J Syst Sci Complex (2019) 32: 997–1018

The Optimal Control of Fully-Coupled Forward-Backward Doubly Stochastic Systems Driven by Itô-Lévy Processes*

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DOI: 10.1007/s11424-018-7300-z
Received: 18 September 2017 / Revised: 27 March 2018

Abstract This paper studies the optimal control of a fully-coupled forward-backward doubly stochastic system driven by Itô-Lévy processes under partial information. The existence and uniqueness of the solution are obtained for a type of fully-coupled forward-backward doubly stochastic differential equations (FBDSDEs in short). As a necessary condition of the optimal control, the authors get the stochastic maximum principle with the control domain being convex and the control variable being contained in all coefficients. The proposed results are applied to solve the forward-backward doubly stochastic linear quadratic optimal control problem.

Keywords Forward-backward doubly stochastic differential equations, Itô-Lévy processes, linear quadratic problem, maximum principle, variational equation.

1 Introduction

The optimal control of forward-backward stochastic differential equations (FBSDEs) has got a lot of attentions over recent years. We can refer to [1] for its widely applications in financial market. The theory of FBSDEs was first developed in the early 90s by [2–4] and others. Wu[5] studied the maximum principle of fully-coupled forward-backward stochastic systems with the control domain being convex. Wu[6] also got the maximum principle of forward-backward stochastic systems in a more general case where the control domain is non-convex. Ji and Wei[7] derived the maximum principle for fully-coupled forward-backward stochastic systems with terminal state constraints. Peng and Wu[8] obtained the existence and uniqueness results

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*This research was supported by the Cultivation Program of Distinguished Young Scholars of Shandong University under Grant No. 2017JQ06, the National Natural Science Foundation of China under Grant Nos. 11671404, 11371374, 61821004, 61633015, the Provincial Natural Science Foundation of Hunan under Grant No. 2017JJ3405.
◇ *This paper was recommended for publication by Editor SUN Jian.*

of fully-coupled forward-backward stochastic differential equations with an arbitrarily large time duration, and given applications of FBSDEs to stochastic optimal control problems and differential games. Wu[9] used the solution of FBSDEs to get the explicit form of the optimal control for LQ problem. Yu[10] obtained the existence and uniqueness result for one kind of forward-backward stochastic differential equations, and he applied the result to the linear-quadratic stochastic optimal control and nonzero-sum differential game of forward-backward stochastic system. Meng[11] established a sufficient and a necessary maximum principle under partial information for a type of FBSDE. In [12], Øksendal and Sulem presented various versions of maximum principle for optimal control of forward-backward stochastic differential equations with jumps. In Wang and Wu[13], a maximum principle for partially observed stochastic recursive optimal control problems was obtained under the assumption that the control domain being non-convex and the forward diffusion coefficients do not contain the control variable. Wang, et al.[14] studied a partial information optimal control problem derived by forward-backward stochastic systems with correlated noises between the system and the observation. Ma and Liu[15] studied the linear-quadratic optimal control problem for partially observed FBSDEs of mean-field type. Wu, et al.[16] researched the optimal control of fully coupled forward-backward stochastic systems with delay and noisy memory. We can refer to the book by Yong and Zhou[17] for details about stochastic control theory.

In order to provide a probabilistic interpretation for the solution of a class of semi-linear stochastic partial differential equations, Pardoux and Peng[18] introduced a new kind of backward stochastic differential equations, which is called backward doubly stochastic differential equations (BDSDEs). Peng and Shi[19] discussed a type of time-symmetric forward-backward stochastic differential equations (FBDSDEs) and established the existence and uniqueness of the solution by the method of continuation under some monotonicity assumptions. Sun and Lu[20] studied the property for solutions of the multi-dimensional BDSDEs with jumps. Recently, the optimal control of BDSDEs has been considered. Han, et al.[21] investigated the optimal control problems for backward doubly stochastic control systems. Zhu, et al.[22] got the maximum principle for backward doubly stochastic systems with jumps. Zhu and Shi[23] also researched the optimal control of backward doubly stochastic systems under partial information. Xu and Han[24] solved a class of doubly stochastic optimal control problems that the state trajectory is described by backward doubly stochastic differential equations with time delay. Zhang and Shi[25] presented the maximum principle for forward-backward doubly stochastic systems with the control domain being non-convex. However, among these literatures, the studies on the optimal control of forward-backward doubly stochastic systems driven by Itô-Lévy processes were few.

In this paper, we investigate the necessary maximum principle of a fully-coupled forward-backward doubly stochastic system with the control domain being convex and the control variable being contained in all coefficients. The rest of the paper is organized as follows. Section 2 begins a general formulation of the stochastic optimal control of FBDSDEs driven by Itô-Lévy processes, and gives some assumptions. In Section 3, we prove the existence and uniqueness of the solution of this type of FBDSDEs. The variational equation and variational inequality are deduced in Section 4. In Section 5, we introduce the adjoint equation and the stochastic Hamiltonian system. Finally, we apply our theoretical results to LQ problem in Section 6.

2 Statement of the Problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the following three mutually independent processes:

1) Suppose that $\{W(t); 0 \leq t \leq T\}$ and $\{B(t); 0 \leq t \leq T\}$ are two standard 1-dimensional Brownian motions defined on $(\Omega, \mathcal{F}, P)$, with values in $\mathbb{R}$.

2) Suppose that $\eta(t)$ is an independent pure jump Lévy martingale. $N(dt, d\theta)$ and $\nu(d\theta)$ denote the jump measure and the Lévy measure of $\eta(\cdot)$, respectively, then $\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta)dt$ is the compensated jump measure of $\eta(\cdot)$. We can write $\eta(t) = \int_0^t \int_{\mathbb{R}_0} \theta \widetilde{N}(dt, d\theta)$ where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\eta$, where for $\{\pi(t)\}$, $\mathcal{F}_{s,t}^\pi = \sigma\{\pi(r) - \pi(s); s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\pi = \mathcal{F}_{0,t}^\pi$. Note that the collection $\{\mathcal{F}_t : t \in [0, T]\}$ is neither increasing nor decreasing, so it does not constitute a filtration. We use the usual inner product $\langle \cdot \rangle$ and norm $||\cdot||$ in $\mathbb{R}^n$.

We adopt the following notations:

$S_{\mathcal{F}}^2([0, T]; \mathbb{R}) := \Big\{v(t, \omega) : v(t, \omega)$ is a 1-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-measurable process which satisfies $\mathbb{E}\left[\sup_{0\leq t\leq T} v(t, \omega)^2\right] < \infty\Big\}$;

$M_{\mathcal{F}}^2([0, T]; \mathbb{R}) := \Big\{v(t, \omega) : v(t, \omega)$ is a 1-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-measurable process which satisfies $\mathbb{E}\left[\int_0^T v(t, \omega)^2 dt\right] < \infty\Big\}$;

$F_{N,\mathcal{F}}^2([0, T]; \mathbb{R}) := \Big\{r(t, \theta, \omega) : r(t, \theta, \omega)$ is a 1-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-measurable process which satisfies $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} r(t, \theta, \omega)^2 \nu(d\theta) dt\right] < \infty\Big\}$;

$L_{\nu(\cdot)}^2(\mathbb{R}) := \big\{r(\theta) : r(\theta)$ is a 1-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-measurable process which satisfies $||r||_\nu = \big(\int_{\mathbb{R}_0} r(\theta)^2 \nu(d\theta)\big)^{\frac{1}{2}} < \infty$ a.s.$\big\}$;

$L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}) := \big\{\xi : \xi$ is a 1-dimensional $\mathcal{F}_T$-measurable random variable which satisfies $\mathbb{E}\left[\xi^2\right] < \infty\big\}$;

$\mathbb{M}^2 := \left[M_{\mathcal{F}}^2([0, T]; \mathbb{R})\right]^4 \times F_{N,\mathcal{F}}^2([0, T]; \mathbb{R})$.

For a given $v \in M_{\mathcal{F}}^2([0, T]; \mathbb{R})$, we can define the forward Itô's integral $\int_0^\cdot v(s) \overrightarrow{d} W(s)$ and the backward Itô's integral $\int_\cdot^T v(s) \overleftarrow{d} B(s)$ (see [18] for details).

We will need the following extension of Itô's formula.

Lemma 2.1 *Let $\alpha \in S_{\mathcal{F}}^2([0, T]; \mathbb{R})$, $\beta, \sigma, \delta \in M_{\mathcal{F}}^2([0, T]; \mathbb{R})$, $\gamma \in L_{\nu(\cdot)}^2(\mathbb{R})$ be such that*

$$
\begin{aligned}
\alpha(t) =& \alpha(0) + \int_0^t \beta(s) ds + \int_0^t \sigma(s) \overleftarrow{d} B(s) + \int_0^t \delta(s) \overrightarrow{d} W(s) \\
&+ \int_0^t \int_{\mathbb{R}_0} \gamma(s, \theta) \widetilde{N}(\overrightarrow{d} s, d\theta), \quad 0 \leq t \leq T.
\end{aligned}
$$

Then

$$\begin{aligned}\alpha(t)^2 &= \alpha(0)^2 + 2\int_0^t \alpha(s)\beta(s)ds + 2\int_0^t \alpha(s)\sigma(s)\overleftarrow{d}B(s)\\ &\quad + 2\int_0^t \alpha(s)\ \delta(s)\overrightarrow{d}W(s) + 2\int_0^t\int_{\mathbb{R}_0} \alpha(s)\gamma(s,\theta)\widetilde{N}(\overrightarrow{d}s, d\theta)\\ &\quad - \int_0^t \sigma(s)^2 ds + \int_0^t \delta(s)^2 ds + \int_0^t\int_{\mathbb{R}_0} \gamma(s,\theta)^2\nu(d\theta)ds,\\ \mathbb{E}\left[\alpha(t)^2\right] &= \mathbb{E}\left[\alpha(0)^2\right] + 2\mathbb{E}\left[\int_0^t \alpha(s)\beta(s)ds\right] - \mathbb{E}\left[\int_0^t \sigma(s)^2 ds\right]\\ &\quad + \mathbb{E}\left[\int_0^t \delta(s)^2 ds\right] + \mathbb{E}\left[\int_0^t\int_{\mathbb{R}_0} \gamma(s,\theta)^2\nu(d\theta)ds\right].\end{aligned}$$

Proof We can adopt the similar steps of Lemma 1.3 in [14], and we omit the details here. ∎

Let $\mathcal{U}$ be a nonempty convex set of $\mathbb{R}$. Let $\mathcal{G}_t$ be a sub-sigma algebra of $\mathcal{F}_t$, i.e., $\mathcal{G}_t \subset \mathcal{F}_t$. For example, we could have $\mathcal{G}_t = \mathcal{F}_t^W$ be the information available to the controller at time t. We say a control process $u(\cdot) : \Omega \times [0,\, T] \to \mathcal{U}$ is admissible if it is $\mathcal{G}_t$-adapted and $u(\cdot) \in M_{\mathcal{F}}^2([0,T],\mathbb{R})$. Denote the set of all admissible control processes by $\mathcal{U}_{ad}$.

