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Guaranteed Cost Finite-Time Control of Fractional-Order Nonlinear Positive Switched Systems with D-Perturbations via MDADT*

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Abstract This paper is concerned with the problem of guaranteed cost finite-time control of fractional-order nonlinear positive switched systems (FONPSS) with D-perturbation. Firstly, the proof of the positivity of FONPSS with D-perturbation is given, the definition of guaranteed cost finite-time stability is firstly given in such systems. Then, by constructing linear copositive Lyapunov functions and using the mode-dependent average dwell time (MDADT) approach, a static output feedback controller is constructed, and sufficient conditions are derived to guarantee that the corresponding closed-loop system is guaranteed cost finite-time stable (GCFTS). Such conditions can be easily solved by linear programming. Finally, an example is provided to illustrate the effectiveness of the proposed method.

Keywords Finite-time stability, fractional-order nonlinear positive switched systems, guaranteed cost control, linear programming, mode-dependent average dwell time.

1 Introduction

Recently, positive switched systems have been paid much attention in control fields. Some remarkable results about such systems have been published in recent years[1−3]. However, these studies mainly focused on integer order derivative. In many practical applications, fractional derivative is more feasible than integer calculations and it is increasingly used to model the behavior of real systems, such as fractional-order circuit[4], fractional-order biological system[5], signal processing[6], electrical machines[7], and so on. During the last decades, fractional-order positive systems have been highlighted by many researchers, and some meaningful results have

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been obtained[8−11]. Among them, only a few results were concerned with fractional-order positive switched systems (FOPSS)[10,11]. [10] studied the controllability of FOPSS for fixed switching sequence. [11] discussed the problem of state-dependent switching control of FOPSS. The above results were involved in asymptotic stability, which dealt with the behavior of a system within a sufficiently long (or in infinite) time interval.

Nevertheless, in fact, it is necessary to maintain the state under some bounds during, at least, a specific time interval. In [12], Doroto has firstly proposed finite time stability (FTS) for linear deterministic systems, which means that, given a bound on the initial condition, the system state does not exceed a certain threshold during a specified time interval. There have been some meaningful results about FTS of positive switched systems[13−17]. Among these results, it should be pointed out that only [16, 17] have investigated the FTS problem of fractional-order linear positive switched systems (FOLPSS) with average dwell time (ADT). The main reason is that fractional order systems have some different features compared with ordinary systems, such as the solution process and stability criterion, which directly result in biggish difficulty of the research of FOPSS. These make the FTS analysis and controller design of FONPSS interesting but full of challenge.

On the other hand, when designing controllers, it is desirable to ensure a satisfactory system performance within a specify time interval. One possible approach to this problem is the guaranteed cost finite-time control. It has the advantage of providing an upper bound on a given system performance index and thus the system performance degradation incurred by the uncertainties or time delays is guaranteed to be less than this bound[18,19]. So it is necessary to study the design problem of guaranteed cost finite-time controller[20,21]. For FOPSS, only [17] considered the guaranteed cost finite-time control of FOLPSS with ADT. As we know, compared with FOLPSS, the theory of FONPSS is less developed, because it is difficult to define the positivity of an FONPSS, and few effective control techniques with respect to such systems are proposed. Furthermore, compared with ADT approach, MDADT technique allows that every subsystem has its own ADT to make the individual properties of each subsystem unneglected, which is more applicable and less conservative. When the nonlinearity and MDADT happen simultaneously in the FOPSS, the guaranteed cost finite-time controller design problem will be more complex.

Moreover, in reality, due to the existence of external disturbance, tool wearing, modeling error, and parameter fluctuation during hardware implementation, almost all the systems contain perturbations. A typical perturbation for positive switched systems is D-perturbation[22]. It may lead to system performance deterioration, even instability. Therefore, the effect of D-perturbation must be taken into account in analyzing and implementing finite time controller scheme.

Motivated by the above discussions, in this paper, the problem of guaranteed cost finite-time control of FONPSS with D-perturbation via MDADT is investigated. The main contribution of this paper can be summarized as follows: (i) The proof of the positivity of FONPSS with D-perturbation is given. (ii) A new cost function is firstly proposed, which can utilize the characteristics of nonnegative states of FONPSS. Then the definition of guaranteed cost finite time stability is also given. (iii) By using co-positive type Lyapunov function and MDADT approach, a static output feedback controller is designed and sufficient conditions are obtained to guarantee the corresponding closed-loop system is GCFTS. Such conditions can be easily solved by linear programming.

The rest of this paper is organized as follows. In Section 2, problem statements and necessary lemmas are given. Main results are given in Section 3. A numerical example is provided in Section 4. Section 5 concludes this paper.

Notations Throughout this paper, $A \succ 0$ ($\succeq 0, \prec 0, \preceq 0$) means that $a_{ij} > 0$ ($\geq 0, < 0, \leq 0$), which is applicable to a vector. $A \succ B$ ($A \succeq B$) means that $A-B \succ 0$ ($A-B \succeq 0$); The symbols R, R^n, and $R^{n\times n}$ denote the set of real numbers, the space of the vectors of n-tuples of real numbers, the space of $n \times n$ matrices with real numbers, respectively. R^n_+ is the n-dimensional non-negative (positive) vector space. Matrix $D \in [\underline{D}, \overline{D}]$ means that $d_{ij} \in [\underline{d}_{ij}, \overline{d}_{ij}]$. A^{T} denotes the transpose of matrix A. $\emptyset$ denotes an empty set. I represents the identity matrix. Matrices are assumed to have compatible dimensions for calculating if their dimensions are not explicitly stated.

2 Preliminaries and Problem Statements

2.1 Fractional-Order Calculus

Fractional-order integral or derivative plays an important role in modern science. There are three commonly used definitions of the fractional-order integro-differential operator: Grunwald-Letnikov, Riemann-Liouville and Caputo definitions. In this paper, we mainly use Riemann-Liouville and Caputo fractional order operators for our study. The uniform formula of a fractional integral with $\alpha \in (0,1)$ is defined as

$$ {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{1} $$

where $\Gamma(\alpha)$ denotes the Gamma function with non-integer arguments, $f(t)$ is an arbitrary integrable function. For $0 < \alpha < 1$, the Riemann-Liouville (RL) definition of fractional derivatives is given as

$$ {}^{RL}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{2} $$

and Caputo definition of fractional derivatives is given as

$$ {}^{C}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{3} $$

where $f(t)$ is an arbitrary integrable function, ${}_{t_0}D_t^{-\alpha}$ is the fractional integral of order α on $[t_0, t]$, $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-t} t^{\alpha-1} dt$. ${}^{RL}_{t_0}D_t^{\alpha}$ and ${}^{RL}_{t_0}D_t^{\alpha}$ represent Riemann-Liouville and Caputo fractional derivatives of order α of $f(t)$ on $[t_0, t]$, respectively. We mainly use these two fractional-order operators in this paper. From the above two definitions, we can obtain the following relation between them:

$$ {}^{RL}_{t_0}D_t^{\alpha} f(t) = {}^{C}_{t_0}D_t^{\alpha} f(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(t_0), \tag{4} $$

Lemma 2.1 (see [16]) *Let $\alpha \in (0,1)$, if $f(0) \geq 0$, then* ${}_{t_0}^{RL}D_t^{\alpha} f(t) \leq {}_{t_0}^{C}D_t^{\alpha} f(t)$.

