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Consensus of Linear Multi-Agent Systems with Persistent Disturbances via Distributed Output Feedback*

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Abstract This paper addresses the consensus problem of general linear multi-agent systems with persistent disturbances by distributed output feedback. Suppose that states of agents can not be obtained directly. Several estimators are designed to observe states of agents and the unknown disturbances. A protocol is proposed to drive all agents achieve consensus. Based on the method of model transformation and the property of permutation matrix, sufficient conditions for consensus are obtained in terms of linear matrix inequalities. Finally, simulations are given to show the effectiveness of presented results.

Keywords Consensus, distributed observer, leaderless, multi-agent systems, persistent disturbances.

1 Introduction

Over past several decades, the distributed coordination control problem of multi-agent systems has been an active research area in the field of systems and control theory. A lot of meaningful results have been reported[1]. The core problem of distributed coordination control is consensus problem, and more and more scholars from various fields, such as control, computer science, physics and mathematics, are paying tremendous attention to this problem. The results reported until now focus primarily on the design methods of control protocols and stability analysis of multi-agent systems with first-order, second-order and mixed-order dynamics[2–7]. However, there are more complex dynamical behaviors in actual physical systems, which cannot be modeled by the low order dynamics. Hence, it is important to investigate the consensus problem of linear multi-agent systems. Recently, many results about consensus have been extended to linear multi-agent systems[8–13] and they assumed that there is no any external disturbance along with system input.

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In reality, agents are usually under uncertain environments, where exist various external disturbances such as actuator bias, measurement or/and calculation errors, the variation of communication links and noises. This may lead to the divergency of the closed-loop systems. Hence, it is an important issue to study the consensus problems for multi-agent systems with external disturbances. There are two kinds of methods to contend with these situations. One is the stochastic analysis method, and the other one is the robust H_∞ control method. By using the tools of stochastic analysis, some sufficient conditions for mean-square consensus were obtained[14–16]. On the other hand, the H_∞ consensus problems were solved by using the tools of robust H_∞ control[17–20] and some sufficient conditions were derived for making all agents achieve consensus with H_∞ performance. By investigating recent literatures, the dynamic behaviors of agents and uncertainties are two key factors which effect the stability of the multi-agent systems. In physical systems, agents may share the linear dynamics and be affected by persistent disturbances[21, 22]. Therefore, it is meaningful to investigate the consensus problem for multi-agent systems with persistent disturbances, which is a challenging problem. However, to the best of our knowledge, few works have considered both difficulties so far. The authors of [23] solved such problems for leader-following multi-agent systems, but their analysis method is invalid for leaderless networks due to the singularity of system matrix, and the time-varying consensus state can not be found. In [24, 25], the consensus problems were solved for multi-agent systems with persistent disturbances by state feedback, while the case for absence of state information has not been considered.

In this paper, we investigate the robust consensus problem of general linear leaderless multi-agent systems with unknown persistent disturbances when the neighbors's state information cannot be obtained directly. Several observers are designed to estimate the disturbances and states by using output information of systems. A distributed protocol is proposed to force the multi-agent systems to achieve robust consensus. By the method of model transformation and the property of permutation matrix, the consensus problems are first changed into the problems of simple matrices stability, which are easier to understand and even more essential. Then the sufficient conditions for robust consensus are obtained. Finally, the effectiveness of presented results is demonstrated by simulations. Compared with the existing works, this paper has the following advantages: 1) In contrast to the consensus problem of linear multi-agent systems[8–13, 16], the disturbances were considered in this paper, hence, the results of this paper is more valuable in applications. 2) In contrast to the existing results[14–20], where the effect of external disturbance can only be decreased and the system cannot achieve fully consensus, in this paper, the closed-loop systems can achieve precise consensus. 3) In contrast to the existing results[22, 23], where the leader following consensus problems have been solved, this paper extends these results to leaderless networks, this is not a trivial work due to the singularity of closed-loop system matrix, and the consensus state is obtained too. 4) In contrast to our previous works in [24, 25], where state feedback controllers were adopted to solve the consensus problem, this paper deals with the output feedback situations.