Consider a stochastic system where the state is governed by the following fully-coupled FBDSDE driven by Itô-Lévy processes:

$$\begin{cases} dy(t) &= f(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),\omega)dt\\ &\quad + g(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),\omega)\overrightarrow{d}W(t) - z(t)\overleftarrow{d}B(t)\\ &\quad + \int_{\mathbb{R}_0} \gamma(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),\theta,u(t),\omega)\widetilde{N}(\overrightarrow{d}t, d\theta), \quad t\in[0,T],\\ y(0) &= x,\\ dY(t) &= -F(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),\omega)dt\\ &\quad - G(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),\omega)\overleftarrow{d}B(t)\\ &\quad + Z(t)\overrightarrow{d}W(t) + \int_{\mathbb{R}_0} K(t,\theta)\widetilde{N}(\overrightarrow{d}t, d\theta), \quad t\in[0,T],\\ Y(T) &= \xi, \end{cases} \tag{1}$$

where $(y,Y,z,Z,K(\cdot,\cdot)) \in \mathbb{R}^5$, $x \in \mathbb{R}$ is a given constant. $T \geq 0$ is a given fixed time duration.

$$\begin{aligned} &F : [0,T]\times\mathbb{R}^5\times\mathcal{U}\times\Omega \to \mathbb{R},\\ &f : [0,T]\times\mathbb{R}^5\times\mathcal{U}\times\Omega \to \mathbb{R},\\ &G : [0,T]\times\mathbb{R}^5\times\mathcal{U}\times\Omega \to \mathbb{R},\\ &g : [0,T]\times\mathbb{R}^5\times\mathcal{U}\times\Omega \to \mathbb{R},\\ &\gamma : [0,T]\times\mathbb{R}^5\times\mathcal{U}\times\Omega \to \mathbb{R}. \end{aligned}$$

The solution $(y(t),Y(t),z(t),Z(t),K(t,\cdot))$ corresponding to $u(t)$ is called the state trajectory.

Suppose that the performance functional is given by:

$$J(u) = \mathbb{E}\left[\int_0^T l(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), \omega)dt + h_1(y(T), \omega) + h_2(Y(0))\right],$$

where

$$l : [0,T] \times \mathbb{R}^5 \times \mathcal{U} \times \Omega \to \mathbb{R}, \quad h_1 : \mathbb{R} \times \Omega \to \mathbb{R}, \quad h_2 : \mathbb{R} \to \mathbb{R}.$$

The optimal control problem is to find $u^*(\cdot) \in \mathcal{U}_{ad}$ such that

$$J(u^*) = \inf_{u \in \mathcal{U}_{ad}} J(u). \tag{2}$$

In the following, the dependence on ω is suppressed for simplicity.

Denote

$$\zeta = \begin{pmatrix} y \\ Y \\ z \\ Z \\ K \end{pmatrix}, \qquad A(t,\zeta) = \begin{pmatrix} -F \\ f \\ -G \\ g \\ \gamma \end{pmatrix}(t,\zeta).$$

We assume that

(A1) For each $\zeta \in \mathbb{R}^5$, $A(\cdot,\zeta)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-measurable process defined on $[0,T]$ with $A(\cdot,0) \in \mathbb{M}^2$.

(A2) $F, f, G, g, \gamma, l, h_1, h_2$ are twice Fréchet differentiable with respect to (y, Y, z, Z, K, u). They and all their derivatives up to the second order are bounded by a constant c.

(A3) The following inequality is the key condition:

$$\begin{aligned}
&-(F(t,\zeta) - F(t,\zeta'))(y - y') + (f(t,\zeta) - f(t,\zeta'))(Y - Y') - (G(t,\zeta) - G(t,\zeta'))(z - z') \\
&+ (g(t,\zeta) - g(t,\zeta'))(Z - Z') + \int_{\mathbb{R}_0} (\gamma(t,\zeta) - \gamma(t,\zeta'))(K(\cdot,\theta) - K'(\cdot,\theta))\nu(d\theta) \\
= &\langle A(t,\zeta) - A(t,\zeta'), \zeta - \zeta'\rangle \leq -\mu||\zeta - \zeta'||^2, \qquad \forall \zeta, \zeta' \in \mathbb{R}^5, \qquad \forall t \in [0,T],
\end{aligned}$$

where μ is a positive constant.

3 The Existence and Uniqueness of the Solution of FBDSDEs

In this section, we give the existence and uniqueness of the solution of FBDSDEs. Our main result is as follows:

Theorem 3.1 *Under Assumptions* (A1)–(A3)*, for each* $x \in \mathbb{R}$*, and random variable* $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$*, the following FBDSDE has a unique solution* $(y(t), Y(t), z(t), Z(t), K(t,\cdot)) \in$

$\mathbb{M}^2$:

$$\begin{cases} dy(t) &= f(t,y(t),Y(t),z(t),Z(t),K(t,\cdot))dt \\ &\quad +g(t,y(t),Y(t),z(t),Z(t),K(t,\cdot))\overrightarrow{d}W(t) - z(t)\overleftarrow{d}B(t) \\ &\quad +\int_{\mathbb{R}_0}\gamma(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),\theta)\widetilde{N}(\overrightarrow{d}t,d\theta), \quad t\in[0,T], \\ y(0) &= x, \\ dY(t) &= -F(t,y(t),Y(t),z(t),Z(t),K(t,\cdot))dt \\ &\quad -G(t,y(t),Y(t),z(t),Z(t),K(t,\cdot))\overleftarrow{d}B(t) \\ &\quad +Z(t)\overrightarrow{d}W(t)+\int_{\mathbb{R}_0}K(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta), \quad t\in[0,T], \\ Y(T) &= \xi. \end{cases} \tag{3}$$

In order to prove the above theorem, we need the following two lemmas. Consider the following family of FBDSDEs parameterized by $\alpha\in[0,1]$,

$$\begin{cases} dy(t) &= [f^\alpha(t,U(t,\cdot))+f_0(t)]dt+[g^\alpha(t,U(t,\cdot))+g_0(t)]\overrightarrow{d}W(t)-z(t)\overleftarrow{d}B(t) \\ &\quad +\int_{\mathbb{R}_0}[\gamma^\alpha(t,U(t,\cdot),\theta)+\gamma_0(t,\theta)]\widetilde{N}(\overrightarrow{d}t,d\theta), \quad t\in[0,T], \\ y(0) &= x, \\ dY(t) &= -[F^\alpha(t,U(t,\cdot))+F_0(t)]dt-[G^\alpha(t,U(t,\cdot))+G_0(t)]\overleftarrow{d}B(t) \\ &\quad +Z(t)\overrightarrow{d}W(t)+\int_{\mathbb{R}_0}K(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta), \quad t\in[0,T], \\ Y(T) &= \xi+(1-\alpha)y(T), \end{cases} \tag{4}$$

where $U(t,\cdot)=(y(t),Y(t),z(t),Z(t),K(t,\cdot))$, $(F_0(t),f_0(t),G_0(t),g_0(t),\gamma_0(t,\cdot))\in\mathbb{M}^2$ are given processes. And for any given $\alpha\in[0,1]$,

$$\begin{aligned} &f^\alpha(t,U(t,\cdot))=\alpha f(t,U(t,\cdot))-(1-\alpha)Y(t), \qquad F^\alpha(t,U(t,\cdot))=\alpha F(t,U(t,\cdot))+(1-\alpha)y(t), \\ &g^\alpha(t,U(t,\cdot))=\alpha g(t,U(t,\cdot))-(1-\alpha)Z(t), \qquad G^\alpha(t,U(t,\cdot))=\alpha G(t,U(t,\cdot))+(1-\alpha)z(t), \\ &\gamma^\alpha(t,U(t,\cdot),\theta)=\alpha\gamma(t,U(t,\cdot),\theta)-(1-\alpha)K(t,\cdot). \end{aligned}$$

When $\alpha=1$, $(F_0(t),f_0(t),G_0(t),g_0(t),\gamma_0(t,\cdot))=0$, (4) is reduced to (3). The following lemma gives an estimate for the existence interval of (4) with respect to $\alpha\in[0,1]$.