2.2 Fractional-Order Nonlinear Positive Switched Systems

Consider the following FONPSS:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = D_1 A_{\sigma(t)} f(x(t)) + D_2 B_{\sigma(t)} u(t), \\ y(t) = D_3 C_{\sigma(t)} f(x(t)), \end{cases} \tag{5}$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ and $y(t) \in R^s$ represent the control input and output, respectively. $f(x) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))^{\mathrm{T}} \in R^n$, where $f_i(x_i)$ is a class of continuous function. ${}_{t_0}^{C}D_t^{\alpha}$ denotes Caputo fractional-order derivative. $\sigma(t) : [0, \infty) \to \underline{S} = \{1, 2, \cdots, S\}$ is the switching signal, and it is continuous from the right everywhere for a switching, S is the number of subsystems; Perturbations $D_1 \in [\underline{D}_1, \overline{D}_1]$, $D_2 \in [\underline{D}_2, \overline{D}_2]$ and $D_3 \in [\underline{D}_3, \overline{D}_3]$ with $\overline{D}_1 \succeq \underline{D}_1 \succeq 0$, $\overline{D}_2 \succeq \underline{D}_2 \succeq 0$ and $\overline{D}_3 \succeq \underline{D}_3 \succeq 0$, where matrices $\overline{D}_1, \underline{D}_1, \overline{D}_2, \underline{D}_2, \overline{D}_3, \underline{D}_3$ are all diagonal. $\forall p \in \underline{S}$, A_p, B_p and C_p are constant matrices with appropriate dimensions, p denotes the pth systems and t_q denotes the qth switching instant.

Next, we will present some definitions, lemmas and inequalities for the FONPSS (5) for our further study.

Definition 2.2 (see [17]) The system (5) is said to be positive if for any switching signals $\sigma(t)$, any initial conditions $x(t_0) \succeq 0$, the corresponding trajectory satisfies $x(t) \succeq 0$ and $y(t) \succeq 0$ for all $t \succeq 0$.

Definition 2.3 (see [13]) A matrix A is called a Metzler matrix if the off-diagonal entries of matrix A are non-negative.

Definition 2.4 (see [14]) For any switching signal $\sigma(t)$ and any $t_2 \geq t_1 \geq 0$, let $N_{\sigma p}(t_1, t_2)$ denote the switching numbers that the pth subsystem is activated over the interval $[t_1, t_2)$ and $T_p(t_1, t_2)$ denote the total running time of the pth subsystem over the interval $[t_1, t_2)$. If there exist $N_{0p} \geq 0$ and $T_{\omega p} > 0$ such that

$$N_{\sigma p}(t_1, t_2) \leq N_{0p} + \frac{T_p(t_1, t_2)}{T_{\omega p}}, \quad \forall\, t_2 \geq t_1 \geq 0, \quad \forall\, p \in \underline{S}, \tag{6}$$

then $T_{\omega p}$ and N_{0p} are called MDADT and mode-dependent chattering bounds, respectively. Generally, we choose $N_{0p} = 0$.

Assumption 2.5 The nonlinear continuous function $f(x)$ lies in sector fields satisfying

$$m_1 x_i^2 \leq f_i(x_i) x_i \leq m_2 x_i^2 \tag{7}$$

for $x_i \in R$ and $i = 1, 2, \cdots, n$, where $0 < m_1 \leq m_2$, $f_i(0) = 0$ and $\forall x_i \geq 0, f(x_i) \geq 0$.

Remark 2.6 The system model (5) is a more general form. Especially, if $m_1 = m_2 = 1$ (it means $f_i(x_i) = x_i$) and $D_1 = D_2 = D_3 = I$, then the system (5) is transformed into FOPSS, such as [11, 16, 17].

Lemma 2.7 *The system* (5) *is positive if and only if* $D_1 A_p$, $\forall p \in \underline{S}$ *are Metzler matrices and* $\forall p \in \underline{S}$, $D_2 B_p \succeq 0$, $D_3 C_p \succeq 0$.

Lemma 2.8 *Let* $\overline{A}_p = D_1 A_p$, $\overline{B}_p = D_2 B_p$, $\overline{C}_p = D_3 C_p$, D_i *are all diagonal and* $D_i \succeq 0$, $i = 1, 2, 3$. *We have the following facts:*

1) A_p *are Metzler matrices* $\Rightarrow \overline{A}_p$ *are Metzler matrices,* $\forall p \in \underline{S}$.

2) B_p(*or* C_p) $\succeq 0 \Rightarrow \overline{B}_p$(*or* $\overline{C}_p$) $\succeq 0$, $\forall p \in \underline{S}$.

Proof 1) A_p are Metzler matrices, according to Definition 2.3, we have $a_{pjk} = \begin{cases} <0, & j=k, \\ \geq 0, & j \neq k, \end{cases}$ $p \in \underline{S}, j, k = 1, 2, \cdots, n$. $D_i \succeq 0$ and D_i are all diagonal. We can easily get $d_{ijk} = \begin{cases} >0, & j=k, \\ 0, & j \neq k, \end{cases}$ $i = 1, 2, 3$, where $a_{pjk}(d_{ijk})$ is in the jth row and kth column of $A_p(D_i)$. Therefore, $\overline{a}_{pjk} = d_{1jk} \cdot a_{pjk} = \begin{cases} <0, & j=k, \\ \geq 0, & j \neq k, \end{cases}$ where $\overline{a}_{pjk}$ is in the jth row and kth column of $\overline{A}_p$. According to Definition 2.3, $\overline{A}_p$ are also Metzler matrices.

2) $B_p \succeq 0$, we have $b_{pjk} \geq 0$, where b_{pjk} is in the jth row and kth column of B_p, $p \in \underline{S}, j, k = 1, 2, \cdots, n$. $\overline{b}_{2jk} = d_{2jk} \cdot b_{pij} \geq 0, \forall j, k = 1, 2, \cdots, n$, so we can conclude $\overline{B_p}$ are nonnegative. Similar to the above process, $\overline{C}_p$ are also nonnegative. ∎

Therefore, the system (5) can be rewritten as:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = \overline{A}_{\sigma(t)} f(x(t)) + \overline{B}_{\sigma(t)} u(t), \\ y(t) = \overline{C}_{\sigma(t)} f(x(t)). \end{cases} \tag{8}$$

Lemma 2.9 *The system* (8) *is positive under any switching signals if and only if* $\overline{A}_p$ *are Metzler matrices,* $\overline{B}_p \succeq 0$ *and* $\overline{C}_p \succeq 0$, $\forall p \in \underline{S}$.

Proof The proof of Lemma 2.9 is similar to the Theorem 1 in [8], the proof process is omitted. ∎

From the above, we can conclude the system (8) is positive. In other words, we can conclude the system (5) is positive.

Definition 2.10 (see [15]) For given time constant T_f and vectors $\delta \succ \varepsilon \succ 0$, the system (5) is said to be FTS with respect to (δ, ε, T_f, $\sigma(t)$), if

$$x^{\mathrm{T}}(t_0)\delta \leq 1 \Rightarrow x^{\mathrm{T}}(t)\varepsilon \leq 1, \quad \forall\, t \in [0, T_f]. \tag{9}$$

Definition 2.11 Define the cost function of the system (5) as follows:

$$\begin{aligned} J &= {}_0D_{T_f}^{-\alpha}(x^{\mathrm{T}}(t)R_1 + u^{\mathrm{T}}(t)R_2)dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T_f} (T_f - t)^{\alpha-1}(x^{\mathrm{T}}(t)R_1 + u^{\mathrm{T}}(t)R_2)dt, \end{aligned} \tag{10}$$

where $R_1 \succ 0$ and $R_2 \succ 0$ are two given vectors.