2 Preliminaries

2.1 Graph Theory

To solve the coordination problems, graph theory is useful[26]. Consider a multi-agent system consists of N agents. Regarding N agents as N nodes, let $V = \{v_i, i = 1, 2, \cdots, N\}$ be the set of nodes and the undirected graph $\mathcal{G} = \{V, \varepsilon\}$ represent communication topology of N agents, where $\varepsilon \subset V \times V$ is the set of edges of the graph. If $(v_i, v_j) \in \varepsilon$, we call v_i is a neighbor of v_j. Let $N_i = \{\ j \mid (v_i, v_j) \in \varepsilon\}$ denotes the set of labels of those agents who are neighbors of agent i $(i = 1, 2, \cdots, N)$. Let $A = [a_{ij}] \in R^{N\times N}$ be the weighted adjacency matrix of the graph $\mathcal{G}$, which is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(i, j) \in \varepsilon$, otherwise, $a_{ij} = 0$ for $i \neq j$, and $D = \mathrm{diag}\{d_1, d_2, \cdots, d_N\} \in R^{N\times N}$ be its degree matrix, where $d_i = \sum_{j=1}^{N} a_{ij}$. Then the Laplacian of the weighted graph is defined as $L = D - A$, which is symmetric. A path that connects v_i and v_j in the graph $\mathcal{G}$ is a sequence of distinct vertices $v_{i_0}, v_{i_1}, \cdots, v_{i_m}$, where $v_{i_0} = v_i, v_{i_m} = v_j$ and $(v_{i_r}, v_{i_{r+1}}) \in \varepsilon, 0 \leq r \leq m - 1$. If there exist a path between any two vertices v_i and v_j $(i \neq j)$, then the graph is said to be connected.

2.2 System Model

Consider a network of N agents with general linear dynamics. The dynamics of agents are given as follows:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + B[u_i(t) + d_i(t) + w_i], \\ y_i(t) = Cx_i(t), \quad i = 1, 2, \cdots, N, \end{cases} \tag{1}$$

where $x_i \in \mathcal{R}^n, u_i \in \mathcal{R}^q$, $d_i(t) \in \mathcal{R}^q$, $w_i \in \mathcal{R}^q$ and $y_i \in \mathcal{R}^r$ are the state, control input, external disturbance, unknown constant disturbance and control output of agent i, respectively. And matrices A, B, C have appropriate dimensions, $N \geq 3$.

Definition 1 The robust consensus of multi-agent system (1) is said to be achieved by protocols $u_i(t),\ i = 1, 2, \cdots, N$ if for any initial condition, we have

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall i, j = 1, 2, \cdots, N.$$

The objective of this paper is to give the designing method of output feedback control protocol for driving all agents achieve consensus. To continue, we make the following assumptions[23].

Assumption 1 The pair (A, B) is stabilizable and the pair (A, C) is detectable.

Assumption 2[23] The external disturbances subject to the following two conditions: 1) $d_i(t) \in L_2[0, \infty)$, and 2) $\lim_{t\to\infty} \dot{d}_i(t) = 0$.

Assumption 3 The communication topology G is undirected and connected.

Remark 1 By Assumption 1, it is easy to see that there exist $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times r}$ (see [27] for details), such that $F = A - GC$, and F is Hurwitz.

Remark 2 In this paper, several observers will be proposed to estimate the states and disturbances for all agents. So Assumption 1 is necessary. In Assumption 2, the external disturbances $d_i(t)$ are assumed to be unknown but have limited energy and w_i is an unknown constant disturbance[21], which might be presented in real physical systems, for example, in the control of the torque of motors[29], slipping can be approximately modeled as unknown constant disturbance. In reality, we need to regard the entirety $d_i(t)+w_i$ as the external disturbances, such as the external disturbances in the MAGnetic LEViation (MAGLEV) suspension system[30]. Hence, Assumption 2 is validated. When $d_i(t)=0$, the external disturbances degrade into the constant disturbances, the corresponding multi-agent system becomes a special case of this paper.