Lemma 3.2 *Under Assumptions* (A1)–(A3)*, if for some* $\alpha_0\in[0,1)$*, and for each* $x\in\mathbb{R}$*,* $\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R})$*,* $(F_0(t),f_0(t),G_0(t),g_0(t),\gamma_0(t,\cdot))\in\mathbb{M}^2$*,* (4) *has a unique solution, then there exists a positive constant* δ_0 *such that for each* $\alpha\in[\alpha_0,\alpha_0+\delta_0]$*,* (4) *also has a unique solution* $(y(t),Y(t),z(t),Z(t),K(t,\cdot))\in\mathbb{M}^2$*.*

Proof Since for each $x\in\mathbb{R}$, $\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R})$, $(F_0(t),f_0(t),G_0(t),g_0(t),\gamma_0(t,\cdot))\in\mathbb{M}^2$, there exists a unique solution to (4) for $\alpha=\alpha_0$. Let δ be a positive number which is less than 1. It is easy to see that for each $\overline{U}(t,\cdot)=(\overline{y}(t),\overline{Y}(t),\overline{z}(t),\overline{Z}(t),\overline{K}(t,\cdot))\in\mathbb{M}^2$, there exists a unique solution $U(t,\cdot) = (y(t), Y(t), z(t), Z(t), K(t,\cdot)) \in \mathbb{M}^2$ satisfying the following equation:

$$\begin{cases} dy(t) &= \big[f^{\alpha_0}(t,U(t,\cdot)) + \delta\big(f(t,\overline{U}(t,\cdot)) + \overline{Y}(t)\big) + f_0(t)\big]dt \\ &\quad + \big[g^{\alpha_0}(t,U(t,\cdot)) + \delta\big(g(t,\overline{U}(t,\cdot)) + \overline{Z}(t)\big) + g_0(t)\big]\overrightarrow{d}W(t) - z(t)\overleftarrow{d}B(t) \\ &\quad + \int_{\mathbb{R}_0}\big[\gamma^{\alpha_0}(t,U(t,\cdot),\theta) + \delta\big(\gamma(t,\overline{U}(t,\cdot),\theta) + \overline{K}(t,\cdot)\big) + \gamma_0(t,\theta)\big]\widetilde{N}(\overrightarrow{d}t,d\theta), \\ y(0) &= x, \\ dY(t) &= -\big[F^{\alpha_0}(t,U(t,\cdot)) + \delta\big(F(t,\overline{U}(t,\cdot)) - \overline{y}(t)\big) + F_0(t)\big]dt + Z(t)\overrightarrow{d}W(t) \\ &\quad - \big[G^{\alpha_0}(t,U(t,\cdot)) + \delta\big(G(t,\overline{U}(t,\cdot)) - \overline{z}(t)\big) + G_0(t)\big]\overleftarrow{d}B(t) + \int_{\mathbb{R}_0} K(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta), \\ Y(T) &= \xi + (1-\alpha_0)y(T) - \delta\overline{y}(T). \end{cases}$$

We now proceed to prove there exists a small enough δ independent of α_0 such that the mapping defined by $U(t,\cdot) = I_{\alpha_0+\delta}(\overline{U}(t,\cdot)) : \mathbb{M}^2 \to \mathbb{M}^2$ is contractive. Let

$$\overline{U}'(t,\cdot) = (\overline{y}'(t), \overline{Y}'(t), \overline{z}'(t), \overline{Z}'(t), \overline{K}'(t,\cdot)),$$

$$U'(t,\cdot) = (y'(t), Y'(t), z'(t), Z'(t), K'(t,\cdot)) = I_{\alpha_0+\delta}(\overline{U}'(t,\cdot)),$$

$$\begin{aligned} \Delta\overline{U}(t,\cdot) &= \overline{U}(t,\theta) - \overline{U}'(t,\theta) = (\overline{y}(t)-\overline{y}'(t), \overline{Y}(t)-\overline{Y}'(t), \overline{z}(t)-\overline{z}'(t), \overline{Z}(t)-\overline{Z}'(t), \overline{K}(t,\cdot)-\overline{K}'(t,\cdot)) \\ &= (\Delta\overline{y}(t), \Delta\overline{Y}(t), \Delta\overline{z}(t), \Delta\overline{Z}(t), \Delta\overline{K}(t,\cdot)), \end{aligned}$$

$$\begin{aligned} \Delta U(t,\cdot) &= U(t,\cdot) - U'(t,\cdot) = (y(t)-y'(t), Y(t)-Y'(t), z(t)-z'(t), Z(t)-Z'(t), K(t,\cdot)-K'(t,\cdot)) \\ &= (\Delta y(t), \Delta Y(t), \Delta z(t), \Delta Z(t), \Delta K(t,\cdot)). \end{aligned}$$

Using Itô's formula to $\Delta y(t)\Delta Y(t)$ and noting the fact that $\mathbb{E}[\Delta y(0)] = 0$, we can obtain:

$$\begin{aligned} &\mathbb{E}\left[\Delta y(T)\big((1-\alpha_0)\Delta y(T) - \delta\Delta\overline{y}(T)\big)\right] \\ =& -\mathbb{E}\left[\int_0^T \Delta y(t)\Big(F^{\alpha_0}(t,U(t,\cdot)) - F^{\alpha_0}(t,U'(t,\cdot)) + \delta\big(F(t,\overline{U}(t,\cdot)) - F(t,\overline{U}'(t,\cdot)) - \Delta\overline{y}(t)\big)\Big)dt\right] \\ &+ \mathbb{E}\left[\int_0^T \Delta Y(t)\Big(f^{\alpha_0}(t,U(t,\cdot)) - f^{\alpha_0}(t,U'(t,\cdot)) + \delta\big(f(t,\overline{U}(t,\cdot)) - f(t,\overline{U}'(t,\cdot)) + \Delta\overline{Y}(t)\big)\Big)dt\right] \\ &- \mathbb{E}\left[\int_0^T \Delta z(t)\Big(G^{\alpha_0}(t,U(t,\cdot)) - G^{\alpha_0}(t,U'(t,\cdot)) + \delta\big(G(t,\overline{U}(t,\cdot)) - G(t,\overline{U}'(t,\cdot)) - \Delta\overline{z}(t)\big)\Big)dt\right] \\ &+ \mathbb{E}\left[\int_0^T \Delta Z(t)\Big(g^{\alpha_0}(t,U(t,\cdot)) - g^{\alpha_0}(t,U'(t,\cdot)) + \delta\big(g(t,\overline{U}(t,\cdot)) - g(t,\overline{U}'(t,\cdot)) + \Delta\overline{Z}(t)\big)\Big)dt\right] \\ &+ \mathbb{E}\Bigg[\int_0^T\int_{\mathbb{R}_0} \Delta K(t,\theta)\Big(\gamma^{\alpha_0}(t,U(t,\cdot),\theta) - \gamma^{\alpha_0}(t,U'(t,\cdot),\theta) + \delta\big(\gamma(t,\overline{U}(t,\cdot),\theta) - \gamma(t,\overline{U}'(t,\cdot),\theta) \\ &\qquad + \Delta\overline{K}(t,\cdot)\big)\Big)\nu(d\theta)dt\Bigg] \\ \le& \mathbb{E}\left[\int_0^T \big[(\alpha_0 - \mu\alpha_0 - 1)||\Delta U(t,\cdot)||^2 + \frac{\delta(c+1)}{2}||\Delta U(t,\cdot)||^2 + \frac{\delta(c+1)}{2}||\Delta\overline{U}(t,\cdot)||^2\big]dt\right]. \end{aligned}$$

Then we can derive that

$$\begin{aligned}&\left[\lambda-\frac{\delta(c+1)}{2}\right]\mathbb{E}\left[\int_0^T||\Delta U(t,\cdot)||^2dt\right]\\ \leq&\frac{\delta(c+1)}{2}\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right]+\frac{\delta}{2}\mathbb{E}[\Delta y(T)^2]+\frac{\delta}{2}\mathbb{E}[\Delta\overline{y}(T)^2],\end{aligned}$$

where $\lambda=\min\{1,\mu\}$. From Itô's formula, we get:

$$\begin{aligned}&\mathbb{E}\left[\Delta y(T)^2\right]+\mathbb{E}\left[\int_0^T\Delta z(t)^2dt\right]\\ =&2\mathbb{E}\left[\int_0^T\Delta y(t)\Big(f^{\alpha_0}(t,U(t,\cdot))-f^{\alpha_0}(t,U'(t,\cdot))+\delta\big(f(t,\overline{U}(t,\cdot))-f(t,\overline{U}'(t,\cdot))+\Delta\overline{Y}(t)\big)\Big)dt\right]\\ &+\mathbb{E}\left[\int_0^T\Big(g^{\alpha_0}(t,U(t,\cdot))-g^{\alpha_0}(t,U'(t,\cdot))+\delta\big(g(t,\overline{U}(t,\cdot))-g(t,\overline{U}'(t,\cdot))+\Delta\overline{Z}(t)\big)\Big)^2dt\right]\\ &+\mathbb{E}\Bigg[\int_0^T\int_{\mathbb{R}_0}\Big(\gamma^{\alpha_0}(t,U(t,\cdot),\theta)-\gamma^{\alpha_0}(t,U'(t,\cdot),\theta)+\delta\big(\gamma(t,\overline{U}(t,\cdot),\theta)-\gamma(t,\overline{U}'(t,\cdot),\theta)\\ &\qquad+\Delta\overline{K}(t,\cdot)\big)\Big)^2\nu(d\theta)dt\Bigg]\\ \leq&\beta\,\mathbb{E}\left[\int_0^T||\Delta U(t,\cdot)||^2dt\right]+\delta^2\beta\,\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right],\end{aligned}$$

where β is a constant which depends only on the constant c, and $\beta>1$. Thus, we have

$$\left[\lambda-\frac{\delta(c+\beta+1)}{2}\right]\mathbb{E}\left[\int_0^T||\Delta U(t,\cdot)||^2dt\right]\leq\frac{\delta(c+\beta+1)}{2}\left(\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right]+\mathbb{E}\left[\Delta\overline{y}(T)^2\right]\right),$$

$$\mathbb{E}\left[\Delta y(T)^2\right]\leq\left[\frac{\beta\delta(c+\beta+1)}{2\lambda-\delta(c+\beta+1)}+\delta^2\beta\right]\left(\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right]+\mathbb{E}\left[\Delta\overline{y}(T)^2\right]\right).$$