Remark 2.12 It should be noted that the proposed cost function is different from the non-positive systems[18−20], it is for the first time introduced in FONPSS. This definition provides a more useful description, because it takes full advantage of the characteristics of nonnegative states of FONPSS. Especially, if $\alpha = 1$, this definition is turned into the definition of cost function in positive switched system[21].

Now we gives the definition of GCFTS for the FONPSS (5).

Definition 2.13 (see [17]) For a given time constant T_f and vectors $\delta \succ \varepsilon \succ 0$. Consider the system (5), if there exist a feedback control law $u(t)$ and a positive scalar J^* such that the closed-loop system is FTS with the respect to $(\delta, \varepsilon, T_f, \sigma(t))$ and the cost function satisfies $J \le J^*$, then the closed-loop FONPSS is called GCFTS, where J^* is a guaranteed cost value and $u(t)$ is a guaranteed cost finite time controller.

Lemma 2.14 (C_p inequality) *For $0 < a < 1$ and any positive real numbers $x_1, x_2, \cdots, x_k$,*

$$\sum_{k=1}^{n} x_k^a \le n^{1-a}\Big(\sum_{k=1}^{n} x_k\Big)^a.$$

The aim of this paper is to design a static output feedback controller $u(t) = K_{\sigma(t)}y(t)$ and a class of switching signals $\sigma(t)$ for FONPSS (5) such that the corresponding closed-loop system is GCFTS.

3 Main Results

3.1 Guaranteed Cost Finite-Time Stability Analysis

In this subsection, we will focus on the problem of GCFTS for NFOPSS (5) with $u(t) \equiv 0$.

Theorem 3.1 *Consider the system (5) with $u(t) \equiv 0$. Given positive constants T_f, λ_p, vectors $\delta \succ \varepsilon \succ 0$ and $R_1 \succ 0$, if there exist positive constants ξ_1, ξ_2, μ_p, and positive vectors v_p, $p \in \underline{S}$, such that the following inequalities hold:*

$$m_2 A_p^{\mathrm{T}} \overline{D}_1 v_p + R_1 \preceq \lambda_p v_p, \tag{11}$$

$$\xi_1 \varepsilon \prec v_p \prec \xi_2 \delta, \tag{12}$$

$$v_p \preceq \mu_p v_q, \tag{13}$$

$$v_p \prec R_1, \tag{14}$$

$$\frac{[\alpha T_f(\lambda - 1)]}{\Gamma(\alpha + 1)} < \ln \frac{\xi_1}{\xi_2}, \tag{15}$$

where $\forall p \in \underline{S}$, $v_p = [v_{p1}, v_{p2}, \cdots, v_{pn}]^{\mathrm{T}}$, $\lambda = \max_{p\in\underline{S}}\{\lambda_p\}$, $T_\omega = \min_{p\in\underline{S}}\{T_{\omega p}\}$, $\lambda_p > 1, \mu_p \ge 1$, then under the following MDADT scheme

$$T_\alpha > T_\alpha^* = T_f\bigg(\ln \mu_p + \frac{(\lambda - 1)(1 - \alpha)}{\Gamma(\alpha + 1)}\bigg)\bigg/\bigg(\ln \frac{\xi_1}{\xi_2} - \frac{[\alpha T_f(\lambda - 1)]}{\Gamma(\alpha + 1)}\bigg), \tag{16}$$

the FONPSS (5) is GCFTS with respect to $(\delta, \varepsilon, T_f, \sigma(t))$ and the guaranteed cost value of the system (5) with $u(t) = 0$ is given by

$$\begin{aligned} J &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(x^{\mathrm{T}}(s)R_1 + u^{\mathrm{T}}(s)R_2)ds \\ &\le J^* \\ &= \xi_2 \mu^{\frac{T_f}{T_\omega}} + \lambda \xi_2 \mu^{\frac{T_f}{T_\omega}} \frac{1}{\alpha\Gamma(\alpha)} \exp\bigg\{\frac{T_f}{T_\omega}\bigg(\ln\mu + \frac{(1-\alpha)(\lambda-1)}{\Gamma(\alpha+1)}\bigg) \\ &\quad +(\lambda - 1)\frac{(1-\alpha) + \alpha T_f}{\Gamma(\alpha + 1)}\bigg\}(T_\omega)^{\alpha - 1} T_f. \end{aligned} \tag{17}$$

Proof Constructing the multiple linear type Lyapunov-Krasovskii functional for the system (5) as follows:

$$V_{\sigma(t)}(t,x(t)) = x^{\mathrm{T}}(t)v_{\sigma(t)}, \tag{18}$$

where $v_p \in R_+^n$, $\forall p \in \underline{S}$.

Denote $t_0, t_1, \cdots$ as the switching instants over the interval $[0, T_f]$. Along the trajectory of the system (5) with $u(t) \equiv 0$, we have

$${}_{t_0}^{C}D_t^{\alpha} V_{\sigma(t)}(t,x(t)) = f^{\mathrm{T}}(x(t))A_{\sigma(t)}^{\mathrm{T}} D_1 v_{\sigma(t)}. \tag{19}$$

From (11), we obtain

$$\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha} V_{\sigma(t)}(t,x(t)) + x^{\mathrm{T}}(t)R_1 \\
&= f^{\mathrm{T}}(x(t))A_{\sigma(t)}^{\mathrm{T}} D_1 v_{\sigma(t)} + x^{\mathrm{T}}(t)R_1 \\
&\le m_2 x^{\mathrm{T}}(t)A_{\sigma(t)}^{\mathrm{T}} \overline{D}_1 v_{\sigma(t)} + x^{\mathrm{T}}(t)R_1 \\
&\le \lambda_p x^{\mathrm{T}}(t)v_{\sigma(t)} \\
&\le \lambda_p V_{\sigma(t)}(t,x(t)).
\end{aligned} \tag{20}$$

Taking the fractional integral ${}_{t_0}^{C}D_t^{-\alpha}$ to both sides of (20) during the period $[t_k, t)$ for $t \in [t_k, t_{k+1})$ leads to

$$\begin{aligned}
&V_{\sigma(t)}(t,x(t)) \\
&\le V_{\sigma(t_k)}(t_k,x(t_k)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1 ds.
\end{aligned} \tag{21}$$

From (14) and (18), (21) can be rewritten as

$$\begin{aligned}
&V_{\sigma(t)}(t,x(t)) \\
&\le V_{\sigma(t_k)}(t_k,x(t_k)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1 ds \\
&\le V_{\sigma(t_k)}(t_k,x(t_k)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}x^{\mathrm{T}}(s)v_{\sigma(t)} ds \\
&= V_{\sigma(t_k)}(t_k,x(t_k)) + \frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds.
\end{aligned} \tag{22}$$