Remark 3 In this paper, we assume that the communication topology G is undirected and connected. Our future work will focus on how to extend our results to the case of jointly connected and directed topologies and how to apply our results to real physical systems.

3 Main Results

In order to solve the robust consensus problem of System (1), we need the following lemmas.

Lemma 1[26] *Let G be a graph on N nodes with Laplacian L. Let $\lambda_1,\lambda_2,\cdots,\lambda_N$ be the eigenvalues of L and $\lambda_1\le\lambda_2\le\cdots\le\lambda_N$. Then $\lambda_1=0$ and $\mathbf{1}=[1,1,\cdots,1]^{\mathrm T}\in\mathcal{R}^N$ is the eigenvector associated with zero eigenvalue. Moreover, $\lambda_2>0$ if G is connected.*

From Lemma 1, there exists an orthogonal matrix $U=[\frac{1}{N}\mathbf{1}_N,\overline{U}],\overline{U}\in\mathcal{R}^{N\times(N-1)}$ such that $U^{\mathrm T}LU=\Lambda:=\mathrm{diag}\{\lambda_1,\lambda_2,\cdots,\lambda_N\}$.

Lemma 2[28] *Consider the linear systems $\dot z(t)=A_1z(t)+B_1v(t)$. If A_1 is Hurwitz, and $v(t)$ is bounded and satisfies that $\lim_{t\to\infty}v(t)=0$, then $\lim_{t\to\infty}z(t)=0$.*

First, we propose the following estimators:

$$\begin{aligned}
&\dot{\widetilde{x}}_i(t)=F\widetilde{x}_i(t)+B[u_i(t)+\widetilde{d}_i(t)+\widetilde{w}_i(t)]+Gy_i(t)+H_1\bigg(\sum_{j=1}^N a_{ij}[\overline{y}_i(t)-\overline{y}_j(t)]\bigg),\\
&\overline{y}_i(t)=C\widetilde{x}_i-y_i(t),\\
&\dot{\widetilde{d}}_i(t)=H_2\bigg(\sum_{j=1}^N a_{ij}[\overline{y}_i(t)-\overline{y}_j(t)]\bigg)-l\widetilde{d}_i(t),\\
&\dot{\widetilde{w}}_i(t)=H_3\bigg(\sum_{j=1}^N a_{ij}[\overline{y}_i(t)-\overline{y}_j(t)]\bigg),
\end{aligned}\tag{2}$$

where $\widetilde{x}_i\in\mathcal{R}^n,\widetilde{d}_i\in\mathcal{R}^q,\widetilde{w}_i\in\mathcal{R}^q$ are the estimated values of x_i,d_i,w_i, and $\overline{y}_i\in\mathcal{R}^r$ is the observed output. F,G are defined in Remark 1, H_1,H_2,H_3 are feedback gain matrices with appropriate dimensions, and $l>0$ is a feedback gain constant.

To solve the consensus problem of the multi-agent network (1), we propose the following protocol:

$$u_i(t)=\widetilde{K}\sum_{j=1}^N a_{ij}[\widetilde{x}_i(t)-\widetilde{x}_j(t)]-\widetilde{d}_i(t)-\widetilde{w_i}(t),\quad i=1,2,\cdots,N,\tag{3}$$

where $\widetilde{K}$ is the feedback gain matrix with appropriate dimension.