Let $\delta_0=\frac{2\lambda}{(8\beta+1)(c+\beta+1)}$, then for any $\delta\in[0,\delta_0]$,

$$\begin{aligned}&\mathbb{E}\left[\int_0^T||\Delta U(t,\cdot)||^2dt\right]\leq\frac{1}{8\beta}\left(\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right]+\mathbb{E}\left[\Delta\overline{y}(T)^2\right]\right),\\ &\mathbb{E}\left[\Delta y(T)^2\right]\leq\frac{3}{8}\left(\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right]+\mathbb{E}\left[\Delta\overline{y}(T)^2\right]\right).\end{aligned}$$

It is clearly to see that

$$\mathbb{E}\left[\int_0^T||\Delta U(t,\cdot)||^2dt\right]+\mathbb{E}\left[\Delta y(T)^2\right]\leq\frac{1}{2}\left(\mathbb{E}\left[\int_0^T||\Delta\overline{U}(t,\cdot)||^2dt\right]+\mathbb{E}\left[\Delta\overline{y}(T)^2\right]\right).$$

Thus, for each fixed $\delta\in[0,\delta_0]$, the mapping $I_{\alpha_0+\delta}$ is contractive, which has a unique fixed point $U(t,\cdot)=(y(t),Y(t),z(t),Z(t),K(t,\cdot))\in\mathbb{M}^2$. We can see that $U(t,\cdot)$ is the solution of (4) for $\alpha=\alpha_0+\delta,\delta\in[0,\delta_0]$. The proof is completed. ▮

When $\alpha = 0$, Equation (4) is reduced to the following simple form:

$$\begin{cases} dy(t) &= [f_0(t) - Y(t)]dt + [g_0(t) - Z(t)]\overrightarrow{d}W(t) - z(t)\overleftarrow{d}B(t) \\ &\quad + \int_{\mathbb{R}_0}[\gamma_0(t,\theta) - K(t,\theta)]\widetilde{N}(\overrightarrow{d}t, d\theta), \\ y(0) &= x, \\ dY(t) &= -[y(t) + F_0(t)]dt - [z(t) + G_0(t)]\overleftarrow{d}B(t) + Z(t)\overrightarrow{d}W(t) \\ &\quad + \int_{\mathbb{R}_0} K(t,\theta)\widetilde{N}(\overrightarrow{d}t, d\theta), \\ Y(T) &= \xi + y(T). \end{cases} \tag{5}$$

We have the following lemma:

Lemma 3.3 *For any $x \in \mathbb{R}, \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, $(F_0(t), f_0(t), G_0(t), g_0(t), \gamma_0(t,\cdot)) \in \mathbb{M}^2$, (5) has a unique solution $(y(t), Y(t), z(t), Z(t), K(t,\cdot)) \in \mathbb{M}^2$.*

Proof Uniqueness: Assume that there exists $(y'(t), Y'(t), z'(t), Z'(t), K'(t,\cdot)) \in \mathbb{M}^2$ satisfying the equation (5). We denote

$$\begin{aligned} &\Delta y(t) = y(t) - y'(t), \quad \Delta Y(t) = Y(t) - Y'(t), \\ &\Delta z(t) = z(t) - z'(t), \quad \Delta Z(t) = Z(t) - Z'(t), \\ &\Delta K(t,\cdot) = K(t,\cdot) - K'(t,\cdot). \end{aligned}$$

Applying Itô's formula to $\Delta y(t)\Delta Y(t)$, we get

$$0 \le \mathbb{E}\left[\Delta y(T)^2\right] = -\mathbb{E}\left[\int_0^T \Delta y(t)^2 + \Delta Y(t)^2 + \Delta z(t)^2 + \Delta Z(t)^2 + ||\Delta K(t,\cdot)||_\nu^2 dt\right].$$

Thus, the solution of Equation (5) is unique in $\mathbb{M}^2$.

Existence: We consider the following backward doubly stochastic differential equation:

$$\begin{cases} d\overline{Y}(t) = -[f_0(t) - \overline{Y}(t) + F_0(t)]dt - G_0(t)\overleftarrow{d}B(t) + [2\overline{Z}(t) - g_0(t)]\overrightarrow{d}W(t) \\ \qquad\quad + \int_{\mathbb{R}_0}[2\overline{K}(t,\theta) - \gamma_0(t,\theta)]\widetilde{N}(\overrightarrow{d}t, d\theta), \\ \overline{Y}(T) \ = \xi. \end{cases} \tag{6}$$

According to [20], there exists a unique solution $(\overline{Y}(t), \overline{Z}(t), \overline{K}(t,\cdot))$ solving (6).

We now consider the following forward doubly stochastic differential equation:

$$\begin{cases} d\overline{y}(t) = [f_0(t) - \overline{Y}(t) - \overline{y}(t)]dt - \overline{z}(t)\overleftarrow{d}B(t) + [g_0(t) - \overline{Z}(t)]\overrightarrow{d}W(t) \\ \qquad\quad + \int_{\mathbb{R}_0}[\gamma_0(t,\theta) - \overline{K}(t,\theta)]\widetilde{N}(\overrightarrow{d}t, d\theta), \\ y(0) \ = x. \end{cases} \tag{7}$$

The above equation has a unique solution $(\overline{y}(t), \overline{z}(t))$. We can refer to [22] for details. Let $y(t) = \overline{y}(t), Y(t) = y(t) + \overline{Y}(t)$, $Z(t) = \overline{Z}(t)$, $z(t) = \overline{z}(t)$, $K(t,\cdot) = \overline{K}(t,\cdot)$, then (5) has a solution $(y(t), Y(t), z(t), Z(t), K(t,\cdot))$. We get the existence. ∎

Now, we are going to prove Theorem 3.1:

Proof Uniqueness: We can derive the result from Lemma 3.2 and Lemma 3.3 obviously. Existence: By Lemma 3.3, for any $x \in \mathbb{R}$, $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, $(F_0(t), f_0(t), G_0(t), g_0(t), \gamma_0(t,\cdot)) \in \mathbb{M}^2$, (5) has a solution $(y(t), Y(t), z(t), Z(t), K(t,\cdot)) \in \mathbb{M}^2$.

We can derive from Lemma 3.2 that there exists a positive constant δ_0, which depends only on c, μ, such that (4) has a unique solution for $\alpha = \delta \in [0, \delta_0]$, thus, we can repeat this process for N times with $1 \le N\delta_0 < 1 + \delta_0$. In particular, Equation (4) for $\alpha = 1$ with $(F_0(t), f_0(t), G_0(t), g_0(t), \gamma_0(t,\cdot)) = 0$ has a unique solution in $\mathbb{M}^2$. The theorem is proved. ∎

4 Variational Equation and Variational Inequality

Suppose that $(y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t))$ is the solution to the optimal control problem (2). For any $v(t) \in \mathcal{U}_{ad}$ satisfying $u(t) + v(t) \in \mathcal{U}_{ad}$, by the convexity of $\mathcal{U}_{ad}$, we get for all $0 < \rho \le 1, u^\rho(t) = u(t) + \rho v(t) \in \mathcal{U}_{ad}$. Let $(y^\rho(t), Y^\rho(t), z^\rho(t), Z^\rho(t), K^\rho(t,\cdot))$ be the trajectory of (1) corresponding to $u^\rho(t)$.

For convenience, we use the following notations in this section:

$$\begin{aligned}
&\Phi(t) = \Phi(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t)),\\
&\Phi^\rho(t) = \Phi(t, y^\rho(t), Y^\rho(t), z^\rho(t), Z^\rho(t), K^\rho(t,\cdot), u^\rho(t)),\\
&\Phi(t, u^\rho(t)) = \Phi(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u^\rho(t)),
\end{aligned}$$

where $\Phi = F, f, G, g, \gamma$ and their derivatives with respect to (y, Y, z, Z, K, u) respectively.

Lemma 4.1 *Under Assumptions* (A1)–(A3)*, we have*

$$\begin{aligned}
&\lim_{\rho\to 0} \mathbb{E}\left[\int_0^T \left(y^\rho(t) - y(t)\right)^2 dt\right] = 0, \qquad \lim_{\rho\to 0} \mathbb{E}\left[\int_0^T \left(Y^\rho(t) - Y(t)\right)^2 dt\right] = 0,\\
&\lim_{\rho\to 0} \mathbb{E}\left[\int_0^T \left(z^\rho(t) - z(t)\right)^2 dt\right] = 0, \qquad \lim_{\rho\to 0} \mathbb{E}\left[\int_0^T \left(Z^\rho(t) - Z(t)\right)^2 dt\right] = 0,\\
&\lim_{\rho\to 0} \mathbb{E}\left[\int_0^T ||K^\rho(t,\cdot) - K(t,\cdot)||_\nu^2 dt\right] = 0.
\end{aligned}$$

Proof Denote $\overline{y}(t) = y^\rho(t) - y(t)$ and similarly for $\overline{Y}(t), \overline{z}(t), \overline{Z}(t), \overline{K}(t,\cdot)$, then

$$\begin{cases}
d\overline{y}(t) &= \big(f^\rho(t) - f(t)\big)dt + \big(g^\rho(t) - g(t)\big)\overrightarrow{d}W(t)\\
&\quad -\overline{z}(t)\overleftarrow{d}B(t) + \displaystyle\int_{\mathbb{R}_0} \big(\gamma^\rho(t,\theta) - \gamma(t,\theta)\big)\widetilde{N}(\overrightarrow{d}t, d\theta),\\
\overline{y}(0) &= 0,\\
d\overline{Y}(t) &= -\big(F^\rho(t) - F(t)\big)dt - \big(G^\rho(t) - G(t)\big)\overleftarrow{d}B(t)\\
&\quad +\overline{Z}(t)\overrightarrow{d}W(t) + \displaystyle\int_{\mathbb{R}_0} \overline{K}(t,\theta)\widetilde{N}(\overrightarrow{d}t, d\theta),\\
\overline{Y}(T) &= 0.
\end{cases}$$