According to the properties of Gamma function $\Gamma(\alpha)$ and the Gronwall-bellman inequality in [6], for $t \in [t_k, t_{t_k+1})$, we have

$$\begin{aligned} V_{\sigma(t)}(t,x(t)) &\le V_{\sigma(t_k)}(t_k,x(t_k)) + \frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds \\ &\le V_{\sigma(t_k)}(t_k,x(t_k))\exp\left\{\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds\right\} \\ &= V_{\sigma(t_k)}(t_k,x(t_k))\exp\left\{\frac{\lambda_{\sigma(t_k)}-1}{\alpha\Gamma(\alpha)}(t-t_k)^{\alpha}\right\} \\ &= V_{\sigma(t_k)}(t_k,x(t_k))\exp\left\{\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right\}. \end{aligned} \tag{23}$$

For $t \in [t_k, t_{k+1})$, $V_{\sigma(t_k)} \le \mu_{\sigma(t_k)}V_{\sigma(t_k^-)}(t_k^-, x(t_k^-))$, is easily obtained from (13) and (18). According to $\exp\{\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\} > 0$, we have

$$V_{\sigma(t)}(t,x(t)) \le \mu_{\sigma(t_k)}V_{\sigma(t_k^-)}(t_k^-,x(t_k^-))\exp\left\{\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right\}. \tag{24}$$

Then, for $t \in [t_0, T_f)$, we obtain from (23) and (24) that

$$\begin{aligned} V_{\sigma(t)}(t,x(t)) &\le V_{\sigma(t_k)}(t_k,x(t_k))\exp\left\{\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right\} \\ &\le \mu_{\sigma(t_k)}V_{\sigma(t_k^-)}(t_k^-,x(t_k^-))\exp\left\{\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right\} \\ &\le \mu_{\sigma(t_k)}V_{\sigma(t_{k-1})}(t_{k-1},x(t_{k-1}))\exp\left\{\left[\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right.\right. \\ &\quad \left.\left.+\frac{\lambda_{\sigma(t_{k-1})}-1}{\Gamma(\alpha+1)}(t_k-t_{k-1})^{\alpha}\right]\right\} \\ &\le \mu_{\sigma(t_k)}\mu_{\sigma(t_{k-1})}V_{\sigma(t_{k-1}^-)}(t_{k-1}^-,x(t_{k-1}^-)))\exp\left\{\left[\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right.\right. \\ &\quad \left.\left.+\frac{\lambda_{\sigma(t_{k-1})}-1}{\Gamma(\alpha+1)}(t_k-t_{k-1})^{\alpha}\right]\right\} \le \cdots \\ &\le \left(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\right)V_{\sigma(t_0)}(t_0,x(t_0))\exp\left\{\left[\frac{\lambda_{\sigma(t_k)}-1}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}\right.\right. \\ &\quad \left.\left.+\frac{\lambda_{\sigma(t_{k-1})}-1}{\Gamma(\alpha+1)}(t_k-t_{k-1})^{\alpha}+\cdots+\frac{\lambda_{\sigma(t_0)}-1}{\Gamma(\alpha+1)}(t_1-t_0)^{\alpha}\right]\right\}. \end{aligned} \tag{25}$$

Let $\lambda = \max_{p\in\underline{S}}\{\lambda_p\}$, we have

$$\begin{aligned} V_{\sigma(t)}(t,x(t)) \le & \left(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\right)V_{\sigma(t_0)}(t_0,x(t_0)) \\ & \cdot\exp\left\{\frac{\lambda-1}{\Gamma(\alpha+1)}[(t-t_k)^{\alpha}+(t_k-t_{k-1})^{\alpha}+\cdots+(t_1-t_0)^{\alpha}]\right\}. \end{aligned} \tag{26}$$

From Definition 2.8 and Lemma 2.14, for $t \in [0, T_f]$, we have

$$
\begin{aligned}
V_{\sigma(t)}(t, x(t)) \leq & \left(\prod_{p=1}^{S} \mu_p^{N_{\sigma p}(0,t)} \right) V_{\sigma(t_0)}(t_0, x(t_0)) \exp\left\{ \frac{\lambda - 1}{\Gamma(\alpha+1)} [(t - t_k)^\alpha \right. \\
& \left. + (t_k - t_{k-1})^\alpha + \cdots + (t_1 - t_0)^\alpha] \right\} \\
\leq & \left(\prod_{p=1}^{S} \mu_p^{\frac{T_p(0,t)}{T_{\omega p}}} \right) V_{\sigma(t_0)}(t_0, x(t_0)) \exp\left\{ \frac{\lambda - 1}{\Gamma(\alpha+1)} [(t - t_k)^\alpha \right. \\
& \left. + (t_k - t_{k-1})^\alpha + \cdots + (t_1 - t_0)^\alpha] \right\} \\
\leq & \, \mathrm{e}^{\sum_{p=1}^{S} \frac{\ln \mu_p}{T_{\omega p}} T_p(0,t)} V_{\sigma(t_0)}(t_0, x(t_0)) \exp\left\{ \frac{\lambda - 1}{\Gamma(\alpha+1)} [(t - t_k)^\alpha \right. \\
& \left. + (t_k - t_{k-1})^\alpha + \cdots + (t_1 - t_0)^\alpha] \right\} \\
\leq & \, \mathrm{e}^{\sum_{p=1}^{S} \frac{\ln \mu_p}{T_{\omega p}} T_p(0,T_f)} V_{\sigma(0)}(0, x(0)) \exp\left\{ \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[\frac{T_p(0, T_f)}{T_{\omega p}}^{1-\alpha} T_f^\alpha \right] \right\} \\
= & \, V_{\sigma(0)}(0, x(0)) \exp\left\{ \sum_{p=1}^{S} \frac{\ln \mu_p}{T_{\omega p}} T_p(0, T_f) + \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[\frac{T_p(0, T_f)}{T_{\omega p}}^{1-\alpha} T_f^\alpha \right] \right\}. \quad (27)
\end{aligned}
$$

According to Young's inequality in [16], (27) can be rewritten as

$$
\begin{aligned}
& V_{\sigma(t)}(t, x(t)) \\
\leq & \, V_{\sigma(t_0)}(t_0, x(t_0)) \exp\left\{ \sum_{p=1}^{S} \frac{\ln \mu_p}{T_{\omega p}} T_p(0, t) + \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[(1 - \alpha) \frac{T_p(0, t)}{T_{\omega p}} + \alpha T_f \right] \right\} \\
\leq & \, V_{\sigma(t_0)}(t_0, x(t_0)) \exp\left\{ \sum_{p=1}^{S} \frac{\ln \mu_p}{T_{\omega p}} T_p(0, t) + \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[(1 - \alpha) \frac{\ln \mu_p}{T_{\omega p}} \frac{T_p(0, t)}{\ln \mu_p} + \alpha T_f \right] \right\}. \quad (28)
\end{aligned}
$$

Let $\beta = \max_{p \in \underline{S}} \{ \frac{\ln \mu_p}{T_{\omega p}} \}$, we have

$$
V_{\sigma(t)}(t, x(t)) \leq V_{\sigma(t_0)}(t_0, x(t_0)) \exp\left\{ \beta T_f + \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[(1 - \alpha) \beta \frac{T_f}{\ln \mu_p} + \alpha T_f \right] \right\}. \quad (29)
$$

From (11), (18) and (29), for $t \in [0, T_f)$, we have

$$
V_{\sigma(t)}(t, x(t)) \geq \xi_1 x^{\mathrm{T}}(t) \varepsilon, \quad (30)
$$

$$
V_{\sigma(t)}(t, x(t)) \leq \xi_2 x^{\mathrm{T}}(t_0) \delta \exp\left\{ \beta T_f + \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[(1 - \alpha) \beta \frac{T_f}{\ln \mu_p} + \alpha T_f \right] \right\}. \quad (31)
$$

Combining (30) with (31), we obtain

$$
x^{\mathrm{T}}(t) \varepsilon \leq \frac{\xi_2}{\xi_1} \{ x^{\mathrm{T}}(t_0) \delta \} \exp\left\{ \beta T_f + \frac{\lambda - 1}{\Gamma(\alpha+1)} \left[(1 - \alpha) \beta \frac{T_f}{\ln \mu_p} + \alpha T_f \right] \right\}. \quad (32)
$$

Substituting (16) into (32), one has

$$
x^{\mathrm{T}}(t) \varepsilon < 1. \quad (33)
$$

From Definition 2.10, we conclude that the system (5) with $u(t)=0$ is FTS with respect to $(\delta,\varepsilon,T_f,\sigma(t))$.