Define $w=[w_1^{\mathrm T},w_2^{\mathrm T},\cdots,w_N^{\mathrm T}]^{\mathrm T},\widetilde{x}(t)=[\widetilde{x}_1(t)^{\mathrm T},\widetilde{x}_2(t)^{\mathrm T},\cdots,\widetilde{x}_N(t)^{\mathrm T}]^{\mathrm T},\widetilde{d}(t)=[\widetilde{d}_1(t)^{\mathrm T},\ \widetilde{d}_2(t)^{\mathrm T},\cdots,\widetilde{d}_N(t)^{\mathrm T}]^{\mathrm T},\widetilde{w}(t)=[\widetilde{w}_1(t)^{\mathrm T},\widetilde{w}_2(t)^{\mathrm T},\cdots,\widetilde{w}_N(t)^{\mathrm T}]^{\mathrm T},\ x(t)=[x_1(t)^{\mathrm T},x_2(t)^{\mathrm T},\cdots,x_N(t)^{\mathrm T}]^{\mathrm T}$. Let $\varepsilon_x(t)=[\varepsilon_{x1}(t)^{\mathrm T},\varepsilon_{x2}(t)^{\mathrm T},\cdots,\varepsilon_{xN}(t)^{\mathrm T}]^{\mathrm T}=\widetilde{x}(t)-x(t),\varepsilon_d(t)=[\varepsilon_{d1}(t)^{\mathrm T},\varepsilon_{d2}(t)^{\mathrm T},\cdots,\varepsilon_{dN}(t)^{\mathrm T}]^{\mathrm T}=$

$\widetilde{d}(t)-d(t), \varepsilon_w(t)=[\varepsilon_{w1}(t)^{\mathrm{T}}, \varepsilon_{w2}(t)^{\mathrm{T}}, \cdots, \varepsilon_{wN}(t)^{\mathrm{T}}]^{\mathrm{T}}=\widetilde{w}(t)-w$. Noted that $F=A-GC$, we have

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}_x(t) \\ \dot{\varepsilon}_d(t) \\ \dot{\varepsilon}_w(t) \end{bmatrix} = \overline{M}^w \begin{bmatrix} x(t) \\ \varepsilon_x(t) \\ \varepsilon_d(t) \\ \varepsilon_w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -I_N\otimes I_q \\ 0 \end{bmatrix} (\dot{d}(t)+ld(t)), \tag{4}$$

where

$$\overline{M}^w = \begin{bmatrix} I_N\otimes A+L\otimes B\widetilde{K} & L\otimes B\widetilde{K} & -I_N\otimes B & -I_N\otimes B \\ 0 & I_N\otimes F+L\otimes H_1C & I_N\otimes B & I_N\otimes B \\ 0 & L\otimes H_2C & -I_N\otimes lI_q & 0 \\ 0 & L\otimes H_3C & 0 & 0 \end{bmatrix}.$$

Let

$$\begin{bmatrix} \overline{x}(t) \\ \overline{\varepsilon}_x(t) \\ \overline{\varepsilon}_d(t) \\ \overline{\varepsilon}_w(t) \end{bmatrix} = \begin{bmatrix} U^{\mathrm{T}}\otimes I_n & 0 & 0 & 0 \\ 0 & U^{\mathrm{T}}\otimes I_n & 0 & 0 \\ 0 & 0 & U^{\mathrm{T}}\otimes I_q & 0 \\ 0 & 0 & 0 & U^{\mathrm{T}}\otimes I_q \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon_x(t) \\ \varepsilon_d(t) \\ \varepsilon_w(t) \end{bmatrix}. \tag{5}$$

Then System (4) can be changed into the following form:

$$\begin{bmatrix} \dot{\overline{x}}(t) \\ \dot{\overline{\varepsilon}}_x(t) \\ \dot{\overline{\varepsilon}}_d(t) \\ \dot{\overline{\varepsilon}}_w(t) \end{bmatrix} = \widetilde{M}^w \begin{bmatrix} \overline{x}(t) \\ \overline{\varepsilon}_x(t) \\ \overline{\varepsilon}_d(t) \\ \overline{\varepsilon}_w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -U^{\mathrm{T}}\otimes I_q \\ 0 \end{bmatrix} (\dot{d}(t)+ld(t)), \tag{6}$$

where

$$\widetilde{M}^w = \begin{bmatrix} I\otimes A+\Lambda\otimes B\widetilde{K} & \Lambda\otimes B\widetilde{K} & -I_N\otimes B & -I_N\otimes B \\ 0 & I_N\otimes F+\Lambda\otimes H_1C & I_N\otimes B & 0 \\ 0 & \Lambda\otimes H_2C & -I_N\otimes lI_q & 0 \\ 0 & \Lambda\otimes H_3C & 0 & 0 \end{bmatrix}.$$