Using Itô's formula to $\overline{y}(t)\overline{Y}(t)$, it follows that

$$
\begin{aligned}
0 \leq & \mathbb{E}\left[\int_0^T \langle A(t,\zeta)-A(t,\zeta'),\zeta-\zeta'\rangle\, dt\right] \\
& + \mathbb{E}\left[\int_0^T \left|\overline{y}(t)\big(F(t,u^\rho(t))-F(t)\big)\right| dt\right] + \mathbb{E}\left[\int_0^T \left|\overline{Y}(t)\big(f(t,u^\rho(t))-f(t)\big)\right| dt\right] \\
& + \mathbb{E}\left[\int_0^T \left|\overline{z}(t)\big(G(t,u^\rho(t))-G(t)\big)\right| dt\right] + \mathbb{E}\left[\int_0^T \left|\overline{Z}(t)\big(g(t,u^\rho(t))-g(t)\big)\right| dt\right] \\
& + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \left|\overline{K}(t,\theta)\big(\gamma(t,u^\rho(t),\theta)-\gamma(t,\theta)\big)\right| \nu(d\theta)dt\right] \\
\leq & -\frac{\mu}{2}\mathbb{E}\left[\int_0^T \{\overline{y}(t)^2+\overline{Y}(t)^2+\overline{z}(t)^2+\overline{Z}(t)^2+||\overline{K}(t,\cdot)||_\nu^2\}dt\right] + c(\mu)\mathbb{E}\left[\int_0^T c^2\rho^2 v^2 dt\right],
\end{aligned}
$$

where $c(\mu)$ is a constant depending on μ only.

Hence, we obtain the desired results. ∎

Lemma 4.2 *Under Assumptions* (A1)–(A3), *we have:*

$$
\lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\big(y^\rho(t)-y(t)\big)^2\right]=0, \qquad \lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\big(Y^\rho(t)-Y(t)\big)^2\right]=0.
$$

Proof We see that

$$
\begin{aligned}
\overline{y}(t)^2 = & \Bigg(\int_0^t \big(f^\rho(s)-f(s)\big)ds + \int_0^t \big(g^\rho(s)-g(s)\big)\overrightarrow{d}W(s) - \int_0^t \overline{z}(s)\overleftarrow{d}B(s) \\
& + \int_0^t\int_{\mathbb{R}_0} \big(\gamma^\rho(s,\theta)-\gamma(s,\theta)\big)\widetilde{N}(\overrightarrow{d}s,d\theta)\Bigg)^2 \\
\leq & C\Big(\int_0^t \big(f^\rho(s)-f(s)\big)ds\Big)^2 + C\Big(\int_0^t \big(g^\rho(s)-g(s)\big)\overrightarrow{d}W(s)\Big)^2 + C\Big(\int_0^T \overline{z}(s)\overleftarrow{d}B(s)\Big)^2 \\
& + C\Big(\int_t^T \overline{z}(s)\overleftarrow{d}B(s)\Big)^2 + C\Big(\int_0^t\int_{\mathbb{R}_0} \big(\gamma^\rho(s,\theta)-\gamma(s,\theta)\big)\widetilde{N}(\overrightarrow{d}s,d\theta)\Big)^2, \\
\overline{Y}(t)^2 = & \Bigg(\int_t^T \big(F^\rho(s)-F(s)\big)ds + \int_t^T \big(G^\rho(s)-G(s)\big)\overleftarrow{d}B(s) - \int_t^T \overline{Z}(s)\overrightarrow{d}W(s) \\
& - \int_t^T\int_{\mathbb{R}_0} \overline{K}(s,\theta)\widetilde{N}(\overrightarrow{d}s,d\theta)\Bigg)^2 \\
\leq & C\Big(\int_t^T (F^\rho(s)-F(s))ds\Big)^2 + C\Big(\int_t^T (G^\rho(s)-G(s))\overleftarrow{d}B(s)\Big)^2 + C\Big(\int_0^T \overline{Z}(s)\overrightarrow{d}W(s)\Big)^2 \\
& + C\Big(\int_0^T\int_{\mathbb{R}_0} \overline{K}(s,\theta)\widetilde{N}(\overrightarrow{d}s,d\theta)\Big)^2 + C\Big(\int_0^t \overline{Z}(s)\overrightarrow{d}W(s)\Big)^2 \\
& + C\Big(\int_0^t\int_{\mathbb{R}_0} \overline{K}(s,\theta)\widetilde{N}(\overrightarrow{d}s,d\theta)\Big)^2,
\end{aligned}
$$

where C is a generic constant which might be different in different place.

Thus by Burkholder-Davis-Gundy's inequality and Assumption (A2), we get

$$\lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\le t\le T}(y^\rho(t)-y(t))^2\right]=0,\qquad \lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\le t\le T}(Y^\rho(t)-Y(t))^2\right]=0.$$

The proof is finished. ∎

We introduce the following variational equation:

$$\begin{cases} dy^1(t) &= \langle\nabla f(t),I(t)\rangle\,dt+\langle\nabla g(t),I(t)\rangle\,\overrightarrow{d}W(t)-z^1(t)\overleftarrow{d}B(t)\\ &\quad+\displaystyle\int_{\mathbb{R}_0}\langle\nabla\gamma(t,\theta),I(t)\rangle\,\widetilde{N}(\overrightarrow{d}t,d\theta),\\ y^1(0) &=0,\\ dY^1(t) &=-\langle\nabla F(t),I(t)\rangle\,dt-\langle\nabla G(t),I(t)\rangle\,\overleftarrow{d}B(t)+Z^1(t)\overrightarrow{d}W(t)\\ &\quad+\displaystyle\int_{\mathbb{R}_0}K^1(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta),\\ Y^1(T) &=0,\end{cases}\tag{8}$$

where for $\phi=f,g,F,G,\gamma$,

$$\begin{aligned} I(t)&=(y^1(t),Y^1(t),z^1(t),Z^1(t),K^1(t,\cdot),v(t)),\\ \langle\nabla\phi(t),I(t)\rangle&=\phi_y(t)y^1(t)+\phi_Y(t)Y^1(t)+\phi_z(t)z^1(t)+\phi_Z(t)Z^1(t)\\ &\quad+\int_{\mathbb{R}_0}\phi_K(t)K^1(t,\theta)\nu(d\theta)+\phi_u(t)v(t).\end{aligned}$$

Setting $\widetilde{y}(t)=\frac{y^\rho(t)-y(t)}{\rho}-y^1(t)$ and similarly for $\widetilde{Y}(t),\widetilde{z}(t),\widetilde{Z}(t),\widetilde{K}(t,\cdot)$, we have the following two lemmas.

Lemma 4.3 *Under Assumptions* (A1)–(A3)*, we have*

$$\begin{aligned} &\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T\widetilde{y}(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T\widetilde{Y}(t)^2dt\right]=0,\\ &\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T\widetilde{z}(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T\widetilde{Z}(t)^2dt\right]=0,\\ &\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T||\widetilde{K}(t,\cdot)||_\nu^2dt\right]=0.\end{aligned}$$

Proof For $\Phi=f,g,F,G,\gamma$, we denote

$$\Phi^{\widetilde{\rho}}(t)=\Phi(t,y^\rho(t)-\rho\widetilde{y}(t),Y^\rho(t)-\rho\widetilde{Y}(t),z^\rho(t)-\rho\widetilde{z}(t),Z^\rho(t)-\rho\widetilde{Z}(t),K^\rho(t,\cdot)-\rho\widetilde{K}(t,\cdot),u^\rho(t)).$$

Note that $\widetilde{y}(t)$ and $\widetilde{Y}(t)$ satisfy the following equation:

$$\begin{cases} d\widetilde{y}(t) &= \left[\dfrac{f^\rho(t)-f(t)}{\rho} - \langle \nabla f(t), I(t)\rangle\right]dt + \left[\dfrac{g^\rho(t)-g(t)}{\rho} - \langle \nabla g(t), I(t)\rangle\right]\overrightarrow{d}W(t) \\ &\quad -\widetilde{z}(t)\overleftarrow{d}B(t) + \displaystyle\int_{\mathbb{R}_0}\left[\dfrac{\gamma^\rho(t,\theta)-\gamma(t,\theta)}{\rho} - \langle \nabla \gamma(t,\theta), I(t)\rangle\right]\widetilde{N}(\overrightarrow{d}t,d\theta), \\ \widetilde{y}(0) &= 0, \\ d\widetilde{Y}(t) &= -\left[\dfrac{F^\rho(t)-F(t)}{\rho} - \langle \nabla F(t), I(t)\rangle\right]dt - \left[\dfrac{G^\rho(t)-G(t)}{\rho} - \langle \nabla G(t), I(t)\rangle\right]\overleftarrow{d}B(t) \\ &\quad +\widetilde{Z}(t)\overrightarrow{d}W(t) + \displaystyle\int_{\mathbb{R}_0}\widetilde{K}(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta), \\ \widetilde{Y}(T) &= 0. \end{cases} \tag{9}$$

