Optimal series-parallel topology of multi-state system with two failure modes, *Reliability Engineering and System Safety*, 2002, **77**(1): 93–107.

[18] Cui L R, Li H J, and Li J L, Markov repairable systems with history-dependent up and down states, *Communications in Statistics Stochastic Models*, 2007, **23**(4): 665–681.

[19] Wang L Y and Cui L R, Aggregated semi-Markov repairable systems with history-dependent up and down states, *Mathematical and Computer Modelling*, 2011, **53**(5–6): 883–895.

[20] Xing L D, Shrestha A, and Dai Y S, Exact combinatorial reliability analysis of dynamic systems with sequence-dependent failures, *Reliability Engineering and System Safety*, 2011, **96**(10): 1375–1385.

[21] Barros A, BÉRenguer C, and Grall A, Optimization of replacement times using imperfect monitoring information, *IEEE Transactions on Reliability*, 2003, **52**(4): 523–533.

[22] Blokus A, Reliability analysis of large systems with dependent components, *Proceedings of the Europpean Safety and Reliability Conference*, Tri City: Poland, 2005, 205–212.

[23] Levitin G and Amari S V, Optimal load distribution in series-parallel systems, *Reliability Engineering and System Safety*, 2009, **94**(2): 254–260.

[24] Nourelfath M and Yalaoui F, Integrated load distribution and production planning in series-parallel multi-state systems with failure rate depending on load, *Reliability Engineering and System Safety*, 2012, **106**(4): 138–145.

[25] Blokus A, Analysis of component failures dependency influence on system lifetime, *Journal of Polish Safety and Reliability Association: Summer Safety and Reliability Seminars*, 2007, **1**: 17–28.

[26] Li Y, Reliability analysis for a k-out-of-n: G system with redundant dependency and repairmen having multiple vacations, *Applied Mathematics and Computation*, 2012, **218**(24): 11959–11969.

[27] Yu H Y, Chu C B, Châtelet E, et al., Reliability optimization of a redundant system with failure dependencies, *Reliability Engineering and System Safety*, 2007, **92**(12): 1627–1634.

[28] Wang L Y, Jia X, and Zhang J, Reliability evaluation for multi-state Markov repairable systems with redundant dependencies, *Quality Technology and Quantitative Management*, 2013, **10**(3): 277–289.

[29] Hu L M, Yue D Q, and Li J D, Availability analysis and design optimization for a repairable series-parallel system with failure dependencies, *International Journal of Innovative Computing, Information and Control*, 2012, **8**(10): 6693–6705.

[30] Yu H Y, Chu C B, and Châtelet E, Availability optimization of a redundant system through dependency modeling, *Applied Mathematical Modelling*, 2014, **38**(19–20): 4574–4585.

[31] Wang L Y, Zhang J, Chen W, et al., Reliability evaluation of a load-sharing parallel system with failure dependence, *Communications in Statistics-Simulation and Computation*, 2016, **45**(9): 3094–3113.

[32] PÉRez-OcÓN R and Montoro-Cazorla D, A multiple system governed by a quasi-birth- and-death process, *Reliability Engineering and System Safety*, 2004, **84**(2): 187–196.

[33] PÉRez-OcÓN R and Montoro-Cazorla D, A multiple warm standby system with operational and repair times following phase-type distributions, *European Journal of Operational Research*, 2006, **169**(1): 178–188.

[34] Ke J C and Wang K H, Vacation policies for machine repair problem with two type spares, *Applied Mathematical Modeling*, 2007, **31**(5): 880–894.