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The Stochastic Maximum Principle for a Jump-Diffusion Mean-Field Model Involving Impulse Controls and Applications in Finance*

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Abstract This paper establishes a stochastic maximum principle for a stochastic control of mean-field model which is governed by a Lévy process involving continuous and impulse control. The authors also show the existence and uniqueness of the solution to a jump-diffusion mean-field stochastic differential equation involving impulse control. As for its application, a mean-variance portfolio selection problem has been solved.

Keywords Impulse control, jump-diffusion, Markowitz's mean-variance model, stochastic maximum principle.

1 Introduction

In recent years, stochastic impulse control has received considerable research attention due to its wide application in different areas. For example, it has been used to solve the optimal portfolio problem with transaction costs, see Oksendal and Sulem[1] and for optimal control of exchange rates between different currencies, see Cadenillas and Zapatero[2]. A forward-backward system involving impulse control has been studied by Wu and Zhang[3]. The study regarding coupled forward-backward stochastic systems with delay and noisy memory were investigated by Wu, et al.[4]. Optimal impulse control of a mean-reverting inventory with quadratic costs and the mean-field backward doubly stochastic systems were given by Hu, et al.[5] and Wu and Liu[6], respectively. A comprehensive survey of impulse control and its

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applications were given by Yu[7] and Korn[8]. A survey of stochastic differential game with partial information has been conducted by Wu and Liu[9].

The stochastic maximum principle and dynamic programming principle are two major approaches to solve stochastic control problems. The former is a stochastic extension of the Pontryagin maximum principle. In most cases, the optimal impulse control problem is studied through dynamic programming principle. The results show that the value function is a solution of some quasi-variational inequalities. The optimal impulse moments and magnitudes can be solved through a verification theorem, e.g., Cadenillas and Zapatero[2] and Korn[8]. Singular control problems have been researched by many authors. The stochastic maximum principle of singular control was obtained by Cadenillas and Haussmann[10] in which linear dynamics and convex value function were discussed. Bahlai and Chala[11] put forward a nonlinear dynamics with convex state constraint. In these papers, the singular control terms are assumed to be a bounded variation process. However, the impulse control is a piecewise process which is not necessarily increasing. Wu and Zhang[3] proposed a piecewise impulse control process which is not increasing for a forward-backward system. In these research mentioned above, stochastic systems are modeled by stochastic differential equations. Nevertheless, there are also some phenomena in which the system depends on the expected value of the system.

The mean-field models present the complex reactions of particles through the medium. However, for a general mean-field controlled jump-diffusion, where the setting is non-Markovian, the feature of Hamilton-Jacobi-Belman equations depends on the law of iterated expectations on value function. In this case the principle of dynamic programming does not apply in general. The stochastic maximum principle provides a new approach for solving such problem. Lasry and Lions[12] initially got the mean-field model in physics and statistical mechanics and proved that the mean-field system could be decomposed into a series of nonlinear equations. Oksendal and Sulem[1] used a stochastic approach to investigate a particular class of mean-field problems and extended the application of the mean-field models to economics and finance. The application of mean-field type problems can be found in Andersson and Djehiche[13]. Buckdahn, et al.[14] showed that the related theories of backward stochastic differential equations could be served as the foundations for solving the optimal control of mean-field type problems. The maximum principle for a jump-diffusion mean-field model was discussed by Shen and Siu[15]. For more theories and applications of the specific stochastic control problems, see Andersson and Djehiche[13].

In [7], the duality and the convex analysis are used to establish a necessary maximum principle and a sufficient verification theorem. In this paper, we use the same method to show the necessary maximum principle and sufficient verification theorem. In [16], near-optimal conditions in mean-field control models involving continuous and impulse control have been studied. The authors constructed inequality formula through spike variation technique to get the absolutely continuous part of near-optimal while the near-optimal impulse controls were obtained by convex perturbation. We establish a necessary and sufficient stochastic maximum principle using the duality and the convex analysis. Compared with Chighoub, et al.[16], we get the results of the continuous and impulse controls using different techniques. Shen and

Siu[15] established a necessary and sufficient stochastic maximum principle for a mean-field model governed by Lévy process through convex perturbation. The model in present paper is somewhat different from Shen and Siu[15]. The dynamics of the controlled system is driven by

$$\begin{cases} dX(t) = b(t, X(t), E[X(t)], u(t))dt + \sigma(t, X(t), E[X(t)], u(t))dB(t) \\ \qquad\qquad + \int_{R_0} \gamma(t, X(t), E[X(t)], u(t), z)\widetilde{N}(dt, dz) + C(t)d\xi(t), \\ X(0) \ = X_0. \end{cases}$$

$\xi(t) = \sum_{i\geq 0} \xi_i \mathbf{1}_{[\tau_i,T]}$ is a piecewise consumption process. $\{\tau_i\}$ is a fixed sequence of increasing $\mathcal{F}_t$-stopping time. Each ξ_i is an $\mathcal{F}_{\tau_i}$-measurable random variable.

The objective of the controller is to minimize the expected cost functional, which depends on the control inputs to the system

$$J(u(\cdot), \xi(\cdot)) = E\bigg[\int_0^T f(t, X(t), E[X(t)], u(t))dt + g(X(T), E[X(T)]) + \sum_{i\geq 0} l(\tau_i, \xi_i)\bigg].$$

Problems of this type occur in many applications. Shen and Siu[15] considered the ideal condition without interference. In fact, the interference can arise in the system at any time and state in practice, e.g., the impulse control problem. This gap is fulfilled by our present study which reproduces a more practical dynamic system and cost functional. In this paper, we consider a stochastic optimal control problem where the controlled state process is described by a jump-diffusion mean-field model involving impulse controls. Moreover, the existence and uniqueness of the solution of this equation have also been proved. These results are applied to the continuous-time Markowitz's mean-variance portfolio selection model with piecewise consumption process. To illustrate the effect of Poisson jump, impulse control, and Brownian motion on the control performance, we give some examples. Assume the dynamics of wealth are given by $X_1(\cdot), X_2(\cdot), X_3(\cdot), X_4(\cdot)$, respectively.

$$\begin{cases} dX_1(t) = (\alpha - X_1(t))dt, \\ X_1(0) = 1, \end{cases}$$

$$\begin{cases} dX_2(t) = (\alpha - X_2(t))dt + \beta\sqrt{X_2(t)}dB(t), \\ X_2(0) = 1, \end{cases}$$

$$\begin{cases} dX_3(t) = (\alpha - X_3(t))dt + \beta\sqrt{X_3(t)}dB(t) + \gamma\int_{R_0} \widetilde{N}(dt, dz), \\ X_3(0) = 1, \end{cases}$$

$$\begin{cases} dX_4(t) = (\alpha - X_4(t))dt + \beta\sqrt{X_4(t)}dB(t) + \gamma\int_{R_0} \widetilde{N}(dt, dz) - \delta d\xi(t), \\ X_4(0) = 1. \end{cases}$$

As we can see from Figure 1 where we take $\alpha = 10, \beta = 1, \gamma = -0.2, \delta = 1$. We have a more intuitive understanding of the above equations from the perspective of mathematical finance.

