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Mean Square Average Generalized Consensus of Multi-Agent Systems Under Time-Delays and Stochastic Disturbances*

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Abstract Compared with the traditional consensus problem, this paper deals with the mean square average generalized consensus (MSAGC) of multi-agent systems under fixed directed topology, where all agents are affected by stochastic disturbances. Distributed protocol depending on delayed time information from neighbors is designed. Based on Lyapunov stability theory, together with results from matrix theory and Itô's derivation theory, the linear matrix inequalities approach is used to establish sufficient conditions to ensure MSAGC of multi-agent systems. Finally, numerical simulations are provided to illustrate the theoretical results.

Keywords Mean square average generalized consensus, multi-agent systems, stochastic disturbances, time delays.

1 Introduction

In recent years, the problem of consensus of multi-agent systems has attracted compelling attention from various scientific communities owing to its extensive applications in real-world distributed computation, wireless sensor networks, satellite formation, the direction of fish or birds, distributed sensor filtering value, cooperative surveillance, and so on[1–4]. During the past decades, several different kinds of consensus have been investigated, such as, mean-square consensus[5–7], leader-following consensus[8, 9], cluster consensus[10] and quantized consensus[11].

The synchronization of chaotic systems has been intensively studied, which was introduced by Pecora and Carroll[12]. In complete synchronization, the dynamics of two coupled systems totally coincide with each other. In generalized synchronization[13–15], the states of the response system synchronize that of the drive system through a nonlinear smooth functional mapping and

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certain known or unknown functional relation exists between dynamics of two coupled systems which are usually nonidentical. Therefore, generalized synchronization has more applications than complete synchronization. Generalized synchronization has been deeply researched in complex networks[16–19]. On the basis of these studies, the concept of generalized consensus was proposed in [20], which is an extension of many types of standard consensus, where there exists certain functional relations among all agents in network. Generalized consensus may be more practical to describe coherence in biological and physical systems consisting of multiple interacting components.

In real systems, time delays always exist due to finite communication speed, which make it difficult or impossible for a networked agent to obtain timely and accurate information of its neighbors. One important challenge is to deal with the influence of time delays in the inter-agent information flows. The recent work of Liu, et al.[21] studied stochastic consensus seeking with communication delays. Olfati-Saber and Murray[22] proposed the average consensus of multi-agent systems with constant and uniform communication time delays under fixed topology. Moreover, stochastic disturbances are unavoidable in the real world, multi-agent systems will be affected by them from external environment. Some related results have been proposed. Hu, et al.[8] investigated the leader-following consensus of multi-agent systems with stochastic disturbances. Yang, et al.[23] and Zhang and Liu[24] studied the consensus of second-order multi-agent systems with exogenous disturbances. All these results can be seen that to study on the consensus problem for multi-agent systems with time delays and stochastic disturbances is necessary and meaningful. However, there are no results concerning the issue of mean square average generalized consensus (MSAGC) for multi-agent systems with time delays and stochastic disturbances.

In response to the above discussion, this paper investigates the MSAGC of multi-agent systems which are often influenced by time delays and external stochastic disturbances in directed topology. Meanwhile, under the updated distributed protocol, it is meaningful that each agent moves to the average of the functional relation of states of its neighbors and the final stable state of systems is generalized consensus in the mean square, which is mainly different from the traditional methods and will lead to less conservative results. Based on the Lyapunov function and the stochastic theory, sufficient conditions for guaranteeing the MSAGC of multi-agent systems are derived.

The rest of the paper is organized as follows. In Section 2, some preliminaries and the model description of novel protocols are given. The main results of distributed consensus are discussed in Section 3. In Section 4, some numerical examples are given to illustrate the theoretical results. Conclusions are finally drawn in Section 5.

Throughout the paper, let $\mathbb{R}^{N\times N}$ be the $N\times N$ real matrix space. I_N (O_N) represents the N-dimensional identity (zero) matrix, and 1_N (0_N) indicates the N-dimensional column vector with each entry being 1 (0). Superscript "T" stands for matrix transposition. diag$\{\cdot\}$ means diagonal matrix and tr$\{A\}$ denotes the trace of matrix A. For a matrix $\mathit{\Gamma}\in\mathbb{R}^{N\times N}$, $\|\mathit{\Gamma}\|$ represents norm of $\mathit{\Gamma}$, $\otimes$ denotes Kronecker product.