We can rewrite the equation as

$$\begin{cases} d\widetilde{y}(t) &= \dfrac{1}{\rho}\Big[f^\rho(t) - f^{\widetilde{\rho}}(t) + \rho A_f(t)y^1(t) + \rho B_f(t)Y^1(t) + \rho C_f(t)z^1(t) + \rho D_f(t)Z^1(t) \\ &\qquad +\rho\displaystyle\int_{\mathbb{R}_0} E_f(t)K^1(t,\theta)\nu(d\theta) + \rho M_f(t)v(t)\Big]dt \\ &\quad +\dfrac{1}{\rho}\Big[g^\rho(t) - g^{\widetilde{\rho}}(t) + \rho A_g(t)y^1(t) + \rho B_g(t)Y^1(t) + \rho C_g(t)z^1(t) + \rho D_g(t)Z^1(t) \\ &\qquad +\rho\displaystyle\int_{\mathbb{R}_0} E_g(t)K^1(t,\theta)\nu(d\theta) + \rho M_g(t)v(t)\Big]\overrightarrow{d}W(t) - \widetilde{z}(t)\overleftarrow{d}B(t) \\ &\quad +\displaystyle\int_{\mathbb{R}_0}\dfrac{1}{\rho}\Big[\gamma^\rho(t,\theta) - \gamma^{\widetilde{\rho}}(t,\theta) + \rho A_\gamma(t)y^1(t) + \rho B_\gamma(t)Y^1(t) + \rho C_\gamma(t)z^1(t) + \rho D_\gamma(t)Z^1(t) \\ &\qquad +\rho\displaystyle\int_{\mathbb{R}_0} E_\gamma(t)K^1(t,\theta)\nu(d\theta) + \rho M_\gamma(t)v(t)\Big]\widetilde{N}(\overrightarrow{d}t,d\theta), \\ \widetilde{y}(0) &= 0, \\ d\widetilde{Y}(t) &= -\dfrac{1}{\rho}\Big[F^\rho(t) - F^{\widetilde{\rho}}(t) + \rho A_F(t)y^1(t) + \rho B_F(t)Y^1(t) + \rho C_F(t)z^1(t) + \rho D_F(t)Z^1(t) \\ &\qquad +\rho\displaystyle\int_{\mathbb{R}_0} E_F(t)K^1(t,\theta)\nu(d\theta) + \rho M_F(t)v(t)\Big]dt \\ &\quad -\dfrac{1}{\rho}\Big[G^\rho(t) - G^{\widetilde{\rho}}(t) + \rho A_G(t)y^1(t) + \rho B_G(t)Y^1(t) + \rho C_G(t)z^1(t) + \rho D_G(t)Z^1(t) \\ &\qquad +\rho\displaystyle\int_{\mathbb{R}_0} E_G(t)K^1(t,\theta)\nu(d\theta) + \rho M_G(t)v(t)\Big]\overleftarrow{d}B(t) \\ &\quad +\widetilde{Z}(t)\overrightarrow{d}W(t) + \displaystyle\int_{\mathbb{R}_0}\widetilde{K}(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta), \\ \widetilde{Y}(T) &= 0, \end{cases}$$

where for $\phi = f, g, F, G, \gamma$,

$$A_\phi(t) = \int_0^1 \{\phi_y(t, y(t) + \lambda\rho y^1(t), Y(t) + \lambda\rho Y^1(t), z(t) + \lambda\rho z^1(t), Z(t) + \lambda\rho Z^1(t), K(t,\cdot) \\ + \lambda\rho K^1(t,\cdot), u(t) + \lambda\rho v(t)) - \phi_y(t)\}d\lambda,$$

$$
\begin{aligned}
B_\phi(t)=&\int_0^1\{\phi_Y(t,y(t)+\lambda\rho y^1(t),Y(t)+\lambda\rho Y^1(t),z(t)+\lambda\rho z^1(t),Z(t)+\lambda\rho Z^1(t),K(t,\cdot)\\
&+\lambda\rho K^1(t,\cdot),u(t)+\lambda\rho v(t))-\phi_Y(t)\}d\lambda,\\
C_\phi(t)=&\int_0^1\{\phi_z(t,y(t)+\lambda\rho y^1(t),Y(t)+\lambda\rho Y^1(t),z(t)+\lambda\rho z^1(t),Z(t)+\lambda\rho Z^1(t),K(t,\cdot)\\
&+\lambda\rho K^1(t,\cdot),u(t)+\lambda\rho v(t))-\phi_z(t)\}d\lambda,\\
D_\phi(t)=&\int_0^1\{\phi_Z(t,y(t)+\lambda\rho y^1(t),Y(t)+\lambda\rho Y^1(t),z(t)+\lambda\rho z^1(t),Z(t)+\lambda\rho Z^1(t),K(t,\cdot)\\
&+\lambda\rho K^1(t,\cdot),u(t)+\lambda\rho v(t))-\phi_Z(t)\}d\lambda,\\
E_\phi(t)=&\int_0^1\{\phi_K(t,y(t)+\lambda\rho y^1(t),Y(t)+\lambda\rho Y^1(t),z(t)+\lambda\rho z^1(t),Z(t)+\lambda\rho Z^1(t),K(t,\cdot)\\
&+\lambda\rho K^1(t,\cdot),u(t)+\lambda\rho v(t))-\phi_K(t)\}d\lambda,\\
M_\phi(t)=&\int_0^1\{\phi_u(t,y(t)+\lambda\rho y^1(t),Y(t)+\lambda\rho Y^1(t),z(t)+\lambda\rho z^1(t),Z(t)+\lambda\rho Z^1(t),K(t,\cdot)\\
&+\lambda\rho K^1(t,\cdot),u(t)+\lambda\rho v(t))-\phi_u(t)\}d\lambda.
\end{aligned}
$$

Applying Itô's formula to $\widetilde{y}(t)\widetilde{Y}(t)$, we get

$$
\begin{aligned}
&\mu\mathbb{E}\Big[\int_0^T\big(\widetilde{y}(t)^2+\widetilde{Y}(t)^2+\widetilde{z}(t)^2+\widetilde{Z}(t)^2+||\widetilde{K}(t,\cdot)||_\nu^2\big)dt\Big]\\
\le&\frac{\mu}{2}\mathbb{E}\Big[\int_0^T\big(\widetilde{y}(t)^2+\widetilde{Y}(t)^2+\widetilde{z}(t)^2+\widetilde{Z}(t)^2+||\widetilde{K}(t,\cdot)||_\nu^2\big)dt\Big]\\
&+C\mathbb{E}\Big[\int_0^T\big(A_f(t)y^1(t)+B_f(t)Y^1(t)+C_f(t)z^1(t)+D_f(t)Z^1(t)\\
&\qquad+\int_{\mathbb{R}_0}E_f(t)K^1(t,\theta)\nu(d\theta)+M_f(t)v(t)\big)^2dt\Big]\\
&+C\mathbb{E}\Big[\int_0^T\big(A_g(t)y^1(t)+B_g(t)Y^1(t)+C_g(t)z^1(t)+D_g(t)Z^1(t)\\
&\qquad+\int_{\mathbb{R}_0}E_g(t)K^1(t,\theta)\nu(d\theta)+M_g(t)v(t)\big)^2dt\Big]\\
&+C\mathbb{E}\Big[\int_0^T\big(A_F(t)y^1(t)+B_F(t)Y^1(t)+C_F(t)z^1(t)+D_F(t)Z^1(t)\\
&\qquad+\int_{\mathbb{R}_0}E_F(t)K^1(t,\theta)\nu(d\theta)+M_F(t)v(t)\big)^2dt\Big]\\
&+C\mathbb{E}\Big[\int_0^T\big(A_G(t)y^1(t)+B_G(t)Y^1(t)+C_G(t)z^1(t)+D_G(t)Z^1(t)\\
&\qquad+\int_{\mathbb{R}_0}E_G(t)K^1(t,\theta)\nu(d\theta)+M_G(t)v(t)\big)^2dt\Big]\\
&+C\mathbb{E}\Big[\int_0^T\big(A_\gamma(t)y^1(t)+B_\gamma(t)Y^1(t)+C_\gamma(t)z^1(t)+D_\gamma(t)Z^1(t)\\
&\qquad+\int_{\mathbb{R}_0}E_\gamma(t)K^1(t,\theta)\nu(d\theta)+M_\gamma(t)v(t)\big)^2dt\Big].
\end{aligned}
$$

Note that

$$\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T A_\phi(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T B_\phi(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T C_\phi(t)^2dt\right]=0,$$
$$\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T D_\phi(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T ||E_\phi(t)||_\nu^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T M_\phi(t)^2dt\right]=0.$$

Thus, we have

$$\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T \widetilde{y}(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T \widetilde{Y}(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T \widetilde{z}(t)^2dt\right]=0,$$
$$\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T \widetilde{Z}(t)^2dt\right]=0,\quad \lim_{\rho\to 0}\mathbb{E}\left[\int_0^T ||\widetilde{K}(t,\cdot)||_\nu^2dt\right]=0.$$

The proof is finished. ∎

Lemma 4.4 *Under Assumptions* (A1)–(A3), *it holds that*

$$\lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\widetilde{y}(t)^2\right]=0,\qquad \lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\widetilde{Y}(t)^2\right]=0.$$