$X_1(\cdot)$ gives the dynamics of bond price and markets have no risk. $X_2(\cdot)$ describes the dynamic of stock price fluctuation which is represented by Brownian motion and markets have risks. $X_3(\cdot)$ represents that the dynamics of stock price changing rule is affected by the price change of the underlying asset which is discontinuous and represented by Poisson jump. $X_4(\cdot)$ denotes that we can choose the corresponding optimal strategy which is represented by impulse control in order to make the price reaching maximum. The jump-diffusion models provide an adequate description of stock price fluctuation and market risk. So in this paper we consider a general jump-diffusion model.

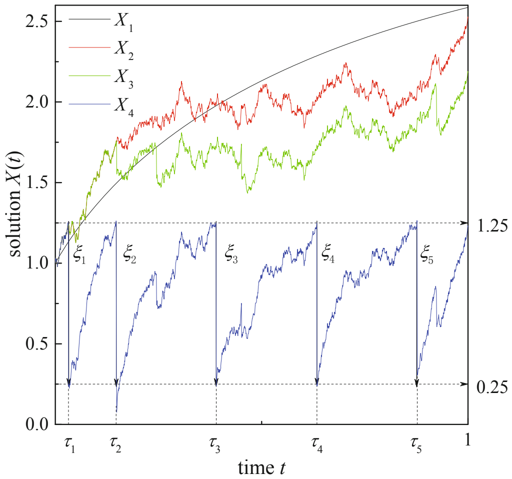

Figure 1 The solution of $X_i, i = 1, 2, 3, 4$

The rest of the paper is organized as follows. Section 2 presents the notations and the existence and uniqueness of solutions with respect to stochastic differential equations (SDEs) of mean-field involving impulse control. In Section 3, we discuss the stochastic maximum principle for the optimal control problem and a verification theorem. In the final section, our results are applied to a Markowitz's mean-variance portfolio selection model.

2 Jump-Diffusion Mean-Field SDE Involving Impulse Control

We fix $T > 0$ as an arbitrarily finite horizon and denote $\mathcal{T} = [0, T]$, $R_0 = R - \{0\}$ and $\mathcal{B}(R_0)$ is the Borel σ-field generated by open subset of R_0, whose closure does not contain the point 0. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e., the right-continuity and the $\mathbb{P}$-completeness of the filtration $\mathbb{F} := \{\mathcal{F}_t | t \in \mathcal{T}\}$. We assume that the Brownian motion and the Possion random measure are stochastically independent under $\mathbb{P}$. $\mathbb{F}$ is the right-continuous, $\mathbb{P}$-complete, natural filtration generated by both Brownian motion and the Possion random measure. We assume that $\mathcal{F}_T = \mathcal{F}$ for convenience.

$S^2(\mathcal{T}, R^n)$: The space of R^n-valued $\mathbb{F}$-adapted càdlàg processes $\{X(t) : t \in [0, T]\}$ such that $E[\sup_{0\le t\le T} |X^2(t)|] < \infty$.

$L^2(\mathcal{F}_T, R^n)$: The space of all the $\mathcal{F}_T$-measurable random variables $X : \Omega \longrightarrow R^n$ such that $E[|X(t)|^2] < \infty$.

$L^2_\beta(\mathcal{T}, R^{n\times d})$: The space of all $R^{n\times d}$-value $\mathbb{F}$-progressively measurable processes $\{v(t) : t \in [0,T]\}$ such that $E[\int_0^T e^{-\beta t}|v(t)|^2 dt] < \infty, \forall\, \beta > 0$.

$L^2(\mathcal{T}, R^{n\times d})$: The space of all $R^{n\times d}$-value $\mathbb{F}$-progressively measurable processes $\{v(t) : t \in [0,T]\}$ such that $E[\int_0^T |v(t)|^2 dt] < \infty$.

$L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}, R^{n\times m})$: The space of $R^{n\times m}$-value $\mathcal{F}_t$-measurable bounded processes.

$L^2(\mathcal{T}, \nu, R^{n\times k})$: The space of all $R^{n\times k}$-value $Leb \otimes \mathcal{B}(R_0)$-measurable processes such that $E\big[\int_0^T \int_{R_0} \mathrm{tr}[v(t,z)\mathrm{diag}(\nu(dz))v(t,z)^{\mathrm{T}}dt]\big] < \infty$.

$L^2_\beta(\mathcal{T}, \nu, R^{n\times k})$: The space of all $R^{n\times k}$-value $Leb \otimes \mathcal{B}(R_0)$-measurable processes such that $E\big[\int_0^T \int_{R_0} e^{-\beta t}\mathrm{tr}[v(t,z)\mathrm{diag}(\nu(dz))v(t,z)^{\mathrm{T}}dt]\big] < \infty, \forall\, \beta > 0$.

$L^2(R_0, \mathcal{B}(R_0), \nu, R^{n\times k})$: The Hilbert space of ν-almost sure equivalence classes formed by the functions from R_0 to the space of $R^{n\times k}$-valued matrices with the norm $|v|_\nu = (\int_{R_0} \mathrm{tr}[v(z)\,\mathrm{diag}(\nu d(z))v(z)^{\mathrm{T}}])^{\frac{1}{2}}$.

$\mathcal{I}(\mathcal{T}, R^n)$: The class of processes $\xi(\cdot) = \sum_{i\geq 0} \xi_i \mathbf{1}_{[\tau_i,T]}$ such that each ξ_i is R^n-valued $\mathcal{F}_{\tau_i}$-measurable random variable, $E[\sum_{i=0}^{\infty}|\eta_i|^2] < \infty$. Assuming $\tau_i \to \infty$ implies that at most finitely many impulses may occur on $\mathcal{T}$.

$$M^2(\mathcal{T}, R^n \times R^{n\times d} \times R^{n\times k}) = S^2(\mathcal{T}, R^k) \times L^2(\mathcal{T}, R^{k\times d}) \times L^2(\mathcal{T}, R^{n\times k}).$$

In the rest of this section, we introduce some conditions that are necessary for the existence and uniqueness of jump-diffusion mean-field stochastic differential equation solution involving impulse control. b, σ, γ are $\mathcal{F}_t$-measurable mappings.

Assumption H$_1$ b, σ, γ are Lipschitz with respect to $x, \overline{x}$ and have a linear growth in $(x, \overline{x})$, i.e., $\exists\, c > 0$ such that $|b(t, x_1, \overline{x}_1) - b(t, x_2, \overline{x}_2)| + |\sigma(t, x_1, \overline{x}_1) - \sigma(t, x_2, \overline{x}_2)| + |\gamma(t, x_1, \overline{x}_1, \cdot) - \gamma(t, x_2, \overline{x}_2, \cdot)|_\nu \leq c(|x_1 - x_2| + |\overline{x}_1 - \overline{x}_2|)$ and $|b(t, x, \overline{x})| + |\sigma(t, x, \overline{x})| + |\gamma(t, x, \overline{x}, \cdot)|_\nu \leq c(1 + |x| + |\overline{x}|)$.

Consider the following stochastic differential equation:

$$\begin{cases} dX(t) = b(t, X(t), E[X(t)])dt + \sigma(t, X(t), E[X(t)])dB(t) \\ \qquad\qquad + \displaystyle\int_{R_0} \gamma(t, X(t), E[X(t)], z)\widetilde{N}(dt, dz) + C(t)d\xi(t), \\ X(0) \ = X_0. \end{cases} \tag{1}$$

Here b, σ, γ are $\mathbb{F}$-measurable functions such that

$b : \Omega \times [0,T] \times R^n \times R^n \to R^n$, $\sigma : \Omega \times [0,T] \times R^n \times R^n \to R^{n\times d}$.