2 Preliminaries and Model Formulation

A directed graph $G = (V, \varepsilon, A)$ consists of a set of vertices $V = \{v_1, v_2, \cdots, v_N\}$, and a set of directed edges $\varepsilon \in V \times V$, and $\mathcal{A} = [a_{ij}]_{N\times N}$ is an adjacency matrix. An edge e_{ij} in graph G is denoted by the ordered pair of vertices (v_j, v_i), where v_j and v_i are called the parent and child vertices, respectively, and $e_{ij} \in \varepsilon$ if and only if $a_{ij} > 0$. For simplicity, denote $G = (V, \varepsilon, A)$ by $G(A)$. A path between nodes v_i and v_j in G is a sequence of edges (v_i, v_{i1}), (v_{i1}, v_{i2}), $\cdots$ (v_{il}, v_j). A directed graph is called strongly connected if and only if there is a directed path between any pair of distinct vertices. The graph Laplacian matrix L of the network is defined by $l_{ii} = \sum_{j=1,j\neq i}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$. For more details of network topology theory, one can see [2, 13, 18, 25].

Classically, the multi-agent system with each agent dynamics is described as

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state of the ith agent and $u_i(t) \in \mathbb{R}^n$ is the control input of the ith agent which is only based on the information of its neighbors. A typical consensus control protocol is $u_i(t) = \alpha \sum_{j=1}^{N} a_{ij}[x_j(t) - x_i(t)]$, where a_{ij} is the (i,j)th entry of the corresponding adjacency matrix, α is the control parameter.

Firstly, the definition of generalized consensus is presented.

Definition 2.1 (see [16]) System (1) is said to reach generalized consensus if for any initial conditions,

$$\lim_{t\to\infty} \|h_j(x_j(t)) - h_i(x_i(t))\| = 0$$

for all $i = 1, 2, \cdots, N$. where $h_i(x_i) : x_i = (x_{i_1}, x_{i_2}, \cdots, x_{i_n}) \in \mathbb{R}^n \to (h_i(x_{i_1}), h_i(x_{i_2}), \cdots, h_i(x_{i_n})) \in \mathbb{R}^n$ are smooth and invertible functions.

If $h_i(x_i) = x_i/k_i$ ($k_i \neq 0$ are constants) for all $i = 1, 2, \cdots, N$, then System (1) achieves linear generalized consensus; if at least one of $h_i(x_i)$ $(i = 1, 2, \cdots, N)$ is nonlinear, then System (1) achieves nonlinear generalized consensus.

Based on Definition 2.1 and [6], we propose the following definition.

Definition 2.2 The agents in the networks are said to reach MSAGC if $E\|h_i(x_i(t))\|^2 < \infty$ and there exists a random variable $x^*(t)$ such that $\lim_{t\to\infty} E\|h_i(x_i(t)) - x^*(t)\|^2 = 0$, for all $i = 1, 2, \cdots, N$.

Consider a model of networks consisting of N agents. The dynamics of the ith agent is described as

$$\dot{x}_i(t) = (h_i'(x_i(t)))^{-1}\sigma_i(h_i(t, x_i(t)))n(t) + u_i(t), \tag{2}$$

where $(h_i'(x_i))^{-1} = \mathrm{diag}((h_i'(x_{i_1}))^{-1}, (h_i'(x_{i_2}))^{-1}, \cdots, (h_i'(x_{i_n}))^{-1})_n$, $\sigma_i(\cdot,\cdot)\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the noise intensity function vector. $n(t)$ is a scalar zero mean Gaussian white noise process. Recall that the time derivative of a Wiener process (Brownian motions) is a white noise process, we have $dw(t) = n(t)dt$.

Correspondingly, we put forward the updating protocol of (2):

$$u_i(t) = \alpha(h_i'(x_i))^{-1}\sum_{j=1}^{N} a_{ij}[h_j(x_j(t-\tau)) - h_i(x_i(t-\tau))]. \tag{3}$$

Substituting (3) into (2) gives

$$\begin{aligned}\dot{x_i}(t) &= (h_i'(x_i(t)))^{-1}\sigma_i(h_i(t, x_i(t)))n(t)\\ &\quad+\alpha(h_i'(x_i(t)))^{-1}\sum_{j=1}^{N} a_{ij}[h_j(x_j(t-\tau)) - h_i(x_i(t-\tau))].\end{aligned} \tag{4}$$

Remark 2.3 The stochastic Brownian motions affect the dynamics of agents. $\sigma_i(h_i(t, x_i(t)))dw(t)$, $i = 1, 2, \cdots, N$, in our model is used to reflect the noises affected by functions $h_i(\cdot)$, which is different from the deterministic cases used in [22, 23].