Let $\overline{x}(t)=[\overline{x}_1(t)^{\mathrm{T}}, \overline{x}^r(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{\varepsilon}_x(t)=[\overline{\varepsilon}_{x1}(t)^{\mathrm{T}}, \overline{\varepsilon}_x^r(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{\varepsilon}_d(t)=[\overline{\varepsilon}_{d1}(t)^{\mathrm{T}}, \overline{\varepsilon}_d^r(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{\varepsilon}_w(t)=[\overline{\varepsilon}_{w1}(t)^{\mathrm{T}}, \overline{\varepsilon}_w^r(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{x}^r(t)=[\overline{x}_2(t)^{\mathrm{T}}, \overline{x}_3(t)^{\mathrm{T}}, \cdots, \overline{x}_N(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{\varepsilon}_x^r(t)=[\overline{\varepsilon}_{x2}(t)^{\mathrm{T}}, \overline{\varepsilon}_{x3}(t)^{\mathrm{T}}, \cdots, \overline{\varepsilon}_{xN}(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{\varepsilon}_d^r(t)=[\overline{\varepsilon}_{d2}(t)^{\mathrm{T}}, \overline{\varepsilon}_{d3}(t)^{\mathrm{T}}, \cdots, \overline{\varepsilon}_{dN}(t)^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{\varepsilon}_w^r(t)=[\overline{\varepsilon}_{w2}(t)^{\mathrm{T}}, \overline{\varepsilon}_{w3}(t)^{\mathrm{T}}, \cdots, \overline{\varepsilon}_{wN}(t)^{\mathrm{T}}]^{\mathrm{T}}$. Then the closed-loop system (6) can be divided into the following two subsystems with no coupling.

$$\begin{aligned} &\dot{\overline{x}}_1(t)=A\overline{x}_1(t)-B\overline{\varepsilon}_{d1}(t)-B\overline{\varepsilon}_{w1}(t), \\ &\dot{\overline{\varepsilon}}_{x1}(t)=F\overline{\varepsilon}_{x1}(t)+B\overline{\varepsilon}_{d1}(t)+B\overline{\varepsilon}_{w1}(t), \\ &\dot{\overline{\varepsilon}}_{d1}(t)=-l\overline{\varepsilon}_{d1}(t)+\frac{1}{N}\sum_{j=1}^{N}(\dot{d}(t)+ld(t)), \\ &\dot{\overline{\varepsilon}}_{w1}(t)=0, \end{aligned} \tag{7}$$

and

$$\begin{bmatrix} \dot{\overline{x}}^r(t) \\ \dot{\overline{\varepsilon}}_x^r(t) \\ \dot{\overline{\varepsilon}}_d^r(t) \\ \dot{\overline{\varepsilon}}_w^r(t) \end{bmatrix} = M^w \begin{bmatrix} \overline{x}^r(t) \\ \overline{\varepsilon}_x^r(t) \\ \overline{\varepsilon}_d^r(t) \\ \overline{\varepsilon}_w^r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\overline{U}^{\mathrm{T}} \otimes I_q \\ 0 \end{bmatrix} (\dot{d}(t) + ld(t)), \tag{8}$$

where

$$M^w = \begin{bmatrix} I \otimes A + \overline{\Lambda} \otimes B\widetilde{K} & \overline{\Lambda} \otimes B\widetilde{K} & -I_N \otimes B & -I_N \otimes B \\ 0 & I_N \otimes F + \overline{\Lambda} \otimes H_1 C & I_N \otimes B & I_N \otimes B \\ 0 & \overline{\Lambda} \otimes H_2 C & -I_N \otimes lI_q & 0 \\ 0 & \overline{\Lambda} \otimes H_3 C & 0 & 0 \end{bmatrix}$$

and $\overline{\Lambda} = \mathrm{diag}\{\lambda_2, \lambda_3, \cdots, \lambda_N\}$.

Next, let we state our main result of this paper.