Next, we will give the fractional-order guaranteed cost value of the system (5).

According to (21) and $V_{\sigma(t_k)}\leq\mu_{\sigma(t_k)}V_{\sigma(t_k^-)}(t_k^-,x(t_k^-))$, for $t\in[t_k,t_{k+1})$, we can obtain

$$\begin{aligned}
&V_{\sigma(t)}(t,x(t))\\
&\leq\mu_{\sigma(t_k)}V_{\sigma(t_k^-)}(t_k^-,x(t_k^-))+\frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds\\
&\leq\mu_{\sigma(t_k)}V_{\sigma(t_{k-1})}(t_{k-1},x(t_{k-1}))+\frac{\lambda_{\sigma(t_{k-1})}\mu_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds\\
&\quad-\frac{\mu_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds+\frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds\leq\cdots\\
&\leq\left(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\right)V_{\sigma(t_0)}(t_0,x(t_0))+\frac{\lambda_{\sigma(t_0)}(\prod_{i=1}^{k}\mu_{\sigma(t_i)})}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds\\
&\quad-\frac{(\prod_{i=1}^{k}\mu_{\sigma(t_i)})}{\Gamma(\alpha)}\int_{t_0}^{t_1}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds\\
&\quad+\frac{\lambda_{\sigma(t_1)}(\prod_{i=1}^{k-1}\mu_{\sigma(t_i)})}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds\\
&\quad-\frac{(\prod_{i=1}^{k-1}\mu_{\sigma(t_i)})}{\Gamma(\alpha)}\int_{t_1}^{t_2}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds+\cdots\\
&\quad+\frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds.
\end{aligned}\tag{34}$$

From (31), let $\varpi=\xi_2\exp\{\beta T_f+\frac{\lambda-1}{\Gamma(\alpha+1)}[(1-\alpha)\beta\frac{T_f}{\ln\mu}+\alpha T_f]\}$, $\mu=\min_{p\in\underline{S}}\{\mu_p\}$, $\lambda=\max_{p\in\underline{S}}\{\lambda_p\}$. For $t\in[0,T_f]$, then (34) can be turned into

$$\begin{aligned}
V_{\sigma(t)}(t,x(t))&\leq\left(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\right)V_{\sigma(t_0)}(t_0,x(t_0))-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{T_f}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds\\
&\quad+\frac{\lambda\prod_{i=1}^{k}\mu_{\sigma(t_i)}}{\Gamma(\alpha)}\Big[\int_{t_k}^{T_f}(T_f-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds+\cdots\\
&\quad+\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}V_{\sigma(t)}(s,x(s))ds\Big]\\
&\leq\left(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\right)V_{\sigma(t_0)}(t_0,x(t_0))-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{T_f}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds\\
&\quad+\frac{\lambda\prod_{i=1}^{k}\mu_{\sigma(t_i)}}{\Gamma(\alpha)}\varpi\Big[\int_{t_k}^{T_f}(T_f-s)^{\alpha-1}ds+\cdots+\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\Big]\\
&=\left(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\right)V_{\sigma(t_0)}(t_0,x(t_0))-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{T_f}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds\\
&\quad+\frac{\lambda\prod_{i=1}^{k}\mu_{\sigma(t_i)}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}[(T_f-t_k)^{\alpha}+\cdots+(t_2-t_1)^{\alpha}+(t_1-t_0)^{\alpha}].
\end{aligned}\tag{35}$$

From $V_{\sigma(t)}(t,x(t)) \geq 0$ and Lemma 2.14, we can get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^{T_f}(t-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds \\
&\leq \Big(\prod_{i=1}^{k}\mu_{\sigma(t_i)}\Big)V_{\sigma(0)}(0,x(0)) \\
&\quad+\frac{\lambda\prod_{i=1}^{k}\mu_{\sigma(t_i)}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}[(T_f-t_k)^\alpha+\cdots+(t_2-t_1)^\alpha+(t_1-t_0)^\alpha] \\
&\leq \Big(\prod_{p=1}^{S}\mu_p^{N_{\sigma p}(0,T_f)}\Big)V_{\sigma(0)}(0,x(0))+\frac{\lambda\prod_{p=1}^{S}\mu_p^{N_{\sigma p}(0,T_f)}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}\Big(\frac{T_f}{T_{\omega p}}\Big)^{1-\alpha}T_f^\alpha \\
&\leq \Big(\prod_{p=1}^{S}\mu_p^{\frac{T_p(0,t)}{T_{\omega p}}}\Big)V_{\sigma(0)}(0,x(0))+\frac{\lambda\prod_{p=1}^{S}\mu_p^{\frac{T_p(0,t)}{T_{\omega p}}}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}\Big(\frac{T_f}{T_{\omega p}}\Big)^{1-\alpha}T_f^\alpha \\
&\leq \mathrm{e}^{\sum_{p=1}^{S}\frac{ln\mu_p}{T_{\omega p}}T_p(0,t)}V_{\sigma(0)}(0,x(0))+\frac{\lambda\exp\{\sum_{p=1}^{S}\frac{ln\mu_p}{T_{\omega p}}T_p(0,t)\}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}\Big(\frac{T_f}{T_{\omega p}}\Big)^{1-\alpha}T_f^\alpha \\
&\leq \mathrm{e}^{\beta T_f}V_{\sigma(0)}(0,x(0))+\frac{\lambda\mathrm{e}^{\beta T_f}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}(T_{\omega p})^{\alpha-1}T_f.
\end{aligned}\tag{36}$$

From (31) and (36), let $T_\omega=\min_{p\in\underline{S}}\{T_{\omega p}\}$, we have

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_{t_0}^{T_f}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds &\leq \mathrm{e}^{\beta T_f}\xi_2x^{\mathrm{T}}(t_0)\delta+\frac{\lambda\mathrm{e}^{\beta T_f}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}(T_{\omega p})^{\alpha-1}T_f \\
&\leq \xi_2\mathrm{e}^{\beta T_f}+\lambda\frac{\mathrm{e}^{\beta T_f}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}(T_\omega)^{\alpha-1}T_f.
\end{aligned}\tag{37}$$