Remark 2.4 Based on the delayed-input approach, delay is considered in the consensus protocol, which is more objective to study is the generalized term in the noise environment. Clearly, if generalized consensus can be achieved, it is natural to require a generalized consensus state $x_0(t) \in \mathbb{R}^n$ of the system (4) satisfied that $h_i(x_i(t)) \to x_0(t)$. Especially, if $h_i(x_i(t)) = x_i(t)/k_i$ for all $i = 1, 2, \cdots, N$, then System (4) is expressed as follows:

$$\dot{x_i}(t) = \alpha k_i \sum_{j=1}^{N} a_{ij}[x_j(t-\tau)/k_j - x_i(t-\tau)/k_i] + k_i\sigma_i(h_i(t, x_i(t)))n(t). \tag{5}$$

Before stating the main results, we also need the following lemmas and assumption.

Lemma 2.5 (see [26]) *Suppose that a directed graph $G(A)$ is strongly connected. Then, its Laplacian matrix L is irreducible and satisfies $L1_N = 0$. Furthermore, there exists a positive vector $\xi = (\xi_1, \xi_2, \cdots, \xi_N)^{\mathrm{T}}$ such that $\xi^{\mathrm{T}}L = 0$ and $\xi^{\mathrm{T}}1_N = 1$.*

Lemma 2.6 (see [25, 27]) *Suppose that $x \in \mathbb{R}^n$, $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$ such that $\mathrm{Rank}(H) = l < n$. Then, the following statements are equivalent:*

1) $x^{\mathrm{T}}Px < 0$, $\forall x \in \{x : Hx = 0, x \neq 0\}$,

2) $H^{\perp^{\mathrm{T}}} P H^{\perp} < 0$, *where $H^{\perp}$ is the kernel of H, i.e., $HH^{\perp} = 0$.*

Assumption 2.7 (see [28]) The noise intensity function vector $\sigma_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \cdots, N$, satisfies the Lipschitz condition, i.e., there exists a constant matrix $\varSigma$ of appropriate dimensions such that the following inequality

$$(\sigma_i(t, u) - \sigma_0(t, v))^{\mathrm{T}}(\sigma_i(t, u) - \sigma_0(t, v)) \leq ||\varSigma(u - v)||^2$$

holds for all $i = 1, 2, \cdots, N$ and $u, v \in R^n$.

3 Main Results

In this section, we analyze the consensus properties of the dynamics of System (4).

Let $\delta_i(t) = h_i(x_i(t)) - x_0(t)$ represent the position vector of the ith agent relative to the weighted average position of all the agents in System (4), where $x_0(t) = \sum_{j=1}^{N} \xi_j h_j(x_j(t))$,

$\xi=(\xi_1,\xi_2,\cdots,\xi_N)^{\mathrm{T}}$ is the positive left eigenvector of Laplacian matrix L associated with its zero eigenvalue, satisfying $\xi^{\mathrm{T}}1_N=1$. Then, one has the following error dynamical system:

$$\begin{aligned}\dot{\delta}_i(t)&=h_i'(x_i(t))\dot{x}_i(t)-\sum_{j=1}^{N}\xi_j h_j'(x_j(t))\dot{x}_j(t)\\&=-\alpha\sum_{i=1}^{N}l_{ij}h_j(x_j(t-\tau))-\sum_{j=1}^{N}\xi_j\Big[-\alpha\sum_{k=1}l_{jk}h_k(x_k(t-\tau))\Big]\\&\quad+\Big[\sigma(h_i(x_i(t)))-\sum_{j=1}\xi_j\sigma(h_j(x_j(t)))\Big]n(t)\\&=-\alpha\sum_{j=1}^{N}l_{ij}(h_j(x_j(t-\tau))-x_0(t-\tau))\\&\quad+\Big[\sigma(h_i(x_i(t)))-\sum_{j=1}\xi_j(h_j(x_j(t)))\Big]n(t).\end{aligned}\tag{6}$$