$\gamma : \Omega \times [0,T] \times R^n \times R^n \times R_0 \to R^{n\times k}$, $C : \Omega \times [0,T] \to R^{n\times m}$.

$B(t) = (B_1(t), B_2(t), \cdots, B_d(t))$ is d-dimensional standard Brownian motion. Suppose $N_i(dt, dz)$, $i = 1, 2, \cdots, k$ are independent Poisson random measures on the product measurable space $(\mathcal{T} \times R, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(R_0))$. Assume $N(dt, dz)$ has the following compensator $\nu(dz)dt = (\nu_1(dz)dt, \nu_2(dz)dt, \cdots, \nu_k(dz)dt)$. $\nu_i(dz)$ is the Lévy measure of the jump amplitude of i-th Poisson random measure. $\nu(dz)$ is k-dimensional Lévy measure. $\widetilde{N}(dt, dz)$: k-dimensional compensated Poisson random measure $\widetilde{N}(dt, dz) = (N_1(dz, dt) - \nu_1(dz)dt, N_2(dz, dt) - \nu_2(dz)dt, \cdots, N_k(dz, dt) - \nu_k(dz)dt)$.

Lemma 2.1 *Let* $C(\cdot) \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}, R^{n\times m})$ *be continuous. Under Condition* H_1 *the SDE*(1) *has a unique solution* $X(\cdot) \in S^2(\mathcal{T}, R^n)$.

Proof If we assume that $\eta(\cdot) \equiv 0$ for any $t \in [0,T]$, Equation (1) becomes a classical SDE with jump without impulses, and the conclusion holds true from the jump-diffusion theory (see [17]).

Let

$$\begin{aligned}
&h(t) = \int_0^t C(s)d\xi(s) = \sum_{\tau_i \le t} C(\tau_i)\xi_i, \quad Y(t) = X(t) - h(t),\\
&\widetilde{b}(t, X(t), E[X(t)]) = b(t, X(t) + h(t), E[X(t) + h(t)]),\\
&\widetilde{\sigma}(t, X(t), E[X(t)]) = \sigma(t, X(t) + h(t), E[X(t) + h(t)]),\\
&\widetilde{\gamma}(t, X(t), E[Y(t)]) = \gamma(t, X(t) + h(t), E[X(t) + h(t)]).
\end{aligned}$$

Then we have

$$\begin{cases}
dY(t) = \widetilde{b}(t, Y(t), E[Y(t)])dt\\
\qquad\qquad + \widetilde{\sigma}(t, Y(t), E[Y(t)])dB(t) + \displaystyle\int_{R_0} \widetilde{\gamma}(t, Y(t), E[Y(t)])\widetilde{N}(dt, dz),\\
Y(0) \ = X_0.
\end{cases} \tag{2}$$

It is easy to verify that $\widetilde{b}(\cdot), \widetilde{\sigma}(\cdot)$ and $\widetilde{\gamma}(\cdot)$ satisfy Assumption H_1.

Step 1 For arbitrary $y(\cdot) \in L^2(\mathcal{T}, R^n)$, consider the following SDE:

$$dY(t) = \widetilde{b}(t, Y(t), E[y(t)])dt + \widetilde{\sigma}(t, Y(t), E[y(t)])dB(t) + \int_{R_0} \widetilde{\gamma}(t, Y(t), E[y(t)])\widetilde{N}(dt, dz). \tag{3}$$

The existence and uniqueness of the solution of SDE (3) are given by Theorem 1.19 in [12].

Step 2 We construct a mapping from SDE (3) into itself, i.e., $l(y(\cdot)) \to Y(\cdot)$ and l is a contractive mapping. Indeed, for any $y_1(\cdot), y_2(\cdot) \in L^2(\mathcal{T}, R^n), Y_1(\cdot) = l(y_1(\cdot)), Y_2(\cdot) = l(y_2(\cdot)), \widehat{y}(\cdot) = y_1(\cdot) - y_2(\cdot), \widehat{Y}(\cdot) = Y_1(\cdot) - Y_2(\cdot)$. Applying Itô's formula to $\mathrm{e}^{-\beta t}|\widehat{Y}(t)|^2$ and taking the mathematical expectation, the result follows from Lipschitz condition,

$$\begin{aligned}
&E\Big[\mathrm{e}^{-\beta t}|\widehat{Y}(t)|^2\Big] + E\Big[\int_0^t \beta \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2\Big]ds\\
&= 2E\Big[\int_0^t \mathrm{e}^{-\beta s}\widehat{Y}^{\mathrm{T}}(s)|\widetilde{b}(s, Y_1(s), E[y_1(s)]) - \widetilde{b}(s, Y_2(s), E[y_2(s)])|ds\Big]\\
&\quad + E\Big[\int_0^t \mathrm{e}^{-\beta s}|\widetilde{\sigma}(s, Y_1(s), E[y_1(s)]) - \widetilde{\sigma}(s, Y_2(s), E[y_2(s)])|^2 ds\Big]\\
&\quad + E\Big[\int_0^t \mathrm{e}^{-\beta s}|\widetilde{\gamma}(s, Y_1(s), E[y_1(s)], \cdot) - \widetilde{\gamma}(s, Y_2(s), E[y_2(s)], \cdot)|_\nu^2 ds\Big]\\
&\le (6c^2 + 1)E\Big[\int_0^t \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2 ds\Big] + 6c^2 E\Big[\int_0^t \mathrm{e}^{-\beta s}|\widehat{y}(s)|^2 ds\Big].
\end{aligned}$$

We get

$$(\beta - 6c^2 - 1)E\Big[\int_0^T \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2 ds\Big] \le 6c^2 E\Big[\int_0^T \mathrm{e}^{-\beta s}|\widehat{y}(s)|^2 ds\Big].$$

Let β be $18c^2+1$. We have

$$E\bigg[\int_0^T \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2ds\bigg] \leq \frac{6c^2}{\beta-6c^2+1}E\bigg[\int_0^T \mathrm{e}^{-\beta s}|\widehat{y}(s)|^2ds\bigg].$$

Since $\frac{6c^2}{\beta-6c^2+1} < 1$, l is a contractive mapping in $L^2_\beta(\mathcal{T},R^n)$. We define $\mathcal{L}_\beta[0,T]$ to be the Banach space $\mathcal{L}_\beta[0,T]=L^2_\beta(\mathcal{T},R^n)$, with the norm $E[\int_0^T \mathrm{e}^{-\beta t}|v(t)|^2dt]$. Since $0<T<\infty$, all the norms $|\cdot|_{\mathcal{L}_\beta[0,T]}$ with different β are equivalent. According to the fixed-point theorem, the mapping has a unique fixed point $Y(\cdot)=l(Y(\cdot))$. The existence and uniqueness of the solution of SDE (3) lead to the existence and uniqueness of the solution of SDE (2). Since $Y(t)=X(t)-h(t)$ is invertible, SDE (1) has a unique solution.