Proof We can also rewrite Equation (9) in the following form:

$$\begin{cases}
d\widetilde{y}(t) = \Big[A_f^1(t)\widetilde{y}(t)+B_f^1(t)\widetilde{Y}(t)+C_f^1(t)\widetilde{z}(t)+D_f^1(t)\widetilde{Z}(t)+\int_{\mathbb{R}_0}E_f^1(t)\widetilde{K}(t,\theta)\nu(d\theta)+M_f^1(t)\Big]dt\\
\quad +\Big[A_g^1(t)\widetilde{y}(t)+B_g^1(t)\widetilde{Y}(t)+C_g^1(t)\widetilde{z}(t)+D_g^1(t)\widetilde{Z}(t)\\
\quad\quad +\int_{\mathbb{R}_0}E_g^1(t)\widetilde{K}(t,\theta)\nu(d\theta)+M_g^1(t)\Big]\overrightarrow{d}W(t)\\
\quad -\widetilde{z}(t)\overleftarrow{d}B(t)+\int_{\mathbb{R}_0}\Big[A_\gamma^1(t)\widetilde{y}(t)+B_\gamma^1(t)\widetilde{Y}(t)+C_\gamma^1(t)\widetilde{z}(t)+D_\gamma^1(t)\widetilde{Z}(t)\\
\quad +\int_{\mathbb{R}_0}E_\gamma^1(t)\widetilde{K}(t,\theta)\nu(d\theta)+M_\gamma^1(t)\Big]\widetilde{N}(\overrightarrow{d}t,d\theta),\\
\widetilde{y}(0) = 0,\\
d\widetilde{Y}(t) = -\Big[A_F^1(t)\widetilde{y}(t)+B_F^1(t)\widetilde{Y}(t)+C_F^1(t)\widetilde{z}(t)+D_F^1(t)\widetilde{Z}(t)\\
\quad\quad +\int_{\mathbb{R}_0}E_F^1(t)\widetilde{K}(t,\theta)\nu(d\theta)+M_F^1(t)\Big]dt\\
\quad -\Big[A_G^1(t)\widetilde{y}(t)+B_G^1(t)\widetilde{Y}(t)+C_G^1(t)\widetilde{z}(t)+D_G^1(t)\widetilde{Z}(t)\\
\quad\quad +\int_{\mathbb{R}_0}E_G^1(t)\widetilde{K}(t,\theta)\nu(d\theta)+M_G^1(t)\Big]\overleftarrow{d}B(t)\\
\quad +\widetilde{Z}(t)\overrightarrow{d}W(t)+\int_{\mathbb{R}_0}\widetilde{K}(t,\theta)\widetilde{N}(\overrightarrow{d}t,d\theta),\\
\widetilde{Y}(T) = 0,
\end{cases}$$

where

$$A^1_\phi(t)=\int_0^1 \phi_y(t,y(t)+\lambda\overline{y}(t),Y(t)+\lambda\overline{Y}(t),z(t)+\lambda\overline{z}(t),Z(t)+\lambda\overline{Z}(t),K(t,\cdot)+\lambda\overline{K}(t,\cdot),u^\rho(t))d\lambda,$$

$$B^1_\phi(t)=\int_0^1 \phi_Y(t,y(t)+\lambda\overline{y}(t),Y(t)+\lambda\overline{Y}(t),z(t)+\lambda\overline{z}(t),Z(t)+\lambda\overline{Z}(t),K(t,\cdot)+\lambda\overline{K}(t,\cdot),u^\rho(t))d\lambda,$$

$$C^1_\phi(t)=\int_0^1 \phi_z(t,y(t)+\lambda\overline{y}(t),Y(t)+\lambda\overline{Y}(t),z(t)+\lambda\overline{z}(t),Z(t)+\lambda\overline{Z}(t),K(t,\cdot)+\lambda\overline{K}(t,\cdot),u^\rho(t))d\lambda,$$

$$D^1_\phi(t)=\int_0^1 \phi_Z(t,y(t)+\lambda\overline{y}(t),Y(t)+\lambda\overline{Y}(t),z(t)+\lambda\overline{z}(t),Z(t)+\lambda\overline{Z}(t),K(t,\cdot)+\lambda\overline{K}(t,\cdot),u^\rho(t))d\lambda,$$

$$E^1_\phi(t)=\int_0^1 \phi_K(t,y(t)+\lambda\overline{y}(t),Y(t)+\lambda\overline{Y}(t),z(t)+\lambda\overline{z}(t),Z(t)+\lambda\overline{Z}(t),K(t,\cdot)+\lambda\overline{K}(t,\cdot),u^\rho(t))d\lambda,$$

$$\begin{aligned}M^1_\phi(t)&=y^1(t)\big(A^1_\phi(t)-\phi_y(t)\big)+Y^1(t)\big(B^1_\phi(t)-\phi_Y(t)\big)+z^1(t)\big(C^1_\phi(t)-\phi_z(t)\big)+Z^1(t)\big(D^1_\phi(t)-\phi_Z(t)\big)\\&\quad+\int_{\mathbb{R}_0}K^1(t,\theta)\big(E^1_\phi(t)-\phi_K(t)\big)\nu(d\theta)\\&\quad+\int_0^1 v(t)\big(\phi_u(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t)+\lambda\rho v(t))-\phi_u(t)\big)d\lambda.\end{aligned}$$

Note that

$$\lim_{\rho\to 0}\mathbb{E}\left[\int_0^T M^1_\phi(t)^2dt\right]=0,$$

and $A^1_\phi(t),B^1_\phi(t),C^1_\phi(t),D^1_\phi(t),E^1_\phi(t)$ are bounded.

Using the similar steps in the proof of Lemma 4.2, we can easily get that

$$\lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\widetilde{y}(t)^2\right]=0,\qquad \lim_{\rho\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\widetilde{Y}(t)^2\right]=0.$$

The proof is finished. ∎

Lemma 4.5 (Variational inequality) *Under Assumptions* (A1)–(A3), *it holds that*

$$\begin{aligned}0\le&\mathbb{E}\bigg[\int_0^T\Big\{l_y(t)y^1(t)+l_Y(t)Y^1(t)+l_z(t)z^1(t)+l_Z(t)Z^1(t)+\int_{\mathbb{R}_0}l_K(t)K^1(t,\theta)\nu(d\theta)\\&+l_u(t)v(t)\Big\}dt\bigg]+\mathbb{E}\left[h_{1y}(y(T))y^1(T)+h_{2Y}(Y(0))Y^1(0)\right].\end{aligned}$$

Proof Since $u(\cdot)$ is the optimal control, by the definition of $J(u)$, we get

$$\begin{aligned}0\le&\rho^{-1}[J(u^\rho)-J(u)]\\=&\rho^{-1}\mathbb{E}\bigg[\int_0^T\big\{l(t,y^\rho(t),Y^\rho(t),z^\rho(t),Z^\rho(t),K^\rho(t,\cdot),u^\rho(t))\\&\qquad-l(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t))\big\}dt\bigg]\\&+\rho^{-1}\mathbb{E}[h_1(y^\rho(T))-h_1(y(T))]+\rho^{-1}\mathbb{E}[h_2(Y^\rho(0))-h_2(Y(0))].\end{aligned}$$

When $\rho \to 0$, we get

$$\begin{aligned}
&\rho^{-1}\mathbb{E}\left[\int_0^T (l^\rho(t) - l(t))dt\right] \to \mathbb{E}\left[\int_0^T \Big\{l_y(t)y^1(t) + l_Y(t)Y^1(t) + l_z(t)z^1(t) + l_Z(t)Z^1(t)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \int_{\mathbb{R}_0} l_K(t)K^1(t,\theta)\nu(d\theta) + l_u(t)v(t)\Big\}dt\right],\\
&\rho^{-1}\mathbb{E}\left[h_1(y^\rho(T) - h_1(y(T))\right] \to \mathbb{E}\left[h_{1y}(y(T))y^1(T)\right],\\
&\rho^{-1}\mathbb{E}\left[h_2(Y^\rho(0) - h_2(Y(0))\right] \to \mathbb{E}\left[h_{2Y}(Y(0))Y^1(0)\right].
\end{aligned}$$

Then the lemma is proved. ∎

5 The Necessary Maximum Principle

To derive the necessary maximum principle in this section, we introduce the following adjoint equation:

$$\begin{cases}
dp(t) &= -H_y(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), p(t), P(t), q(t), Q(t), r(t,\cdot))dt\\
&\quad -H_z(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), p(t), P(t), q(t), Q(t), r(t,\cdot))\overleftarrow{d}B(t)\\
&\quad +q(t)\overrightarrow{d}W(t) + \int_{\mathbb{R}_0} r(t,\theta)\widetilde{N}(\overrightarrow{d}t, d\theta),\\
dP(t) &= -H_Y(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), p(t), P(t), q(t), Q(t), r(t,\cdot))dt - Q(t)\overleftarrow{d}B(t)\\
&\quad -H_Z(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), p(t), P(t), q(t), Q(t), r(t,\cdot))\overrightarrow{d}W(t)\\
&\quad -\int_{\mathbb{R}_0} H_K(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), p(t), P(t), q(t), Q(t), r(t,\cdot))\widetilde{N}(\overrightarrow{d}t, d\theta),\\
p(T) &= h_{1y}(y(T)),\\
P(0) &= -h_{2Y}(Y(0)),
\end{cases} \tag{10}$$

where the Hamiltonian functional $H : [0,T] \times \mathbb{R}^5 \times \mathcal{U} \times \mathbb{R}^5 \to \mathbb{R}$ is defined by:

$$\begin{aligned}
&H(t, y, Y, z, Z, K, u, p, P, q, Q, r)\\
=&f(t, y, Y, z, Z, K, u)p + g(t, y, Y, z, Z, K, u)q - F(t, y, Y, z, Z, K, u)P\\
&- G(t, y, Y, z, Z, K, u)Q + \int_{\mathbb{R}_0} \gamma(t, y, Y, z, Z, K, u, \theta)r\ \nu(d\theta) + l(t, y, Y, z, Z, K, u).
\end{aligned}$$

We can verify that the equation (10) has a unique solution $(p(t), P(t), q(t), Q(t), r(t,\cdot)) \in \mathbb{M}^2$. We then have the main result in this paper.