Theorem 1 *Consider the multi-agent network* (1). *Suppose that Assumptions* 1–3 *hold. If there exist positive definite matrix* $P_1 \in \mathcal{R}^{n \times n}$ *and* $P_2 \in \mathcal{R}^{(n+2q)\times(n+2q)}$ *such that the following two linear matrix inequalities hold:*

$$AP_1 + P_1 A^{\mathrm{T}} - 2BB^{\mathrm{T}} < 0, \tag{9}$$

$$P_2 \overline{F} + \overline{F}^{\mathrm{T}} P_2 - 2\overline{C}^{\mathrm{T}} \overline{C} < 0. \tag{10}$$

Then the robust consensus problem of multi-agent network (1) *can be solved by estimator* (2) *and protocol* (3), *where* $\widetilde{K} = -\tau B^{\mathrm{T}} P_1^{-1}$, $[H_1^{\mathrm{T}}\ H_2^{\mathrm{T}}\ H_3^{\mathrm{T}}]^{\mathrm{T}} = -\mu P_2^{-1}\overline{C}$, $\tau > \frac{1}{\lambda_2}, \mu > \frac{1}{\lambda_2}$, $\overline{F} = \left[\begin{smallmatrix} F & B & B \\ 0 & -lI_q & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ *and* $\overline{C} = [C\ \ 0\ \ 0]$.

Proof Note that the permutation matrix is an orthogonal matrix and the eigenvalues can not be changed when both sides of M^w are multiplied by an orthogonal matrix and its transposition, we have M^w is Hurwitz if and only if $M_i^w, i = 2, 3, \cdots, N$ are Hurwitz, which are equivalent to both $A + \lambda_i B\widetilde{K}$ and

$$F^i := \begin{bmatrix} F + \lambda_i H_1 C & -B & -B \\ \lambda_i H_2 C & -lI_q & 0 \\ \lambda_i H_3 C & 0 & 0 \end{bmatrix} = \overline{F} + \lambda_i \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \overline{C}, \quad i = 1, 2, \cdots, N$$

are Hurwitz. From (9), it is easy to prove that $A + \lambda_i B\widetilde{K}$ is Hurwitz. Thus, we only need to prove F^i is Hurwitz, equivalently, there exists a positive definite matrix $P_2 \in \mathcal{R}^{(n+2q)\times(n+2q)}$ such that $P_2 F^i + F^{iT} P_2 < 0$, i.e.,

$$P_2 \overline{F} + \lambda_i P_2 \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \overline{C} + \overline{F}^{\mathrm{T}} P_2 + \lambda_i \left(\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \overline{C} \right)^{\mathrm{T}} P_2 < 0. \tag{11}$$

Noted that $[H_1^{\mathrm{T}}\ H_2^{\mathrm{T}}\ H_3^{\mathrm{T}}]^{\mathrm{T}} = -\mu P_2^{-1}\overline{C}^{\mathrm{T}}$ and $\mu > \frac{1}{\lambda_2}$, it is easy to prove that (11) can be deduced by linear matrix inequality (10). So M^w is Hurwitz and $\lim_{t\to\infty}\overline{x}^r(t) = 0, \lim_{t\to\infty}\overline{\varepsilon}_x^r(t) = 0$, $\lim_{t\to\infty}\overline{\varepsilon}_d^r(t) = 0$ and $\lim_{t\to\infty}\overline{\varepsilon}_w^r(t) = 0$. According to Lemma 2, we have $\lim_{t\to\infty}\overline{\varepsilon}_{d1}(t) = 0$ and $\overline{\varepsilon}_{w1}(t) = \overline{\varepsilon}_{w1}(0)$ for any $t > 0$. By the invertibility of (5), we have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(U\otimes I_n)\left[\begin{smallmatrix}\overline{x}_1(t)\\ 0\end{smallmatrix}\right] = \frac{1}{N}(\mathbf{1}_N\otimes I_n)\lim_{t\to\infty}\overline{x}_1(t) = \frac{1}{N}(\mathbf{1}_N\otimes I_n)\lim_{t\to\infty}[\mathrm{e}^{At}\frac{1}{N}\sum_{j=1}^N x_j(0) - \int_0^t \mathrm{e}^{A(t-s)}B\frac{1}{N}\sum_{j=1}^N \varepsilon_{wj}(0)ds]$, which implies that $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$. Therefore, the robust consensus problem of multi-agent network (1) can be solved by estimator (2) and protocol (3). ∎