Step 3 According to Cauchy-Schwarz inequality, Doob martingale inequality, Itô's isometry and Burkhölder-Davis-Gundy inequality, we have the following results respectively

$$E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t b(s,X(s),E[X(s)])ds\bigg|^2\bigg] \leq TE\bigg[\int_0^T \Big|b(s,X(s),E[X(s)])\Big|^2 ds\bigg],$$

$$\begin{aligned}E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t \sigma(s,X(s),E[X(s)])dB(s)\bigg|^2\bigg] &\leq 4E\bigg[\bigg|\int_0^T \sigma(s,X(s),E[X(s)])dB(s)\bigg|^2\bigg]\\ &= 4E\bigg[\bigg|\int_0^T \Big|\sigma(s,X(s),E[X(s)])\Big|^2 ds\bigg],\end{aligned}$$

$$E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t\int_{R_0}\gamma(s,X(s),E[X(s)],z)\widetilde{N}(ds,dz)\bigg|^2\bigg] \leq 4E\bigg[\int_0^T \Big|\gamma(s,X(s),E[X(s)],\cdot)\Big|^2_\nu ds\bigg].$$

The result follows from Lipschitz condition and Jensen's inequality

$$\begin{aligned}
&E\bigg[\int_0^T |b(s,X(s),E[X(s)])|^2ds\bigg] + E\bigg[\int_0^T |\sigma(s,X(s),E[X(s)])|^2ds\bigg]\\
&\quad + E\bigg[\int_0^T |\gamma(s,X(s),E[X(s)],\cdot)|^2_\nu ds\bigg]\\
&\leq cE\bigg[\int_0^T |X(s)|^2ds\bigg] + cE\bigg[\int_0^T |b(s,0,0)|^2ds\bigg] + cE\bigg[\int_0^T |\sigma(s,0,0)|^2\bigg]\\
&\quad + cE\bigg[\int_0^T |\gamma(s,0,0,\cdot)|^2_\nu ds\bigg],\\
&E\bigg[\sup_{0\leq t\leq T}|X(t)|^2\bigg] \leq L\bigg\{E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t b(s,X(s),E[X(s)])ds\bigg|^2\bigg]\\
&\quad + E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t \sigma(s,X(s),E[X(s)])dB(s)\bigg|^2\bigg] + E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t C(s)d\xi(s)\bigg|^2 + E[X_0]^2\bigg]\\
&\quad + E\bigg[\sup_{0\leq t\leq T}\bigg|\int_0^t\int_{R_0}\gamma(s,X(s),E[X(s)],z)\widetilde{N}(ds,dz)\bigg|^2\bigg]\bigg\}\\
&\leq L\bigg\{E|X_0|^2 + TE\bigg[\int_0^T |b(s,X(s),E[X(s)])|^2ds\bigg] + 4E\bigg[\int_0^T |\sigma(s,X(s),E[X(s)])|^2ds\bigg]\\
&\quad + 4E\bigg[\int_0^T |\gamma(s,X(s),E[X(s)],\cdot)|^2_\nu ds\bigg] + E\sum_{\tau_i\leq T}[C(\tau_i)\xi_i]^2\bigg\}
\end{aligned}$$

$$\leq L\bigg\{E|X_0|^2+E\bigg[\int_0^T|X(s)|^2ds\bigg]+E\bigg[\int_0^T|b(s,0,0)|^2ds\bigg]+E\bigg[\int_0^T|\sigma(s,0,0)|^2ds\bigg]$$
$$+E\sum_{\tau_i\leq T}[C(\tau_i)\xi_i]^2+E\bigg[\int_0^T|\gamma(s,0,0,\cdot)|_\nu^2ds\bigg]\bigg\}<\infty,$$

where L is a constant which will change from line to line. We have $E[\sup_{0\leq t\leq T}|X(t)|^2]<\infty$, i.e., $X(\cdot)\in S^2(\mathcal{T},R^n)$. ▮

Assumption H$_2$

H$_{2.1}$: $\phi(\cdot,0,0,0,0,0,0)\in L^2(\mathcal{F},R^n)$;

H$_{2.2}$: ϕ is uniformly Lipschitz, i.e., $\exists\ c>0$ such that $\forall\ t,\chi^1=(y^1,z^1,v^1),\chi^2=(y^2,z^2,v^2)$, $\overline{\chi}^1=(\overline{y}^1,\overline{z}^1,\overline{v}^1),\overline{\chi}^2=(\overline{y}^2,\overline{z}^2,\overline{v}^2)$,
$|\phi(t,\chi^1,\overline{\chi}^1)-\phi(t,\chi^2,\overline{\chi}^2)|\leq c(|y^1-y^2|+|z^1-z^2|+|v^1-v^2|_\nu+|\overline{y}^1-\overline{y}^2|+|\overline{z}^1-\overline{z}^2|+|\overline{v}^1-\overline{v}^2|_\nu)$.

Lemma 2.2 *Under Condition* H$_2$*, the following BSDE* (4) *has a unique solution* $(Y(\cdot),Z(\cdot),V(\cdot,\cdot))\in M^2(\mathcal{T},R^n\times R^{n\times d}\times R^{n\times k})$.

$$\begin{cases}dY(t)=-\phi(t,Y(t),Z(t),V(t),E[Y(t)],E[Z(t)],E[V(t,\cdot)])dt+Z(t)dB(t)\\ \qquad\qquad +\displaystyle\int_{R_0}V(t,z)\widetilde{N}(dt,dz)+C(t)d\xi(t),\\ Y(T)\ =\zeta,\end{cases}\tag{4}$$

where ζ is a square-integrable and $\mathcal{F}_T$-measurable random variable.

Proof Lemma 2.2 can be proved by the fixed-point theorem.

Let

$$h(t)=\int_0^tC(s)d\xi(s)=\sum_{\tau_i\leq t}C(\tau_i)\xi_i,\quad \widetilde{Y}(t)=Y(t)-h(t),$$
$$\widetilde{\phi}(t,Y(t),Z(t),V(t),E[Y(t)],E[Z(t)],E[V(t,\cdot)])$$
$$=\phi(t,Y(t)+h(t),Z(t),V(t),E[Y(t)+h(t)],E[Z(t)],E[V(t,\cdot)]),$$
$$\begin{aligned}d\widetilde{Y}(t)&=\widetilde{\phi}(t,\widetilde{Y}(t),Z(t),V(t),E[\widetilde{Y}(t)],E[Z(t)],E[V(t,\cdot)])dt+Z(t)dB(t)\\&\quad+\int_{R_0}V(t,z)\widetilde{N}(dt,dz),\\ \widetilde{Y}(T)&=\zeta-h(T).\end{aligned}\tag{5}$$

According to Theorem 3.1 in [15], the mean-field BSDE (5) has a unique solution. We observe that $\widetilde{Y}(t)=Y(t)-h(t)$ is invertible. Then, we obtain the solution $(Y(\cdot),Z(\cdot),V(\cdot,\cdot))\in M^2(\mathcal{T},R^n\times R^{n\times d}\times R^{n\times k})$. ▮

3 Stochastic Maximum Principle

We call the following problem an optimal control problem of jump-diffusion mean-field model involving impulse control. The state process $X(\cdot) := \{X(t) : t \in [0,T]\}$ is given by a jump-diffusion mean-field stochastic differential equation involving impulse control.

$$\begin{cases} dX(t) = b(t,X(t),E[X(t)],u(t))dt + \sigma(t,X(t),E[X(t)],u(t))dB(t) \\ \qquad\qquad + \displaystyle\int_{R_0} \gamma(t,X(t),E[X(t)],u(t),z)\widetilde{N}(dt,dz) + C(t)d\xi(t), \\ X(0) \ = X_0. \end{cases} \tag{6}$$