Then, the guaranteed cost value of System (5) with $u(t)=0$ is given by

$$\begin{aligned}
J&=\frac{1}{\Gamma(\alpha)}\int_0^{T_f}(t-s)^{\alpha-1}(x^{\mathrm{T}}(s)R_1+u^{\mathrm{T}}(s)R_2)ds \\
&\leq J^* \\
&=\xi_2\mathrm{e}^{\beta T_f}+\lambda\frac{\mathrm{e}^{\beta T_f}}{\Gamma(\alpha)}\frac{\varpi}{\alpha}(T_\omega)^{\alpha-1}T_f \\
&\leq \xi_2\mu^{\frac{T_f}{T_\omega}}+\lambda\xi_2\mu^{\frac{T_f}{T_\omega}}\frac{1}{\alpha\Gamma(\alpha)}\exp\Big\{\frac{T_f}{T_\omega}\Big(\ln\mu+\frac{(1-\alpha)(\lambda-1)}{\Gamma(\alpha+1)}\Big) \\
&\quad+(\lambda-1)\frac{(1-\alpha)+\alpha T_f}{\Gamma(\alpha+1)}\Big\}(T_\omega)^{\alpha-1}T_f.
\end{aligned}\tag{38}$$

According to Definition 2.13, we can conclude that the system (5) is GCFTS with respect to $(\delta,\varepsilon,T_f,\sigma(t))$. Thus, the proof is completed. ∎

Remark 3.2 To achieve guaranteed cost value (38), in (34), we can make t increase to T_f, that is, $-\frac{1}{\Gamma(\alpha)}[\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds+\cdots+\int_{t_k}^{T_f}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds] < -\frac{1}{\Gamma(\alpha)}[\int_{t_0}^{t_1}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds+\int_{t_1}^{t_2}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds+\cdots+\int_{t_k}^{T_f}(T_f-s)^{\alpha-1}x^{\mathrm{T}}(s)R_1ds] = -\frac{1}{\Gamma(\alpha)}\int_{t_0}^{T_f}(T_f-s)x^{\mathrm{T}}(s)R_1ds=-J$. It means guaranteed cost finite-time control of FONPSS is feasible. Instead, if $T_f\to\infty$, then the integral items can not be dealt with, the guaranteed cost control of FONPSS can not be carried out.

Corollary 3.3 *Replace* ${}_{t_0}^{C}D_t^{\alpha}x(t)$ *by* ${}_{t_0}^{RL}D_t^{\alpha}x(t)$ *in Theorem* 3.1*. If the conditions in Theorem* 3.1 *hold, then the FONPSS* (5) *is GCFTS with respect to* $(\delta, \varepsilon, T_f, \sigma(t))$*, and the guaranteed cost value is* (17)*.*

Proof According to (4) and Lemma 2.1, we can obtain

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}V_{\sigma(t)}(t,x(t)) + x^{\mathrm{T}}(t)R_1 &\leq {}_{t_0}^{RL}D_t^{\alpha}V_{\sigma(t)}(t,x(t)) + x^{\mathrm{T}}(t)R_1 \\ &\leq f^{\mathrm{T}}(x(t))A_{\sigma(t)}^{\mathrm{T}}D_1 v_{\sigma(t)} + x^{\mathrm{T}}(t)R_1 \\ &\leq x^{\mathrm{T}}(t)(m_2 A_{\sigma(t)}^{\mathrm{T}}\overline{D}_1 v_{\sigma(t)} + R_1) \\ &\leq \lambda_p V_{\sigma(t)}(t,x(t)). \end{aligned} \tag{39}$$

Similar to the proof process of Theorem 3.1, we can obtain the same results and and the proof is omitted. ▮

3.2 Guaranteed Cost Finite-Time Controller Design

In this section, we focus on the problem of GCFTS controller design of the system (5). A static output feedback controller will be designed to ensure the following system is GCFTS.

Consider the system (5), under the controller $u(t) = K_{\sigma(t)}y(t)$, the corresponding closed-loop system is given by

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) = (D_1 A_{\sigma(t)} + D_2 B_{\sigma(t)} K_{\sigma(t)} D_3 C_{\sigma(t)}) f(x(t)), \\ y(t) = D_3 C_{\sigma(t)} f(x(t)), \quad 0 < \alpha < 1. \end{cases} \tag{40}$$

According to Lemma 2.6, to guarantee the positivity of the system (40), $D_1A_p + D_2B_pK_pD_3C_p$ should be Metzler matrices, $\forall p \in \underline{S}$. Theorem 3.4 gives some sufficient conditions to guarantee that the closed-loop system (40) is GCFTS.

Theorem 3.4 *Consider the FONPSS* (40)*. For given constants* T_f, λ_p *and vectors* $\delta \succ \varepsilon \succ 0$, $R_1 \succ 0$ *and* $R_2 \succ 0$, *if there exist constants* ξ_1, ξ_2, μ_p *and positive vectors* ν_p, $p \in \underline{S}$, *such that* (12), (13), (15) *and the following conditions hold:*

$$\underline{D}_1 A_p + \underline{D}_2 B_p K_p \underline{D}_3 C_p \ \text{and}\ \overline{D}_1 A_p + \overline{D}_2 B_p K_p \overline{D}_3 C_p \ \text{are Metzler matrices},\ K_p \succeq 0, \tag{41}$$

$$m_2 A_p^{\mathrm{T}}\overline{D}_1 v_p + R_1 + f_p + m_2 C_p^{\mathrm{T}}\overline{D}_3 K_p^{\mathrm{T}} R_2 \preceq \lambda_p v_p, \tag{42}$$

$$v_p \prec R_1 + m_1 C_p^{\mathrm{T}}\underline{D}_3 K_p^{\mathrm{T}} R_2, \tag{43}$$

where $\forall p \in \underline{S}$, $f_p = m_2 C_p^{\mathrm{T}}\overline{D}_3 K_p^{\mathrm{T}} B_p^{\mathrm{T}} \overline{D}_2 v_p$, $v_p = [v_{p1}, v_{p2}, \cdots, v_{pn}]^{\mathrm{T}}$, $\lambda = \max_{p\in\underline{S}}\{\lambda_p\}$, $\mu = \max_{p\in\underline{S}}\{\mu_p\}$, $\lambda_p > 1, \mu_p \geq 1$, $T_\omega = \min_{p\in\underline{S}}\{T_{\omega p}\}$, *then under the MDADT scheme* (16)*, the resulting closed-loop system* (40) *is GCFTS with respect to* $(\delta, \varepsilon, T_f, \sigma(t))$*, and the cost value is given by*

$$\begin{aligned} J &= \frac{1}{\Gamma(\alpha)}\int_0^{T_f}(t-s)^{\alpha-1}(x^{\mathrm{T}}(s)R_1 + y^{\mathrm{T}}(s)K_p^{\mathrm{T}}R_2)ds \\ &\leq J^* \\ &= \xi_2\mu^{\frac{T_f}{T_\omega}} + \lambda\xi_2\mu^{\frac{T_f}{T_\omega}}\frac{1}{\alpha\Gamma(\alpha)}\exp\left\{\frac{T_f}{T_\alpha}\left(\ln\mu + \frac{(1-\alpha)(\lambda-1)}{\Gamma(\alpha+1)}\right)\right. \\ &\quad \left. + (\lambda-1)\frac{(1-\alpha)+\alpha T_f}{\Gamma(\alpha+1)}\right\}(T_\omega)^{\alpha-1}T_f. \end{aligned} \tag{44}$$

Proof We first prove the positivity of the resulting closed-loop system(40). By Definition 2.3 and (41), we get $\underline{D}_1 A_p + \underline{D}_2 B_p K_p \underline{D}_3 C_p \preceq D_1 A_p + D_2 B_p K_p D_3 C_p \preceq \overline{D}_1 A_p + \overline{D}_2 B_p K_p \overline{D}_3 C_p$. It means that $D_1 A_p + D_2 B_p K_p D_3 C_p$ are Metzler matrices. According to Lemma 2.8, the system (40) is positive if $\forall p \in \underline{S}$, B_p, C_p are all nonnegative.