Let $\delta(t)=[\delta_1^{\mathrm{T}}(t),\delta_2^{\mathrm{T}}(t),\cdots,\delta_N^{\mathrm{T}}(t)]$. Then, System (6) can be written as

$$d\delta(t)=-\alpha(L\otimes I_n)\delta(t-\tau)dt+\sigma(\delta(t))dw(t).\tag{7}$$

Theorem 3.1 *Suppose that the network $G(A)$ is strongly connected and Assumption* 2.7 *holds. The MSAGC for the system* (4) *is achieved if there exist a scalar $\lambda>0$ and symmetric matrices $P,R\in\mathbb{R}^{N\times N}$ such that $E^{\mathrm{T}}PE>0$, $E^{\mathrm{T}}RE>0$, and the following linear matrix inequalities hold:*

$$P<\lambda I,\tag{8}$$

$$\begin{bmatrix}\lambda E^{\mathrm{T}}\Sigma^{\mathrm{T}}\Sigma E-\alpha E^{\mathrm{T}}L^{\mathrm{T}}PE-\alpha E^{\mathrm{T}}PLE & *\\ \lambda E^{\mathrm{T}}\Sigma^{\mathrm{T}}\Sigma E-\alpha E^{\mathrm{T}}PLE & \lambda E^{\mathrm{T}}\Sigma^{\mathrm{T}}\Sigma E-\frac{\alpha}{\tau}E^{\mathrm{T}}RE\end{bmatrix}<0,\tag{9}$$

where

$$E=\begin{bmatrix}I_{N-1}\\ -\frac{\overline{\xi}^{\mathrm{T}}}{\xi_N}\end{bmatrix}\in\mathbb{R}^{N\times(N-1)},\tag{10}$$

$\overline{\xi}=[\xi_1,\xi_2,\cdots,\xi_{N-1}]^{\mathrm{T}}\in\mathbb{R}^{N-1}$, *and* $\xi=[\xi_1,\xi_2,\cdots,\xi_N]^{\mathrm{T}}$ *is the positive left eigenvector of Laplacian matrix L associated with its zero eigenvalue, satisfying $\xi^{\mathrm{T}}1_N=1$.*

Proof Construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned}V(\delta(t),t)&=\delta^{\mathrm{T}}(t)(P\otimes I_n)\delta(t)\\&\quad+\alpha\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\delta}^{\mathrm{T}}(s)(R\otimes I_n)\dot{\delta}(s)dsd\theta,\end{aligned}\tag{11}$$

where symmetric matrices P and $R\in\mathbb{R}^{N\times N}$ satisfy $E^{\mathrm{T}}PE>0$ and $E^{\mathrm{T}}RE>0$, with

$$E=\begin{bmatrix}I_{N-1}\\ -\frac{\overline{\xi}^{\mathrm{T}}}{\xi_N}\end{bmatrix}\in\mathbb{R}^{N\times(N-1)},$$

where $\overline{\xi} = [\xi_1, \xi_2, \cdots, \xi_{N-1}]^{\mathrm{T}} \in \mathbb{R}^{N-1}$, $\xi = [\xi_1, \xi_2, \cdots, \xi_N]^{\mathrm{T}} \in \mathbb{R}^N$ is the positive left eigenvector of Laplacian matrix L associated with its zero eigenvalue, satisfying $\xi^{\mathrm{T}} 1_N = 1$.

By Itô's differential formula, the stochastic derivative of $V(\delta(t), t)$ along (7) can be obtained as follows:

$$dV(\delta(t), t) = \pounds V(\delta(t), t)dt + 2\delta^{\mathrm{T}}(t)(P \otimes I_n)\sigma(\delta(t))dw(t). \tag{12}$$

Furthermore, we have

$$\begin{aligned}\pounds V(\delta(t), t) = {} & 2\delta^{\mathrm{T}}(t)(P \otimes I_n)[-\alpha(L \otimes I_n)\delta(t-\tau)] + \alpha\tau\dot{\delta}^{\mathrm{T}}(t)(R \otimes I_n)\dot{\delta}(t) \\ & -\alpha \int_{t-\tau}^{t} \dot{\delta}^{\mathrm{T}}(s)(R \otimes I_n)\dot{\delta}(s)ds + \mathrm{tr}[\sigma^{\mathrm{T}}(\delta(t))(P \otimes I_n)\sigma(\delta(t))].\end{aligned} \tag{13}$$