The objective of the controller is to minimize the expected cost function, which depends on the control inputs to the system

$$J(u(\cdot),\xi(\cdot)) = E\bigg[\int_0^T f(t,X(t),E[X(t)],u(t))dt + g(X(T),E[X(T)]) + \sum_{i\geq 0} l(\tau_i,\xi_i)\bigg]. \tag{7}$$

Here $f : [0,T] \times R^n \times R^n \times U \to R, g : R^n \times R^n \to R, l : [0,T] \times U \to R$. Suppose that the controller wants to minimize the cost functional J by choosing an appropriate admissible control $(u(\cdot),\xi(\cdot))$ such that $J(u(\cdot),\xi(\cdot))) = \inf_{(v,\eta)\in\mathcal{A}} J(v(\cdot),\eta(\cdot))$, where $\mathcal{A} = \mathcal{U}\times\mathcal{I}$ is called an admissible control set. u: $\mathcal{T} \times \Omega \to U$ is defined in a non-empty, closed and convex set $\mathcal{U}$ ($\mathcal{U}$: A nonempty convex set of R^n). We require that the control process $\{u(t)|t \in \mathcal{T}\}$ is $\mathbb{F}$-predictable and has right limits.

Assumption H_3

$\mathrm{H}_{3.1}$: b,σ,γ are continuously differentiable, Lipschitz in $(x,\overline{x},u)$, and have a linear growth in $(x,\overline{x},u)$, i.e., for any $(x,\overline{x}_1,u_1)$, $(x_2,\overline{x}_2,u_2)$ $\exists$ c such that: $|b(t,x_1,\overline{x}_1,u_1) - b(t,x_2,\overline{x}_2,u_2)| + |\sigma(t,x_1,\overline{x}_1,u_1) - \sigma(t,x_2,\overline{x}_2,u_2)| + |\gamma(t,x_1,\overline{x}_1,u_1) - \gamma(t,x_2,\overline{x}_2,u_2)|_\nu \leq c(|x_1 - x_2| + |\overline{x}_1 - \overline{x}_2| + |u_1 - u_2|)$ and $|b(t,x,\overline{x},u)| + |\sigma(t,x,\overline{x},u)| + |\gamma(t,x,\overline{x},u)|_\nu \leq c(1 + |x| + |\overline{x}| + |u|)$.

$\mathrm{H}_{3.2}$: l is continuously differentiable in ξ and $l(\tau,\xi) \leq c(1 + |\xi|)$.

The purpose of the rest of this section is to present the stochastic maximum principle for an optimal control problem of jump-diffusion mean-field model involving impulse control. Since $\mathcal{A}$ is convex, a convex perturbation technique will be used to prove a necessary condition for the above optimal control problem. We will give a verification theorem for the necessary condition.

Lemma 3.1 *Under Assumption* H_3*, the mean-field SDE* (6) *has a unique solution* $X(\cdot) \in S^2(\mathcal{T},R^n)$.

Proof The conclusion follows from Lemma 2.1. ▮

The Hamiltonian function is defined as follows

$$\begin{aligned} H(t,x,\overline{x},u,p,q,r) &= f(t,x,\overline{x},u) + b^{\mathrm{T}}(t,x,\overline{x},u)p + \mathrm{tr}[\sigma^{\mathrm{T}}(t,x,\overline{x},u)q] \\ &\quad + \int_{R_0} \mathrm{tr}[\gamma(t,x,\overline{x},u,z)\mathrm{diag}(\nu(dz))r^{\mathrm{T}}(t,z)]. \end{aligned} \tag{8}$$

$H : \Omega \times \mathcal{T} \times R^n \times R^n \times U \times R^n \times R^{n\times d} \times \mathcal{R} \to R$, where $\mathcal{R}$ is the set of functions $r(\omega,t,z)$: $\Omega \times \mathcal{T} \times R_0 \to R^{n\times k}$ such that the integral in (8) and its derivatives with respect to $(x,\overline{x},u)$ converge.

Assumption $\mathbf{H}_4$

$\mathrm{H}_{4.1}$: f, b, σ, γ are continuously differentiable in $(x, \overline{x}, u)$, g is continuously differentiable in $(x, \overline{x})$, l is continuously differentiable in ξ.

$\mathrm{H}_{4.2}$: The derivative of b, σ, γ are bounded. The derivative of f, g, l are bounded by $C(1 + |x| + |\overline{x}| + |u|), C(1 + |x| + |\overline{x}|), C(1 + |\xi|)$, respectively.

We denote $\psi(t) = \psi(t.X(t), E[X(t)], u(t))$, for $\psi = b, \sigma, \gamma, b_x, b_{\overline{x}}, b_u, \sigma_x, \sigma_{\overline{x}}, \sigma_u, \gamma_x, \gamma_{\overline{x}}, \gamma_u, f, f_x, f_u, H(t) = H(t, x, \overline{x}, u, p, q, r)$. We introduce the adjoint equation:

$$\begin{cases} dp(t) = -(\nabla_x H(t) + E[\nabla_{\overline{x}} H(t)])dt + q(t)dB(t) + \displaystyle\int_{R_0} r(t,z)\widetilde{N}(dz, dt), \\ p(T) = \nabla_x g(X(T), E[X(T)]) + E[\nabla_{\overline{x}} g(X(T), E[X(T)])]. \end{cases} \tag{9}$$

It is easy to see that (9) has a unique solution $(p(\cdot), q(\cdot), r(\cdot)) \in M^2(\mathcal{T}, R^n \times R^{n\times d} \times R^{n\times k})$ under Assumption H_4. Let $(u(\cdot), \xi(\cdot) = \sum_{i\geq 0} \xi_i \mathbf{1}_{[\tau_i,T]})$ be the optimal control of the above stochastic optimal control problem. $(v(\cdot), \eta(\cdot) = \sum_{i\geq 0} \eta_i \mathbf{1}_{[\tau_i,T]})$ makes $u(\cdot)+v(\cdot) \in \mathcal{U}, \xi(\cdot)+\eta(\cdot) \in \mathcal{I}$. Since $\mathcal{U}, \mathcal{I}$ are convex, then for arbitrarily $\varepsilon > 0, u^\varepsilon(\cdot) = u(\cdot)+\varepsilon v(\cdot) \in \mathcal{U}, \xi^\varepsilon(\cdot) = \xi(\cdot)+\varepsilon\eta(\cdot) \in \mathcal{I}$. $X^\varepsilon(\cdot)$ represents the corresponding trajectory of $(u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))$. We introduce the following variational equation:

$$\begin{cases} dX^1(t)^{\mathrm{T}} = [X^1(t)^{\mathrm{T}} b_x(t) + E[X^1(t)^{\mathrm{T}}] b_{\overline{x}}(t) + v(t)^{\mathrm{T}} b_u(t)]dt \\ \qquad + [X^1(t)^{\mathrm{T}} \sigma_x(t) + E[X^1(t)^{\mathrm{T}}]\sigma_{\overline{x}}(t) + v(t)^{\mathrm{T}}\sigma_u(t)]dB(t) \\ \qquad + \displaystyle\int_{R_0} [X^1(t-)^{\mathrm{T}}\gamma_x(t) + E[X^1(t)^{\mathrm{T}}]\gamma_{\overline{x}}(t) + v(t)^{\mathrm{T}}\gamma_u(t)]\widetilde{N}(dz, dt) \\ \qquad + C(t)d\eta(t), \\ X^1(0) = 0. \end{cases} \tag{10}$$

By Assumption H_4, Equation (10) has a unique solution. Denote $\widetilde{X}(t) = \frac{X^\varepsilon(t) - X(t)}{\varepsilon} - X^1(t)$.