Theorem 5.1 (The necessary maximum principle) *Suppose that Assumptions* (A1)–(A3) *hold, and let* $(y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t))$ *be the solution to the optimal control problem* (2), $(p(t), P(t), q(t), Q(t), r(t,\cdot))$ *be the corresponding solution of* (10). *Then the maximum principle holds, that is,*

$$\mathbb{E}[H_u(t, y(t), Y(t), z(t), Z(t), K(t,\cdot), u(t), p(t), P(t), q(t), Q(t), r(t,\cdot))|\mathcal{G}_t] = 0.$$

Proof Applying Itô's formula to $p(t)y^1(t)$ and $P(t)Y^1(t)$, and combining with the fact that $y^1(0)=0, Y^1(T)=0$, we get

$$\begin{aligned}
&\mathbb{E}\left[h_{1y}(y(T))y^1(T)+h_{2Y}(Y(0))Y^1(0)\right]\\
=&\mathbb{E}\left[p(T)y^1(T)-P(0)Y^1(0)\right]\\
=&-\mathbb{E}\left[\int_0^T H_y(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))y^1(t)dt\right]\\
&-\mathbb{E}\left[\int_0^T H_z(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))z^1(t)dt\right]\\
&+\mathbb{E}\left[\int_0^T \Big(p(t)\left\langle\nabla f(t),I(t)\right\rangle+q(t)\left\langle\nabla g(t),I(t)\right\rangle+\int_{\mathbb{R}_0} r(t,\theta)\left\langle\nabla\gamma(t,\theta),I(t)\right\rangle\nu(d\theta)\Big)dt\right]\\
&-\mathbb{E}\left[\int_0^T H_Y(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))Y^1(t)dt\right]\\
&-\mathbb{E}\left[\int_0^T H_Z(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))Z^1(t)dt\right]\\
&-\mathbb{E}\Bigg[\int_0^T\int_{\mathbb{R}_0} H_K(t,y(t),Y(t),z(t),Z(t),K(t,\theta),u(t),p(t),P(t),q(t),Q(t),r(t,\theta))\\
&\qquad\cdot K^1(t,\theta)\nu(d\theta)dt\Bigg]\\
&-\mathbb{E}\left[\int_0^T \Big(P(t)\left\langle\nabla F(t),I(t)\right\rangle+Q(t)\left\langle\nabla G(t),I(t)\right\rangle\Big)dt\right].
\end{aligned}$$

Using Lemma 4.5, we have

$$\begin{aligned}
0\leq&\mathbb{E}\Bigg[\int_0^T\Big\{l_y(t)y^1(t)+l_Y(t)Y^1(t)+l_z(t)z^1(t)+l_Z(t)Z^1(t)\\
&\quad+\int_{\mathbb{R}_0} l_K(t)K^1(t,\theta)\nu(d\theta)+l_u(t)v(t)\Big\}dt\Bigg]+\mathbb{E}\left[h_{1y}(y(T))y^1(T)+h_{2Y}(Y(0))Y^1(0)\right]\\
=&\mathbb{E}\left[\int_0^T H_u(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))v(t)dt\right].
\end{aligned}$$

Letting $\alpha(s)=v_{[t,t+\varepsilon)}(s)$ for $\forall v(t)\in\mathcal{U}_{ad}$, we have $\alpha(s)\in\mathcal{U}_{ad}$, then

$$\mathbb{E}\Bigg[\int_t^{t+\varepsilon} H_u(s,y(s),Y(s),z(s),Z(s),K(s,\cdot),u(s),p(s),P(s),q(s),Q(s),r(s,\cdot))v(s)ds\Bigg]\geq 0.$$

Differentiating with respect to ε at $\varepsilon=0$, we get

$$\mathbb{E}\left[H_u(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))v(t)\right]\geq 0,\quad \text{a.s.}$$

And this inequality holds for all $v(t)$ which is $\mathcal{G}_t$-adapted, we have

$$\mathbb{E}[H_u(t,y(t),Y(t),z(t),Z(t),K(t,\cdot),u(t),p(t),P(t),q(t),Q(t),r(t,\cdot))|\mathcal{G}_t]=0.$$

The proof is completed. ∎

6 Application

In this section, we give an example of the optimal control of forward-backward doubly stochastic systems. Consider the optimal control problem, where the state process $(y(t), Y(t))$ is governed by

$$\begin{cases} dy(t) &= A(t)Y(t)dt + B(t)Z(t)\overrightarrow{d}W(t) - z(t)\overleftarrow{d}B(t), \qquad t \in [0,T], \\ y(0) &= x, \\ dY(t) &= -(C(t)y(t) + D(t)v(t))dt + Z(t)\overrightarrow{d}W(t) - N(t)z(t)\overleftarrow{d}B(t), \quad t \in [0,T], \\ Y(T) &= \xi, \end{cases} \tag{11}$$

where $A(t), B(t), C(t), N(t)$ are deterministic functions such that the above equation satisfies the conditions (A1)–(A3). And $v(t)$ is our control process.

The performance functional $J(v)$ is given by

$$J(v) = \frac{1}{2}\mathbb{E}\left[\int_0^T (R(t)y^2(t) + S(t)Y^2(t) + L(t)v^2(t))dt + Fy^2(T) + GY^2(0)\right],$$

where $R(t)$ and $S(t)$ are non-negative deterministic functions, and $L(t)$ is a positive deterministic function. We proceed to find an optimal control $u(t)$, such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v),$$

where $\mathcal{U}_{ad}$ is the collection of all admissible control processes.

The Hamiltonian functional H is reduced to

$$\begin{aligned} H(t,y,Y,z,Z,v,p,P,q,Q) =& \frac{1}{2}(R(t)y^2 + S(t)Y^2 + L(t)v^2) + A(t)Yp + B(t)Zq \\ &- (C(t)y + D(t)v)P - N(t)zQ. \end{aligned}$$

And we can deduce the adjoint equation as follows:

$$\begin{cases} dp(t) &= -(R(t)y(t) - c(t)P(t))dt + N(t)Q(t)\overleftarrow{d}B(t) + q(t)\overrightarrow{d}W(t), \\ p(T) &= Fy(T), \\ dP(t) &= -(A(t)p(t) + S(t)Y(t))dt - B(t)q(t)\overrightarrow{d}W(t) - Q(t)\overleftarrow{d}B(t), \\ P(0) &= -GY(0). \end{cases}$$

Assuming that the information available to the controller is denoted by $\mathcal{F}_t^W$. According to Theorem 5.1, if u is the optimal control, then

$$u(t) = L^{-1}(t)D(t)\mathbb{E}[P(t)|\mathcal{F}_t^W].$$

In the rest of our paper, we try to give a more explicit representation of $u(\cdot)$. We set $\widehat{X}(t) =$

$\mathbb{E}[X(t)|\mathcal{F}_t^W]$, $X = y, Y, z, Z, p, P, q, Q$. By the Lemma 5.4 of [26], $\widehat{p}(\cdot)$ and $\widehat{P}(\cdot)$ satisfy

$$\begin{cases} d\widehat{p}(t) &= -(R(t)\widehat{y}(t) - c(t)\widehat{P}(t))dt + \widehat{q}(t)\overrightarrow{d}W(t), \\ \widehat{p}(T) &= F\widehat{y}(T), \\ d\widehat{P}(t) &= -(A(t)\widehat{p}(t) + S(t)\widehat{Y}(t))dt - B(t)\widehat{q}(t)\overrightarrow{d}W(t), \\ \widehat{P}(0) &= -G\widehat{Y}(0), \end{cases}$$

where $\widehat{y}(\cdot)$ and $\widehat{Y}(\cdot)$ satisfy

$$\begin{cases} d\widehat{y}(t) &= A(t)\widehat{Y}(t)dt + B(t)\widehat{Z}(t)\overrightarrow{d}W(t), \qquad t \in [0,T], \\ \widehat{y}(0) &= x, \\ d\widehat{Y}(t) &= -(C(t)\widehat{y}(t) + D(t)u(t))dt + \widehat{Z}(t)\overrightarrow{d}W(t), \quad t \in [0,T], \\ \widehat{Y}(T) &= \xi. \end{cases} \tag{12}$$

We put

$$\widehat{p}(t) = F(t)\widehat{y}(t), \quad F(T) = F, \quad \widehat{P}(t) = G(t)\widehat{Y}(t), \quad G(0) = -G.$$

By Itô's formula, we have

$$\begin{cases} d\widehat{p}(t) &= (\dot{F}(t)\widehat{y}(t) + F(t)A(t)\widehat{Y}(t))dt + F(t)B(t)\widehat{Z}(t)\overrightarrow{d}W(t), \\ \widehat{p}(T) &= F\widehat{y}(T), \\ d\widehat{P}(t) &= \Big[\dot{G}(t)\widehat{Y}(t) - G(t)(C(t)\widehat{y}(t) + D(t)u(t))\Big]dt + G(t)\widehat{Z}(t)\overrightarrow{d}W(t), \\ \widehat{P}(0) &= -G\widehat{Y}(0). \end{cases}$$

Combining the drift and the diffusion items of $\widehat{p}(t)$ and $\widehat{P}(t)$, we obtain

$$\begin{cases} \dot{F}(t)\widehat{y}(t) + F(t)A(t)\widehat{Y}(t) = -(R(t)\widehat{y}(t) - c(t)\widehat{P}(t)), \\ F(t)B(t)\widehat{Z}(t) = \widehat{q}(t), \\ \dot{G}(t)\widehat{Y}(t) - G(t)(C(t)\widehat{y}(t) + D(t)L^{-1}(t)D(t)\widehat{P}(t)) = -(A(t)\widehat{p}(t) + S(t)\widehat{Y}(t)), \\ G(t)\widehat{Z}(t) = -B(t)\widehat{q}(t). \end{cases}$$

Then it follows that

$$\begin{cases} (C(t)G(t) - A(t)F(t))^2 + (\dot{F}(t) + R(t))(G^2(t)D^2(t)L^{-1}(t) - S(t) - \dot{G}(t)) = 0, \\ F(t)B^2(t) + G(t) = 0, \\ F(T) = F, \\ G(0) = -G. \end{cases} \tag{13}$$

We have the following conclusion:

Proposition 6.1 *If all the assumptions hold, then the optimal control process $u(\cdot)$ of this problem is*

$$u(t) = L^{-1}(t)D(t)G(t)\widehat{Y}(t),$$

where $\widehat{Y}(t)$ is given by (12) *and $G(t)$ satisfies the equation* (13).

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