It follows from Jensen's Inequality[29] that

$$-\alpha \int_{t-\tau}^{t} \dot{\delta}^{\mathrm{T}}(s)(R \otimes I_n)\dot{\delta}(s)ds \le -\frac{\alpha}{\tau}[\delta(t) - \delta(t-\tau)]^{\mathrm{T}}(R \otimes I_n)[\delta(t) - \delta(t-\tau)]. \tag{14}$$

Next, it follows from the condition (8) and Assumption 2.7 that

$$\begin{aligned}\mathrm{tr}(\sigma^{\mathrm{T}}(t, \delta(t))(P \otimes I_n)\sigma(t, \delta(t)) & \le \lambda_{\max}(P \otimes I_n)\mathrm{tr}(\sigma^{\mathrm{T}}(t, \delta(t))\sigma(t, \delta(t))) \\ & \le \lambda\delta^{\mathrm{T}}(t))(\varSigma \otimes I_n)^{\mathrm{T}}(\varSigma \otimes I_n)\delta(t).\end{aligned} \tag{15}$$

Let $\delta(t) - \delta(t-\tau) = \mu(t)$. Then, according to (13) to (15), one gets

$$\begin{aligned}\pounds V(\delta(t), t) \le {} & -2\alpha\delta^{\mathrm{T}}(t)(P \otimes I_n)(L \otimes I_n)\delta(t-\tau) + \alpha\tau\dot{\delta}^{\mathrm{T}}(t)(R \otimes I_n)\dot{\delta}(t) \\ & -\frac{\alpha}{\tau}\mu(t)^{\mathrm{T}}(R \otimes I_n)\mu(t) + \lambda\delta^{\mathrm{T}}(t)(\varSigma \otimes I_n)^{\mathrm{T}}(\varSigma \otimes I_n)\delta(t).\end{aligned} \tag{16}$$

Let $y(t) = [\dot{\delta}^{\mathrm{T}}(t), \delta^{\mathrm{T}}(t), \delta^{\mathrm{T}}(t-\tau), \mu^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then,

$$\pounds V(\delta(t), t) \le y^{\mathrm{T}}(t)(\varOmega \otimes I_n)y(t), \tag{17}$$

where

$$\varOmega = \begin{bmatrix} \alpha\tau R & 0 & 0 & 0 \\ 0 & \lambda\varSigma^{\mathrm{T}}\varSigma & -\alpha PL & 0 \\ 0 & -\alpha L^{\mathrm{T}}P & 0 & 0 \\ 0 & 0 & 0 & -\frac{\alpha}{\tau}R \end{bmatrix}.$$

Furthermore, it follows from the fact $[(1_4^{\mathrm{T}} \otimes \xi^{\mathrm{T}}) \otimes I_n]y(t) = 0$ that $Ay(t) = 0_{(N+4)n}$, where

$$A = \begin{bmatrix} S \\ T \end{bmatrix} \otimes I_n, \quad S = \begin{bmatrix} 0_N & I_N & -I_N & -I_N \end{bmatrix} \in \mathbb{R}^{N \times 4N}, \quad T = [I_4^{\mathrm{T}} \otimes \xi^{\mathrm{T}}] \in \mathbb{R}^{4 \times 4N}.$$

System (4) is asymptotically stable if for all $y(t)$ satisfying $Ay(t) = 0$, so one has

$$y^{\mathrm{T}}(t)(\varOmega \otimes I_n)y(t) < 0. \tag{18}$$

According to Lemma 2.6, $y^{\mathrm{T}}(t)(\Omega \otimes I_n)y(t) < 0$ is equivalent to

$$A^{\perp^{\mathrm{T}}}(\Omega \otimes I_n)A^{\perp} < 0, \tag{19}$$

where

$$A^{\perp} = \begin{bmatrix} 0 & 0 \\ E & E \\ E & 0 \\ 0 & E \end{bmatrix} \otimes I_n. \tag{20}$$