Lemma 3.2

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq T} E[|\widetilde{X}(t)|^2] = 0.$$

Proof Since $\widetilde{X}(t)$ does not depend on the impulse term, the corresponding conclusion is derived from Lemma 4.3 in [15]. ∎

Since $(u(\cdot), \xi(\cdot))$ is an optimal control, it is obvious that

$$\varepsilon^{-1}[J(u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) - J(u(\cdot), \xi(\cdot))] \geq 0.$$

We obtain the following variational inequality.

Lemma 3.3 *Assume $(u(\cdot), \xi(\cdot))$ is an optimal control, then*

$$E\bigg[X^1(T)^{\mathrm{T}}(\nabla_x g(X(T), E[X(T)]) + E[\nabla_{\overline{x}} g(X(T), E[X(T)])]) + \int_0^T \big[X^1(t)^{\mathrm{T}} f_x(t) \\ + E[X^1(t)^{\mathrm{T}}] f_{\overline{x}}(t) + f_u(t)v(t)\big]dt + \sum_{i\geq 0} l_\xi(\tau_i, \xi_i)\eta_i\bigg] \geq 0.$$

Proof From Lemma 3.2, It is easy to see that when $\varepsilon \to 0$,

$$\varepsilon^{-1}E\Big[\sum_{i\geq 0} l(\tau_i,\xi_i^\varepsilon)-l(\tau_i,\xi_i)\Big]=E\Big[\sum_{i\geq 0}\int_0^1 l_\xi(\tau_i,\xi_i+\varepsilon\lambda\eta_i)\Big]\eta_i d\lambda \to E\Big[\sum_{i\geq 0} l_\xi(\tau_i,\xi_i)\eta_i\Big],$$

$$\begin{aligned}
&\varepsilon^{-1}E\Big[\int_0^T f(t,X^\varepsilon(t),E[X^\varepsilon(t)],u^\varepsilon(t))dt\Big]-\varepsilon^{-1}E\Big[\int_0^T f(t,X(t),E[X(t)],u(t))dt\Big]\\
=&E\Big[\int_0^T\int_0^1\Big\{\frac{(X^\varepsilon(t)-X(t))^{\mathrm T}}{\varepsilon}f_x(t,X(t)+\lambda\varepsilon(\widetilde{X}(t)+X^1(t)),E[X(t)],u(t))\\
&+\frac{(E[X^\varepsilon(t)]-E[X(t)])^{\mathrm T}}{\varepsilon}f_{\overline{x}}(t,X(t),E[X(t)+\lambda\varepsilon(\widetilde{X}(t)+X^1(t))],u(t))\\
&+f_u(t,X(t),E[X(t)],u(t)+\lambda\varepsilon v(t))v(t)\Big\}d\lambda dt\Big]\\
\to&E\Big[\int_0^T\big[X^1(t)^{\mathrm T}f_x(t)+E[X^1(t)^{\mathrm T}]f_{\overline{x}}(t)+f_u(t)v(t)\big]dt\Big],
\end{aligned}$$

$$\begin{aligned}
&E\Big[\frac{(X^\varepsilon(t)-X(t))^{\mathrm T}}{\varepsilon}\nabla_x g(X(T),E[X(T)])+\frac{(EX^\varepsilon(t)-E[X(t)])^{\mathrm T}}{\varepsilon}]\nabla_{\overline{x}}g(T,X(T))\Big]\\
\to&E\Big[[X^1(T)]^{\mathrm T}(\nabla_x g(X(T),E[X(T)])+E[\nabla_{\overline{x}}g(X(T),E[X(T)])]\Big].
\end{aligned}$$

Due to the optimality of $(u(\cdot),\xi(\cdot))$, we derive that

$$\begin{aligned}
0\leq&\varepsilon^{-1}[J(u^\varepsilon(\cdot),\xi^\varepsilon(\cdot))-J(u(\cdot),\xi(\cdot))]\\
=&\varepsilon^{-1}E\Big[\int_0^T f(t,X^\varepsilon(t),E[X^\varepsilon(t)],u^\varepsilon(t))dt+g(X^\varepsilon(T),E[X^\varepsilon(T)])+\sum_{i\geq 0}l(\tau_i,\xi_i^\varepsilon)\Big]\\
&-\varepsilon^{-1}E\Big[\int_0^T f(t,X(t),E[X(t)],u(t))dt+g(X(T),E[X(T)])+\sum_{i\geq 0}l(\tau_i,\xi_i)\Big]\\
\to&E\Big[[X^1(T)]^{\mathrm T}(\nabla_x g(X(T),E[X(T)])+E[\nabla_{\overline{x}}g(X(T),E[X(T)])])\\
&+\int_0^T\big[X^1(t)^{\mathrm T}f_x(t)+E[X^1(t)^{\mathrm T}]f_{\overline{x}}(t)+f_u(t)v(t)\big]dt+\sum_{i\geq 0}l_\xi(\tau_i,\xi_i)\eta_i\Big].
\end{aligned}$$

The proof is finished. ▮

Theorem 3.4 *Under Assumptions* $\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3,\mathrm{H}_4$, $(u(\cdot),\xi(\cdot))$ *is an optimal control;* $(p(\cdot), q(\cdot),r(\cdot))$ *is the solution of* (9) *and* $X(\cdot)$ *is the corresponding trajectory. Then* $\forall\, v\in\mathcal{U},\eta\in\mathcal{I}$,

$$\nabla_u H(t,X(t),E[X(t)],u(t),p(t),q(t),r(t,\cdot))(v-u(t))^{\mathrm T}\geq 0,\ \text{ a.e. } t\in\mathcal{T},\mathbb{P}-\text{a.s.} \tag{11}$$

$$E\Big[\sum_{i\geq 0}[l_\xi(\tau_i,\xi_i)+p(\tau_i)C(\tau_i)]\mathbf{1}_{0\leq\tau_i\leq T}(\eta_i-\xi_i)\Big]\geq 0. \tag{12}$$

Proof Using Itô's formula to $p(t)^{\mathrm{T}}X^1(t)$ and combining with Lemma 3.3, we have

$$
\begin{aligned}
&E\Big[X^1(T)^{\mathrm{T}}p(T) - X^1(0)^{\mathrm{T}}p(0)\Big]\\
&= E\Big[\int_0^T X^1(t)^{\mathrm{T}}\big[-\nabla_x H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot)\\
&\quad + E[\nabla_{\overline{x}} H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot))])\big]dt\\
&\quad + p(t)\Big\{\big[X^1(t)^{\mathrm{T}}b_x(t) + E[X^1(t)^{\mathrm{T}}]b_{\overline{x}}(t) + v(t)^{\mathrm{T}}b_u(t)\big]dt + C(t)d\eta(t)\Big\}\\
&\quad + q(t)\big[X^1(t)^{\mathrm{T}}\sigma_x(t) + E[X^1(t)^{\mathrm{T}}]\sigma_{\overline{x}}(t) + v(t)^{\mathrm{T}}\sigma_u(t)\big]dt + \int_{R_0}\Big\{\big[X^1(t-)^{\mathrm{T}}\gamma_x(t)\\
&\quad + E[X^1(t)^{\mathrm{T}}]\gamma_{\overline{x}}(t) + v(t)^{\mathrm{T}}\gamma_u(t)\big]\nu(dz)dt\Big\}r(t,z)\big]\Big]\\
&= E\Big[\int_0^T \big[\nabla_u H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot))v(t) - X^1(t)^{\mathrm{T}}f_x(t) - E[X^1(t)^{\mathrm{T}}]f_{\overline{x}}(t)\\
&\quad - f_u(t)v(t)\big]dt + \sum_{i\geq 0} p(\tau_i)C(\tau_i)\eta_i\Big].
\end{aligned}
$$