Remark 4 Theorem 1 not only presents the consensus conditions, but also gives the ultimate stable state of each agent. Compared with [23], where the consensus problem was considered for multi-agent systems with a leader and the analysis method therein is hard to understand, while this paper considers the multi-agent systems without leaders, the analysis method in [23] is invalid here due to the existence of zero eigenvalue of the closed-loop system matrix. Hence, this paper is different from [23] and a new method is proposed to overcome this difficulty. Besides, the method in this paper can be easily developed to solve the other distributed coordination control problem, such as formation, flocking, and so on.

Remark 5 As we know, it is hard to solve linear matrix inequality (10) in general, maybe we have following concerns. From the proof of Theorem 1, we can see that

$$\overline{F}^i = \begin{bmatrix} F + \lambda_i H_1 C & -B \\ \lambda_i H_2 C & -lI_q \end{bmatrix}$$

is Hurwitz. So there exists a positive definite matrix P_{11}, such that $P_{11}\overline{F}^i + \overline{F}^{i\mathrm{T}}P_{11} < 0$. And then we can choose P_{12} such that $P_{12}^{\mathrm{T}}\left[\begin{smallmatrix}B\\0\end{smallmatrix}\right] + \left[\begin{smallmatrix}B^{\mathrm{T}} & 0\end{smallmatrix}\right]P_{12} < 0$. Then, we can choose $P_{22} > 0$ such that $P_2 = \left[\begin{smallmatrix}P_{11} & P_{12}\\ P_{12}^{\mathrm{T}} & P_{22}\end{smallmatrix}\right] > 0$. It is need to point out that P_{22} always exists. In fact, we can take $P_{22} = \delta I_q, \delta > 0$. Then $P_{11} - P_{12}P_{22}^{-1}P_{12}^{\mathrm{T}} = P_{11} - \frac{1}{\delta}P_{12}P_{12}^{\mathrm{T}} > 0$ always holds if $\delta > 0$ is large enough. So $P_2 > 0$ by Schur Complement Lemma (see [31] for details). At last, we test Inequality (10).

Remark 6 In this paper, we don't need to require that $(A,\ B)$ is controllable. From Inequality (9), it is clear that (A, B) is stabilizable.

4 Simulations

In this section, we present some numerical simulations to illustrate the theoretical results obtained in the previous sections. Figure 1 shows a graph with $N = 5$ nodes, which represents the communication topology of multi-agent systems.

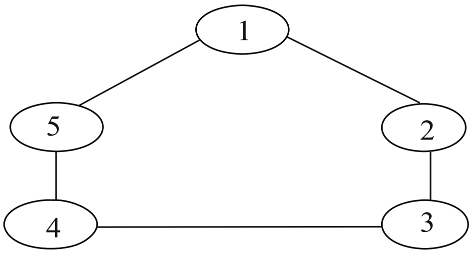

Figure 1 Network with five agents

Suppose that matrices in System (1) are as follows[23]:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

It is easy to obtain $F = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, G = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, by the method in [27], we have $F = A - GC$ and F is Hurtwiz.

By solving the linear matrix inequality (9), we have $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. So we can choose $\widetilde{K} = -\tau \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and take $\tau = 1$, which satisfies $\tau > \frac{1}{\lambda_2}$, where $\lambda_2 = 1.3820$. By solving (10), it is easy to obtain that

$$P_2 = \begin{bmatrix} 1 & -0.4 & -0.3 & -0.1 \\ -0.4 & 0.8 & -0.3 & -0.2 \\ -0.3 & -0.3 & 1.5 & -0.2 \\ -0.1 & -0.2 & -0.2 & 2 \end{bmatrix},$$