Next, we prove the guaranteed cost finite-time stability of the system (40). Consider $K_p \geq 0$, from (42), we have

$$\begin{aligned}
&{}^{C}_{t_0}D_t^{\alpha} V_{\sigma(t)}(t, x(t)) + x^{\mathrm{T}}(t) R_1 + u^{\mathrm{T}}(t) R_2 \\
&= f^{\mathrm{T}}(x(t)) A^{\mathrm{T}}_{\sigma(t)} D_1 v_{\sigma(t)} + x^{\mathrm{T}}(t) R_1 + y^{\mathrm{T}}(t) K_p^{\mathrm{T}} R_2 \\
&\leq m_2 x^{\mathrm{T}}(t) A_p^{\mathrm{T}} \overline{D}_1 v_p + x^{\mathrm{T}}(t) R_1 + m_2 x^{\mathrm{T}}(t) C_p^{\mathrm{T}} \overline{D}_3 K_p^{\mathrm{T}} B_p^{\mathrm{T}} \overline{D}_2 v_p + m_2 x^{\mathrm{T}}(t) C_p^{\mathrm{T}} \overline{D}_3 K_p^{\mathrm{T}} R_2 \\
&= x^{\mathrm{T}}(t)(m_2 A_p^{\mathrm{T}} \overline{D}_1 v_p + R_1 + m_2 C_p^{\mathrm{T}} \overline{D}_3 K_p^{\mathrm{T}} B_p^{\mathrm{T}} \overline{D}_2 v_p + m_2 C_p^{\mathrm{T}} \overline{D}_3 K_p^{\mathrm{T}} R_2) \\
&\leq \lambda_p x^{\mathrm{T}}(t) v_{\sigma(t)} \\
&= \lambda_p V_{\sigma(t)}(t, x(t)).
\end{aligned} \tag{45}$$

Taking the fractional integral ${}^{C}_{t_0}D_t^{-\alpha}$ to both sides of (45) during the period $[t_k, t)$ for $t \in [t_k, t_{k+1})$ leads to

$$\begin{aligned}
V_{\sigma(t)}(t, x(t)) &\leq V_{\sigma(t_k)}(t_k, x(t_k)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} V_{\sigma(t)}(s, x(s)) ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} (x^{\mathrm{T}}(s) R_1 + y^{\mathrm{T}}(s) K_p^{\mathrm{T}} R_2) ds \\
&\leq V_{\sigma(t_k)}(t_k, x(t_k)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} V_{\sigma(t)}(s, x(s)) ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} x^{\mathrm{T}}(s)(R_1 + m_1 C_p^{\mathrm{T}} \underline{D}_3 K_p^{\mathrm{T}}(s) R_2) ds.
\end{aligned} \tag{46}$$

From (43), (46) can be written as

$$\begin{aligned}
V_{\sigma(t)}(t, x(t)) &\leq V_{\sigma(t_k)}(t_k, x(t_k)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} V_{\sigma(t)}(s, x(s)) ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} x^{\mathrm{T}}(s) v_p ds \\
&= V_{\sigma(t_k)}(t_k, x(t_k)) + \frac{\lambda_{\sigma(t_k)-1}}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} V_{\sigma(t)}(s, x(s)) ds.
\end{aligned} \tag{47}$$

Similar to the process of (23)–(33),we easily obtain that the system (40) is FTS. Next, we consider the guaranteed cost value of the system (40).
From (46) and $V_{\sigma(t_k)} \leq \mu_{\sigma(t_k)} V_{\sigma(t_k^-)}(t_k^-, x(t_k^-))$, for $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned}
V_{\sigma(t)}(t, x(t)) &\leq \mu_{\sigma(t_k)} V_{\sigma(t_k^-)}(t_k^-, x(t_k^-)) + \frac{\lambda_{\sigma(t_k)}}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} V_{\sigma(t)}(s, x(s)) ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} (x^{\mathrm{T}}(s) R_1 + y^{\mathrm{T}}(s) K_p^{\mathrm{T}} R_2) ds.
\end{aligned} \tag{48}$$

Similar to the proof process of (34)–(38), the guaranteed cost value is given as

$$\begin{aligned} J &= \frac{1}{\Gamma(\alpha)}\int_0^{T_f}(t-s)^{\alpha-1}(x^{\mathrm{T}}(s)R_1+y^{\mathrm{T}}(s)K_p^{\mathrm{T}}R_2)ds \\ &\le J^* \\ &= \xi_2\mu^{\frac{T_f}{T_\omega}}+\lambda\xi_2\mu^{\frac{T_f}{T_\omega}}\frac{1}{\alpha\Gamma(\alpha)}\exp\left\{\frac{T_f}{T_\alpha}\left(\ln\mu+\frac{(1-\alpha)(\lambda-1)}{\Gamma(\alpha+1)}\right)\right. \\ &\quad \left.+(\lambda-1)\frac{(1-\alpha)+\alpha T_f}{\Gamma(\alpha+1)}\right\}(T_\omega)^{\alpha-1}T_f. \end{aligned} \tag{49}$$

The proof is completed. ∎

Remark 3.5 In Theorem 3.4, the gain matrix $K_p \succeq 0, \forall p \in \underline{S}$ is used. Naturally, when $K_p \preceq 0$, we only replace (41) by the following conditions

$$\underline{D}_1A_p+\overline{D}_2B_pK_p\overline{D}_3C_p \text{ and } \overline{D}_1A_p+\underline{D}_2B_pK_p\underline{D}_3C_p \text{ are Metzler matrices, } K_p\preceq 0, \tag{50}$$

$$R_1+m_2C_p^{\mathrm{T}}K_p^{\mathrm{T}}R_2\succeq 0. \tag{51}$$

Following the proof line of Theorem 3.4, we can also get the resulting closed-loop system (40) is GCFTS with the MDADT scheme (16), and the guaranteed cost value is also given by (44).

Next, an algorithm is presented to obtain the feedback gain matrices K_p, $p\in\underline{S}$.

Algorithm 3.6

Step 1 Input the matrices A_p, B_p, C_p, $\underline{D}_i$, $\overline{D}_i$, R_1 and R_2, $p\in\underline{S}$, $i=1,2,3$.

Step 2 By adjusting the parameter $\lambda_p>0$ and solving (12)–(13), (15) and (42)–(43) via linear programming, we can get the solutions v_p, K_p and f_p.

Step 3 Then, $\widetilde{f}_p=m_2C_p^{\mathrm{T}}\overline{D}_3K_p^{\mathrm{T}}B_p^{\mathrm{T}}\overline{D}_2v_p$ are obtained. If $f_p-\widetilde{f}_p\succeq 0$, $\underline{D}_1A_p+\underline{D}_2B_pK_p\underline{D}_3C_p$ and $\overline{D}_1A_p+\overline{D}_2B_pK_p\overline{D}_3C_p$ are Metzler matrices, then K_p are admissible. Otherwise, return to Step 2.