Adding the same item

$$
E\Big[\int_0^T \big[X^1(t)^{\mathrm{T}}f_x(t) + E[X^1(t)^{\mathrm{T}}]f_{\overline{x}}(t) + f_u(t)v(t)\big]dt + \sum_{i\geq 0} l_\xi(\tau_i,\xi_i)\eta_i\Big]
$$

on both sides and combining Lemma 3.3, we have

$$
\begin{aligned}
&E\Big[\int_0^T \nabla_u H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot))v(t)dt\\
&\quad + \sum_{i\geq 0}[p(\tau_i)C(\tau_i) + l_\xi(\tau_i,\xi_i)]\mathbf{1}_{\{0\leq\tau_i\leq T\}}\eta_i\Big] \geq 0.
\end{aligned}
$$

By arbitrariness of $v(\cdot)$ and $\eta_i, i = 1, 2, \cdots$, we choose $v(\cdot)\equiv 0$ and $\eta_i \equiv 0$, respectively, to get

$$
E\Big[\int_0^T \nabla_u H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot))(\widetilde{v}(t) - u(t))dt\Big] \geq 0, \tag{13}
$$

for all $\widetilde{v}(\cdot) \in \mathcal{U}$, and

$$
E\Big[\sum_{i\geq 0}[p(\tau_i)C(\tau_i) + l_\xi(\tau_i,\xi_i)]\mathbf{1}_{\{0\leq\tau_i\leq T\}}(\eta_i - \xi_i)\Big] \geq 0,
$$

for all $\eta(\cdot) = \sum_{i\geq 0}\eta_i\mathbf{1}_{[\tau_i,T]} \in \mathcal{I}$. Next we refer to the proof in Theorem 1.5 of [6]. For $v \in \mathcal{U}$, defined by B^v the set of $(t,\omega) \in [0,T]\times\Omega$ such that $\nabla_u H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot))(v(t) - u(t))dt < 0$.

Obviously, for each $t \in [0,T]$, $B_t^v \in \mathcal{F}_t$. Let us consider $\widetilde{v} \in \mathcal{U}$ defined by

$$\widetilde{v}(t,\omega) = \begin{cases} v, & \text{if } (t,\omega) \in B^v, \\ u(t,\omega), & \text{otherwise.} \end{cases}$$

If $(Leb \otimes \mathbb{P})(B^v) > 0$, then it follows that

$$E\Big[\int_0^T \nabla_u H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t,\cdot))(\widetilde{v}(t) - u(t))dt\Big] < 0,$$

which contradicts (13). Thus, we get $(Leb \otimes \mathbb{P})(B^v) = 0$, and the conclusion (11), (12) hold. ∎

Theorem 3.5 *Under Assumptions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$, *we assume that* l, g, H *are convex respect to* $\xi, (x, \overline{x})$ *and* $(x, \overline{x}, u)$, *respectively.* $X(\cdot)$ *is the corresponding trajectory of* $(u(\cdot), \xi(\cdot)) \in \mathcal{A}$ *and* $\big(p(\cdot), q(\cdot), r(\cdot,\cdot)\big)$ *is a unique solution of* (9). *If* (11), (12) *hold, then* $(u(\cdot), \xi(\cdot))$ *is an optimal control process.*

Proof For any $(v(\cdot), \eta(\cdot)) \in \mathcal{U} \times \mathcal{I}$, $X^v(\cdot)$ represents the corresponding trajectory of $(v(\cdot), \eta(\cdot))$. Consider $J(v(\cdot), \eta(\cdot)) - J(u(\cdot), \xi(\cdot))$, using the convexity of $g(\cdot), l(\cdot)$, we have

$$\begin{aligned} & J(v(\cdot), \eta(\cdot)) - J(u(\cdot), \xi(\cdot)) \\ &= E\Big[\int_0^T f(t, X^v(t), E[X^v(t)], v(t)) - f(t, X(t), E[X(t)], u(t))dt \\ &\quad + g(X^v(T), E[X^v(T)]) - g(X(T), E[X(T)]) + \sum_{i\geq 0}(l(\tau_i, \eta_i) - l(\tau_i, \xi_i))\Big] \\ &\geq E\Big[\int_0^T f(t, X^v(t), E[X^v(t)], v(t)) - f(t, X(t), E[X(t)], u(t))dt \\ &\quad + (X^v(T) - X(T))^{\mathrm{T}}(\nabla_x g(X(T), E[X(T)]) + E[\nabla_{\overline{x}} g(X(T), E[X(T)])]) + \sum_{i\geq 0} l_\xi(\tau_i, \xi_i)(\eta_i - \xi_i)\Big] \\ &= E\Big[\int_0^T \big[f(t, X^v(t), E[X^v(t)], v(t)) - f(t, X(t), E[X(t)], u(t))\big]dt \\ &\quad + (X^v(T) - X(T))^{\mathrm{T}} p(T) + \sum_{i\geq 0} l_\xi(\tau_i, \xi_i)(\eta_i - \xi_i)\Big]. \end{aligned}$$

Using Itô's formula to $(X^v(T) - X(T))^{\mathrm{T}} p(T)$, the definition of $H(\cdot)$, the convexity of $H(\cdot)$ and (12), we have

$$\begin{aligned} & J(v(\cdot), \eta(\cdot)) - J(u(\cdot), \xi(\cdot)) \\ &\geq E\Big[\int_0^T \Big\{H(t, X^v(t), E[X^v(t)], v(t), p(t), q(t), r(t)) - H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t))\big) \\ &\quad - (X^v(t) - X(t))^{\mathrm{T}}(\nabla_x H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t)) \\ &\quad + E[\nabla_{\overline{x}} H(t, X(t), E[X(t)], u(t), p(t), q(t), r(t))])\Big\}dt + \sum_{i\geq 0}[l_\xi(\tau_i, \xi_i) + p(\tau_i)C(\tau_i)](\eta_i - \xi_i)\Big] \geq 0. \end{aligned}$$

So $(u(\cdot), \xi(\cdot))$ is an optimal control. ∎

4 Application

Suppose we have two kinds of securities in the market for possible investment choices. A risk-free security (e.g., a bond), whose price $S_0(t)$ at time t is given by

$$\begin{cases} dS_0(t) = \rho(t)S_0(t)dt, \\ S_0(0) = S_0. \end{cases}$$

A risk security (e.g., a stock), whose price $S(t)$ at time t is given by

$$\begin{cases} dS(t) = S(t)\Big[\mu(t)dt + \sigma(t)dB(t) + \int_{R_0} \gamma(t,z)\widetilde{N}(dt,dz)\Big], \\ S(0) = S, \end{cases}$$

where $\rho(t) \le \mu(t)$; $\sigma : [0,T] \times R \times R \to R$; $\gamma : [0,T] \times R \times R \times R_0 \to R$. The wealthy dynamics follows

$$\begin{cases} dX(t) = (\rho(t)(X(t) - u(t)) + u(t)\mu(t))dt + u(t)\sigma(t)dB(t) + \int_{R_0} u(t)\gamma(t,z)\widetilde{N}(dt,dz) - d\xi(t), \\ X(0) = \beta. \end{cases}$$

$u(\cdot)$ is a portfolio strategy of agent and $X(t) = X^u(t)$ is the total wealth of the agent at time t corresponding to portfolio strategy $u(\cdot)$. $\xi(t) = \sum_{i\ge 0} \xi_i \mathbf{1}_{[\tau_i,T]}$ is a piecewise consumption process.