and then

$$H_1 = \begin{bmatrix} -1.6395 \\ -1.1003 \end{bmatrix},$$

$H_2 = -0.5813, H_3 = -0.2501$. The external disturbances are selected as: $d_1 = \mathrm{e}^{-0.5t}$, $d_2 = \mathrm{e}^{-0.5t}$, $d_3 = \mathrm{e}^{-0.05t}$, $d_4 = \mathrm{e}^{-0.05t}$, $d_5 = \mathrm{e}^{-0.005t}$ and $w_i = 1, i = 1, 2, \cdots, 5$. With the initial values $x_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x_3 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, x_4 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, x_5 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, Figures 2, 3 and 4 show the state $x_i(t)$, the state error $e_{xi}(t)$ and the disturbance error $e_{di}(t)$ of each agent i respectively. Simulation results agree well with the theoretical results obtained before.

5 Conclusions

In this paper, we mainly investigate the robust consensus problems for general linear multi-agent systems with persistent disturbances. A class of distributed state estimators and disturbances estimators were designed to estimate the disturbances and states by using output information of agents. Based on the estimated information, a new control protocol was proposed to force the multi-agent systems to achieve robust consensus. In the process of consensus analysis, the method of model transformation and the property of permutation matrix were used to change the consensus problem into a simple matrices stability problem, which is easier to understand and even more essential compared with the existing method. Finally, the effectiveness of presented results were demonstrated by simulations. Future work will focus on the real applications of the results given in this paper and the consensus problem of linear multi-agent systems with persistent disturbances, time-delay and directed switching topologies.

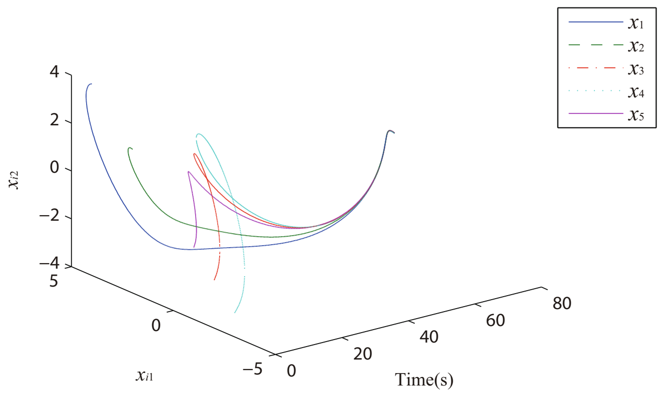

Figure 2 The state trajectories of agents

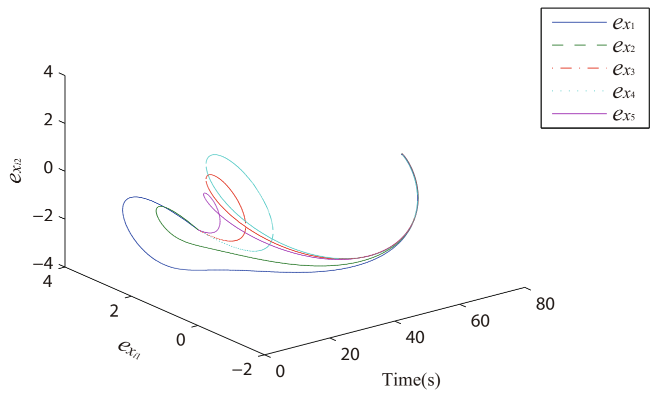

Figure 3 The state error trajectories of agents

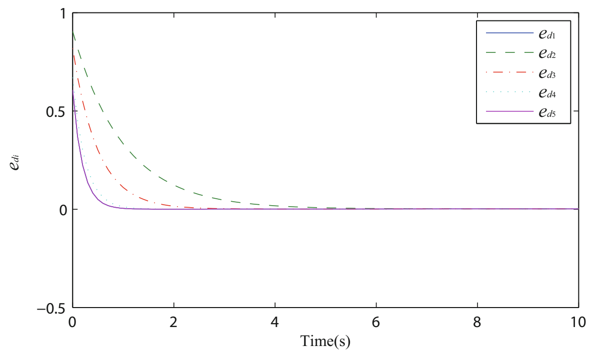

Figure 4 The disturbance error trajectories of agents

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