Thus, Inequality (18) can be rewritten as

$$\begin{bmatrix} \lambda E^{\mathrm{T}}\Sigma^{\mathrm{T}}\Sigma E - \alpha E^{\mathrm{T}}L^{\mathrm{T}}PE - \alpha E^{\mathrm{T}}PLE & * \\ \lambda E^{\mathrm{T}}\Sigma^{\mathrm{T}}\Sigma E - \alpha E^{\mathrm{T}}PLE & \lambda E^{\mathrm{T}}\Sigma^{\mathrm{T}}\Sigma E - \frac{\alpha}{\tau}E^{\mathrm{T}}RE \end{bmatrix} < 0. \tag{21}$$

Therefore, $\pounds V(\delta(t), t) < 0$. Taking the mathematical exception operator of both sides of (12), we have

$$\frac{d\mathbb{E}V(\delta(t))}{dt} \leq 0. \tag{22}$$

It can now be concluded from Lyapunov stability theory that the error system (7) is globally, asymptotically stable in the mean square, which implies that the mean square average generalized consensus in System (4) is achieved. The proof is completed. ▮

4 Simulations

In this section, some numerical simulations are provided to validate the effectiveness of the theoretical results.

Example 4.1 (Linear case) Consider multi-agent system (5) with the topology $G(A_1)$ as in Figure 1. The noise intensity function satisfies the Lipschitz condition, there exists a matrix $\Sigma = \mathrm{diag}\{0.01, 0.02, 0.01, 0.01, 0.02, 0.01, 0.01, 0.02\}$. Let the coupling strength $\alpha = 0.50$. By resorting to some standard software in Matlab, the matrix inequalities in (8) and (9) are solvable with a feasible solution as follows: $\lambda = 1.6988$,

$$P = \begin{bmatrix} 0.4948 & 0.1350 & 0.0456 & -0.0945 & -0.0803 & -0.0250 & 0.0419 & 0.0069 \\ 0.1350 & 0.5256 & 0.1329 & -0.0937 & -0.1259 & -0.0677 & 0.0260 & 0.0110 \\ 0.0456 & 0.1329 & 0.4846 & 0.0360 & -0.1347 & -0.1100 & 0.0133 & 0.0128 \\ -0.0945 & -0.0937 & 0.0360 & 0.4555 & 0.0601 & -0.0691 & 0.0020 & 0.0092 \\ -0.0803 & -0.1259 & -0.1347 & 0.0601 & 0.4718 & 0.0971 & 0.0075 & 0.0130 \\ -0.0250 & -0.0677 & -0.1100 & -0.0691 & 0.0971 & 0.4765 & 0.0590 & 0.0375 \\ 0.0419 & 0.0260 & 0.0133 & 0.0020 & 0.0075 & 0.0590 & 0.1957 & 0.0551 \\ 0.0069 & 0.0110 & 0.0128 & 0.0092 & 0.0130 & 0.0375 & 0.0551 & 0.0363 \end{bmatrix},$$

$$R = \begin{bmatrix} -18.36 & -36.71 & -71.95 & -44.94 & -118.28 & -58.23 & -58.65 & -9.52 \\ -36.71 & -70.26 & -140.11 & -86.08 & -232.76 & -112.67 & -114.76 & -17.77 \\ -71.95 & -140.11 & -260.21 & -179.79 & -399.81 & -219.67 & -205.93 & -43.55 \\ -44.94 & -86.08 & -179.79 & -98.11 & -318.76 & -138.62 & -151.89 & -16.53 \\ -118.28 & -232.76 & -399.81 & -318.76 & -538.15 & -358.65 & -298.58 & -89.87 \\ -58.23 & -112.67 & -219.67 & -138.62 & -358.65 & -177.88 & -178.48 & -29.82 \\ -58.65 & -114.76 & -205.93 & -151.89 & -298.58 & -178.48 & -158.01 & -39.71 \\ -9.52 & -17.77 & -43.55 & -16.53 & -89.87 & -29.82 & -39.71 & -0.05 \end{bmatrix}.$$

Let the linear projective parameters are $k = (k_1, k_2, \cdots, k_8) = (-1.8, 1, 4, 5.5, 1.1, -2, 8, 9)$. The projective position states of all agents are obtained as shown in Figure 2, respectively, with the any given initial conditions $x(0) = (-1, -2.1, 0.6, 4, 1.5, 0.8, 0, 3)$. The stable state are $x(t) \to (-0.6180, 0.3426, 1.3674, 1.8842, 0.3765, -0.6848, 2.7412, 3.0826)$, here the weighted average position $x_0(t) \to 0.3425$ by simulation as $t \to +\infty$. Figure 3 shows the global error $\sqrt{\sum_{i=1}^{7}(\frac{x_i(t)}{k_i} - \frac{x_{i+1}(t)}{k_{i+1}})^2} \to 0$ and the linear MSAGC is achieved.