Assume that:

1) Short selling is allowed;

2) The trading strategies are self-financing.

The investor selects an investment strategy and a consumption strategy to minimize the variation and maximize the expected function. The cost functional is given by

$$J(u(\cdot),\xi(\cdot)) = \frac{a}{2}\mathrm{Var}[X(T)] - E[X(T)] + E\Big[\frac{S}{2}\sum_{0\le\tau_i\le T}\xi_i^2 + \int_0^T \frac{1}{2}Q(t)u^2(t)dt\Big],$$

where a is a constant, $Q(t)$ is a deterministic function.

$$J(u(\cdot),\xi(\cdot)) = E\Big[\int_0^T \frac{1}{2}Q(t)u^2(t)dt + \Big[\frac{a}{2}X^2(T) - X(T)\Big] - \frac{a}{2}[E[X(T)]]^2 + \frac{S}{2}\sum_{0\le\tau_i\le T}\xi_i^2\Big].$$

As we can see from (7), $g(x,\overline{x}) = \frac{a}{2}x^2 - x - \frac{a}{2}\overline{x}^2, l(\tau_i,\xi_i) = \frac{S}{2}\xi_i^2, f(t,x,\overline{x},u) = \frac{1}{2}Q(t)u^2(t)$.

g is not convex in $\overline{x}$, but we have the following corollary.

Corollary 4.1 *If the convex condition is satisfied in expected sense, i.e., the following inequality holds, for any* $X_1, X_2 \in L^2(\mathcal{F}_T, R^n)$, $E[g(X_1, E[X_1]) - g(X_2, E[X_2])] \le E\big[(X_1 - X_2)^{\mathrm{T}}\{\nabla_x g(X_1, E[X_1]) + \nabla_{\overline{x}} g(X_1, E[X_1])\}\big]$, *the maximum principle is still valid.*

Proof This proof can refer to Corollary 4.1 in [15]. ∎

Define the Hamiltonian equation:

$$H(t,x,u,p,q,r) = [\rho(t)(x-u) + u\mu(t)]p + u\sigma(t)q + \int_{R_0} u\gamma(t,z)\nu(dz)r + \frac{1}{2}Q(t)u^2.$$

The corresponding adjoint equation is as follows

$$\begin{cases} dp(t) = -\rho(t)p(t)dt + q(t)dB(t) + \int_{R_0} r(t,z)\widetilde{N}(dz,dt), \\ p(T) = aX(T) - 1 - aE[X(T)]. \end{cases}$$

Then, by Theorem 3.4 and Corollary 4.1 we have

$$u(t) = -\frac{[\mu(t)-\rho(t)]p(t) + \sigma(t)q(t) + \int_{R_0}\gamma(t,z)\nu(dz)r(t,z)}{Q(t)}, \tag{14}$$

$$\xi(t) = \frac{1}{S}\sum_{0\le\tau_i\le T} p(\tau_i). \tag{15}$$

We give three schematics of the solution $p(t)$, the continuous control $u(\cdot)$ and the impulse control $\xi(\cdot)$ as shown in Figures 2–4. The involved parameters are given by: $T=10, Q(t)=1, \mu(t)-\rho(t)=0.5, \rho(t)=\rho, q(t)=q, r(t)=r, \sigma(t)=\gamma(t)=0, S=-1$. We assume $\Delta\tau = 0.5; 1; 2, \tau_i = i\Delta\tau$ $(i=1,2,\cdots,\frac{10}{\Delta\tau}-1)$. (14) and (15) are simplified as follows:

$$u(t) = -0.5p(t), \quad \xi(t) = -\sum_{0\le\tau_i\le T} p(\tau_i).$$

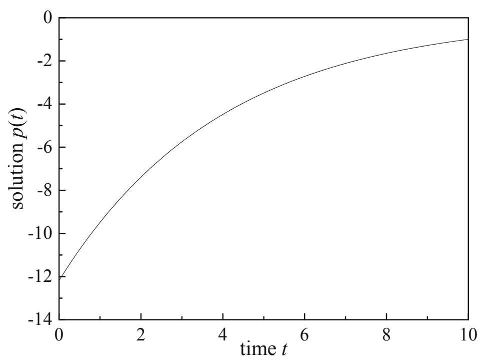

Figure 2 The solution of adjoint equation

From Figure 2, we observe that larger time t leads to larger values of $p(\cdot)$. On the other hand, as we can see from Figure 3, larger $\Delta\tau$ leads to smaller optimal consumption $\xi(\cdot)$. Figure 4 shows that larger time t leads to smaller portfolio $u(\cdot)$ in risky security.

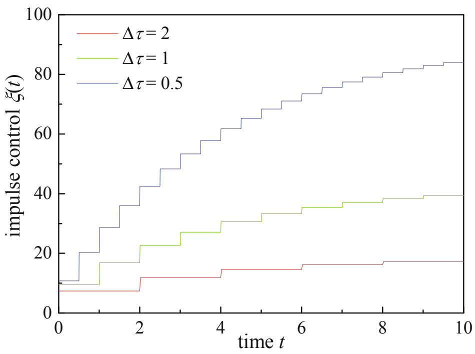

Figure 3 The optimal consumption $\xi(t)$ with different $\Delta\tau$

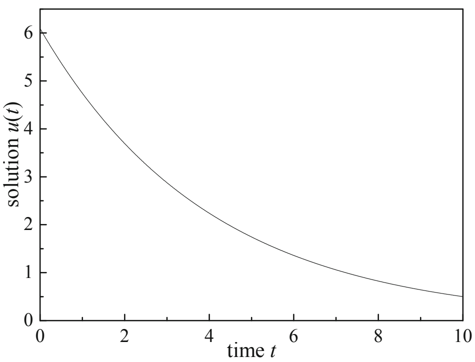

Figure 4 The optimal control $u(t)$

5 Conclusion

We have showed the existence and uniqueness of the solution to a jump-diffusion mean-field stochastic differential equation involving impulse control. We also give a necessary maximum principle and a sufficient verification theorem for the continuous control and impulse control. Finally, a mean-variance portfolio selection problem has been solved based on the theoretical results in this study.

In this paper, the control domain is assumed to be convex. It demands more complicated technique to solve non-convex domain case for the optimal control problems of jump-diffusion mean-field model involving impulse control. This will be studied in our future work.

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