Remark 3.7 From Algorithm 3.6, the outcome largely depends on the selection of λ_p. However, there is not a general approach to choose the value of λ_p. In this paper, λ_p should be selected small by experience.

4 Numerical Example

In this section, an example will be given to illustrate the effectiveness of the proposed method. Consider the system (5) with the parameters as follows:

$$A_1=\begin{bmatrix}-2&0\\0&-1\end{bmatrix},\quad B_1=\begin{bmatrix}2&0\\0&1\end{bmatrix},\quad C_1=\begin{bmatrix}0.8&0\\0&0.6\end{bmatrix},$$

$$\underline{D}_1=\begin{bmatrix}0.4&0\\0&0.6\end{bmatrix},\quad \underline{D}_2=\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix},$$

$$A_2=\begin{bmatrix}-2&0\\0&-2\end{bmatrix},\quad B_2=\begin{bmatrix}1.2&0\\0&0.8\end{bmatrix},\quad C_2=\begin{bmatrix}0.6&0\\0&0.4\end{bmatrix},$$

$$\overline{D}_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad \overline{D}_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\underline{D}_3 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad \overline{D}_3 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix},$$

$$\delta = \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} 0.02 \\ 0.01 \end{bmatrix},$$

where $f_i(x_i(t)) = x_i(t) + \frac{x_i(t)}{x_i^2(t)+2}$, then we get $m_1 = 1$, $m_2 = 2$.

Let $\alpha = 0.8$, $\mu_1 = 1.06$, $\mu_2 = 1.08$, $\lambda = 1.1$, $\lambda_2 = 1.12$. Solving the inequalities in Theorem 3.4 by linear programming, we have

$$v_1 = \begin{bmatrix} 1.0040 \\ 0.6573 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0.9746 \\ 0.6340 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0.8711 \\ 0.5427 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1.1347 \\ 0.4323 \end{bmatrix},$$

$$\xi_1 = 7.3720, \quad \xi_2 = 1.4311,$$

$$K_1 = \begin{bmatrix} 3.1145 \ 2.6216 \\ 3.1145 \ 2.6216 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2.6788 \ 2.5784 \\ 2.6788 \ 2.5784 \end{bmatrix}.$$

It is easy to verify that $\underline{D}_1 A_p + \underline{D}_2 B_p K_p \underline{D}_3 C_p$ and $\overline{D}_1 A_p + \overline{D}_2 B_p K_p \overline{D}_3 C_p$ are Metzler matrices for each $p \in \underline{S}$. Then, according to (20), we can obtain $T^*_{\omega 1} = 1.2623, T^*_{\omega 2} = 1.3821$. Choosing $T_{\omega 1} = 1.3 > T^*_{\omega 1}$ and $T_{\omega 2} = 1.4 > T^*_{\omega 2}$. Under the static output feedback controller, the simulation results are shown in Figures 1–4. The initial conditions of the system (5) are $x(0) = [0.6\ 0.4]^{\mathrm{T}}$, which satisfies $x^{\mathrm{T}}(0)\delta \leq 1$. Figure 1 shows the state trajectories of the closed-loop system (5). The state trajectories of the closed-loop system with ADT are shown in Figure 2. The switching signal $\sigma(t)$ with MDADT is depicted in Figure 3. Figure 4 plots the evolution of $x^{\mathrm{T}}(t)\varepsilon$ of System (5). The cost value is $J^* = 135.4144$, which can be obtained by (44).

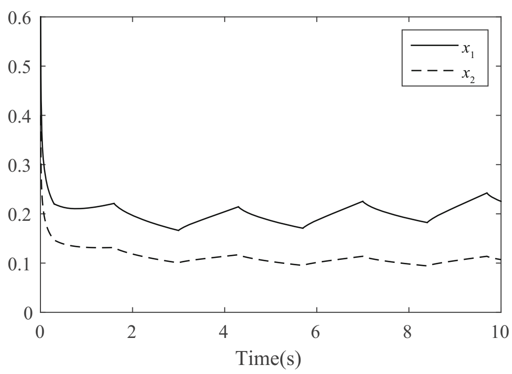

Figure 1 State trajectories of open-loop system (5)

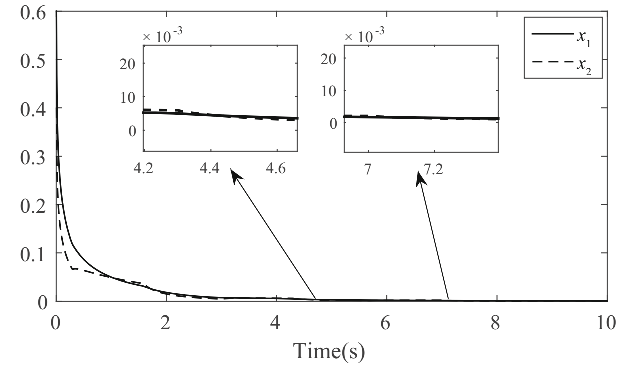

Figure 2 State trajectories of closed-loop system (5)

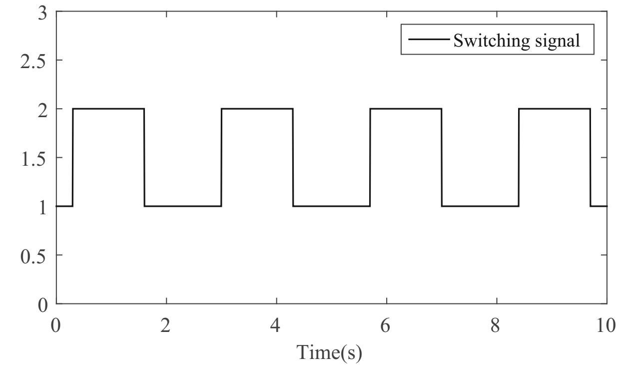

Figure 3 Switching signal of System (5) with MDADT

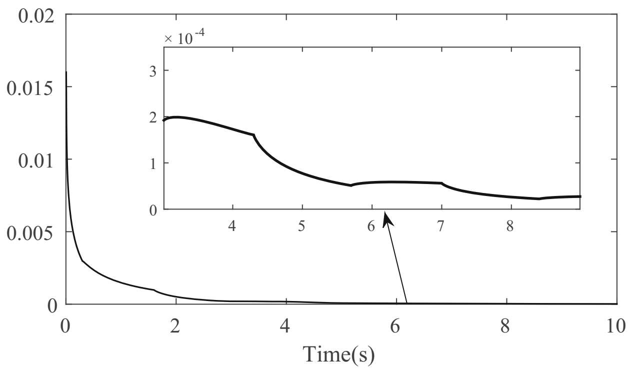

Figure 4 The evolution of $x^{\mathrm{T}}(t)\varepsilon$ of System (5)

5 Conclusions

This paper has investigated the problem of guaranteed cost finite-time control for FONPSS with D-perturbation. A novel fractional-order guaranteed cost function is extended to FONPSS. By using MDADT approach and constructing multiple linear copositive Lyapunov functions, a static output feedback controller is designed, then a series of switching signals and some sufficient conditions are obtained to guarantee that the closed-loop system is GCFTS. Such sufficient conditions can be solved by linear programming. Finally, an example is given to show the effectiveness of the proposed method.

Time delays usually occur in many practical systems and may result in system performance deterioration, even instability. In our further work, we will extend the proposed method to FOPSS with time-varying delays.

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