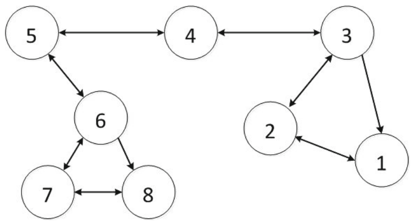

Figure 1 Communication topology $G(A_1)$

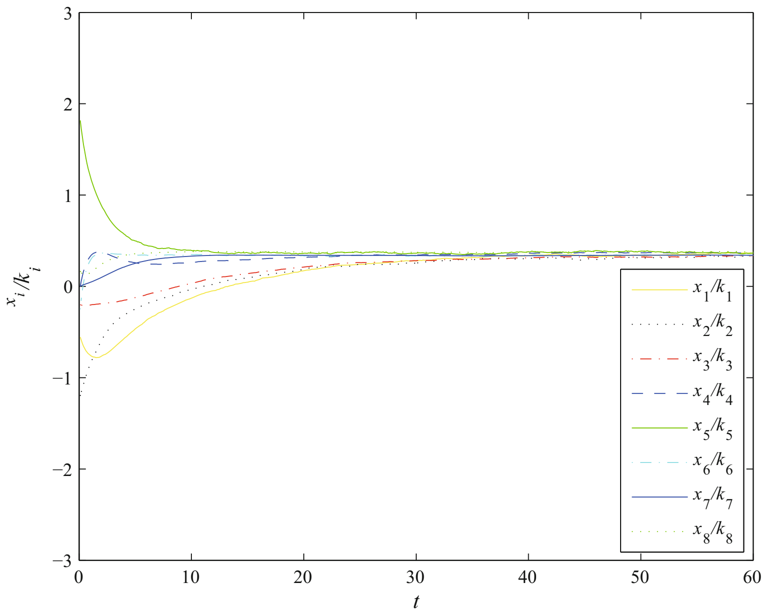

Figure 2 Linear MSAGC is achieved for Example 4.1

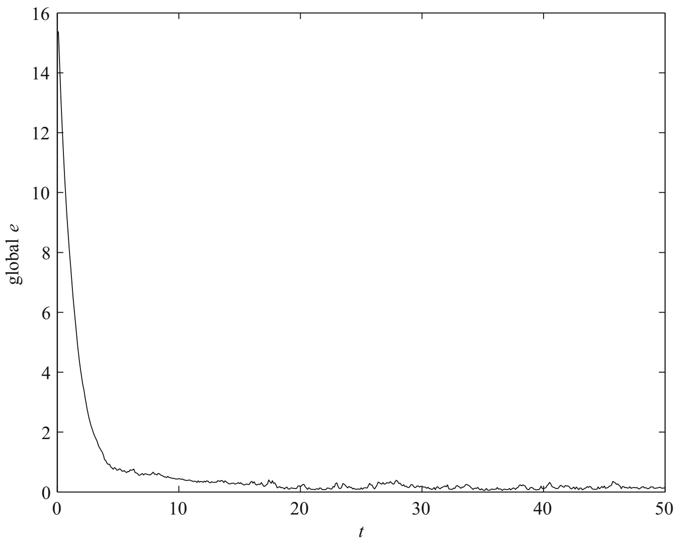

Figure 3 The evolution of global error $e = \sqrt{\sum_{i=1}^{7}(\frac{x_i(t)}{k_i} - \frac{x_{i+1}(t)}{k_{i+1}})^2}$

Example 4.2 (Nonlinear case) Consider multi-agent system (4) with topology $G(A_2)$ as in Figure 4. Suppose that there is a virtue leader labeled V in System (4). Let $x^* = 4$, $h_1(x) = 4x$, $h_2(x) = x^2$, $h_3(x) = \frac{4}{3}x$, $h_4(x) = 2x - 4$, $h_5(x) = \sqrt{x+11}$, Figure 5 shows that the stable state are $x(t) \to (h_1^{-1}(x^*), h_2^{-1}(x^*), h_3^{-1}(x^*), h_4^{-1}(x^*), h_5^{-1}(x^*)) = (1, 2, 3, 4, 5)$ as $t \to +\infty$, with the any given initial conditions $x(0) = (-1, 0.6, -0.3, 0, 1.2)$. Figure 6 shows the global error

$$\sqrt{\sum_{i=1}^{4}(h_i(x_i(t)) - h_{i+1}(x_{i+1}(t)))^2} \to 0,$$

the nonlinear MSAGC is achieved.

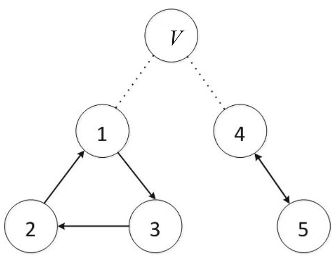

Figure 4 Communication topology $G(A_2)$

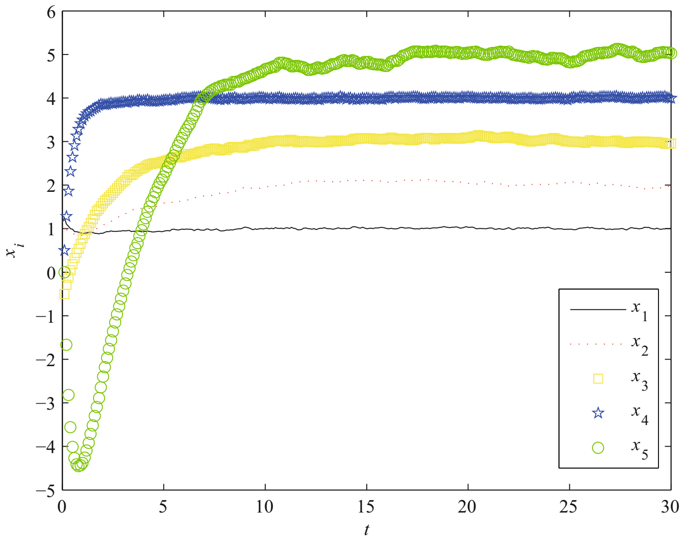

Figure 5 Nonlinear MSAGC is achieved for Example 4.2

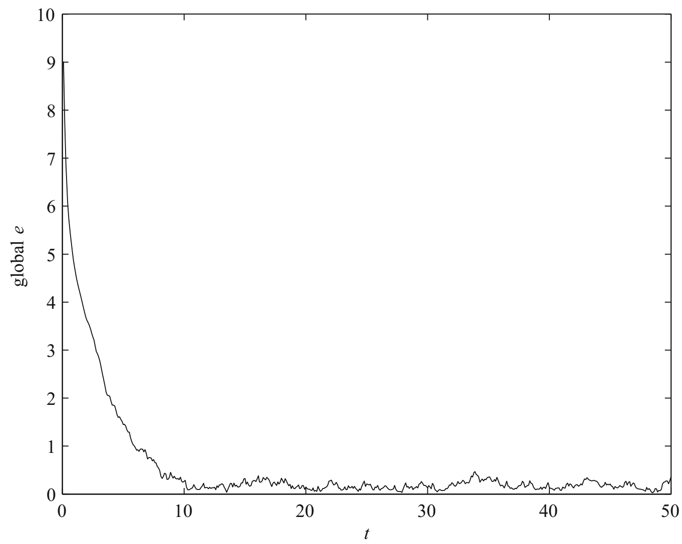

Figure 6 The evolution of global error $e = \sqrt{\sum_{i=1}^{4}(h_i(x_i(t)) - h_{i+1}(x_{i+1}(t)))^2}$

5 Conclusions

In summary, we have given a new concept of MSAGC in continuous-time multi-agent systems. External stochastic disturbances and time delays increase the difficulty of stability analysis in some ways. Then the MSAGC problem is investigated based on Lyapunov stability theory and the stochastic theory. Sufficient conditions are established in terms of a set of linear matrix inequalities. The mean square stability of the error dynamics is shown to guarantee generalized consensus of our model. Finally, simulation examples of linear and nonlinear cases have been given to illustrate the generalization of the theoretical results. Our future works will focus on the MSAGC of more practical models, such as second-order multi-agent systems. Continued research would be desirable.

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