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On Designing Consistent Extended Kalman Filter*

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Abstract This paper studies the consistency of the extended Kalman filter (EKF) for a kind of nonlinear systems. Based on the EKF algorithm, the authors propose the quasi-consistent EKF (QCEKF) as well as the tuning law for its parameters. The quasi-consistency of the proposed algorithm is proved. Finally, the feasibility of the algorithm is illustrated by the numerical simulation on an orbit determination example.

Keywords Extended Kalman filter (EKF), consistency, nonlinear system.

1 Introduction

Filtering is one of the most pervasive tools for estimating the systems' states by the measurement which is contaminated by random noise. For linear systems, Kalman proposed the Kalman filter (KF) in 1960[1], which is an optimal minimum mean square error estimator. Besides the state estimate, the KF algorithm algorithm can also provide the mean square error of the estimation by the covariance matrix P_k, which can be calculated by the filter. Hence, P_k can be evaluate the filtering accuracy in real time. For linear systems, KF has been widely used[2–4]. For nonlinear systems, the extended Kalman filter (EKF) has been widely applied[5–7]. In the EKF algorithm, the nonlinear functions are linearized at the current state estimate. Due to the errors introduced during linearization, EKF is usually a biased estimator and the matrix P_k in EKF is no longer the mean square error of the estimation. In some special cases, the estimation error of the EKF algorithm may diverge[8].

Although EKF has been widely used, little previous work focuses on theoretical analysis of its estimation error for general nonlinear systems. When the initial estimation error and

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the covariance of measurement noise are small enough, [8, 9] proved the boundedness of the estimation error of EKF. [10] proved the convergence of EKF in the condition that the initial estimation error is small enough and the linearized systems is observable at the current estimate. References [11, 12] proposed a new filter based on EKF for nonlinear systems with incomplete information and proved the stability of the proposed filter by assuming bounded covariance of the estimation. However, the assumptions in [8–12] are usually too strict to be satisfied in practice, and few results in [8–12] can evaluate the boundary of the filtering error in real time.

On the other hand, in the past few years, lots of researches have shown that EKF may produce inconsistent estimates due to the error introduced in linearization process. A state estimator is consistent if it is unbiased and its actual mean square errors equal to the one calculated by the filter[13]. Consistency is a fundamental criterion for evaluating the performance of an estimator. In practice, since the true system state is unknown, if an estimator is inconsistent, then the errors of the estimates are unknown and the estimate results may be unreliable. The consistency of EKF, which is an essential issue, has not been investigated in depth. The inconsistent problem of the EKF-based simultaneous localization and mapping (SLAM) algorithm are studied in [14–20]. In [20], the cause of the inconsistency was analyzed and a framework for improving the consistency of EKF-base SLAM was proposed.

In this paper, the cause of the inconsistency of EKF for a kind of general nonlinear systems is investigated and a novel framework for designing a quasi-consistency EKF (QCEKF) for general nonlinear systems is proposed. A state estimator is quasi-consistent if its actual mean square error is smaller or equal to the one calculated by the filter. It is proved that the proposed QCEKF can assure the actual mean square error is no larger than the bound calculated by the filter on time. Hence the boundary of the filtering error can be evaluated in real time. Moreover, the initial estimation error and the covariance of measurement noise are no longer to be assumed being small enough by using QCEKF.

The paper is organized as follows. Section 2 provides a brief introduction about the traditional EKF. In Section 3, the QCEKF algorithm is proposed for a kind of general nonlinear systems, and the quasi-consistency of the proposed algorithm is proved. In Section 4, the method to tune the parameters for QCEKF based orbit determination algorithm is presented, and the feasibility of QCEKF is illustrated by the numerical simulation. The concluding remarks are given in Section 5.

2 The Traditional Extended Kalman Filter

Consider the following nonlinear system:

$$\begin{cases} X_{k+1} = f(X_k), \\ y_k = g(X_k) + v_k, \end{cases} \qquad k = 0, 1, \cdots, \tag{1}$$

where $X_k \in R^n$ is the state vector, and $y_k \in R^m$ is the measurement vector. $v_k \in R^m$ is a zero-mean white noise processes with covariances being R_k. The nonlinear function f and g are assumed to be C^1-function with proper dimensions. The initial state X_0 has an unbiased estimator $\widehat{X}_0$, which is independent of v_k, and the covariance is known:

$$E[\widehat{X}_0] = X_0, \quad E[(\widehat{X}_0 - X_0)(\widehat{X}_0 - X_0)^{\mathrm{T}}] = P_0.$$

To obtain an estimate of X_k via the measurements y_k, the traditional EKF for the system (1) is given as follows[21]:

$$\begin{cases} \overline{X}_{k+1} = f(\widehat{X}_k), \\ \overline{P}_{k+1} = A_k P_k A_k^{\mathrm{T}}, \\ K_k = \overline{P}_{k+1} C_{k+1}^{\mathrm{T}} (C_{k+1} \overline{P}_{k+1} C_{k+1}^{\mathrm{T}} + R_{k+1})^{-1}, \\ \widehat{X}_{k+1} = \overline{X}_{k+1} + K_k (y_{k+1} - g(\overline{X}_{k+1})), \\ P_{k+1} = (I - K_k C_{k+1}) \overline{P}_{k+1} (I - K_k C_{k+1})^{\mathrm{T}} + K_k R_{k+1} K_k^{\mathrm{T}}, \end{cases} \tag{2}$$

where $A_k = \frac{\partial f}{\partial X}|_{\widehat{X}_k}$, $C_{k+1} = \frac{\partial g}{\partial X}|_{\overline{X}_{k+1}}$, $\widehat{X}_k$ is the filtering value at time-step k, $\overline{X}_k$ is the predicted value at time-step k.

If the system (1) is linear, that is, $f(X_k) = A_k X_k$, $g(X_k) = C_k X_k$, where A_k and C_k are constant matrices, then the EKF (2) is equivalent to KF, which is an optimal minimum mean square error estimator and is also consistency.

Definition 2.1 (see [13]) A state estimator (filter) is consistent if its state estimate errors satisfy

$$E[\widehat{X}_k - X_k] = 0, \quad E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] = P_k.$$

In other words, it is unbiased and its actual mean square error equals to the one calculated by the filter.

Since the filter gain K_k is calculated based on the calculated error covariance P_k, the consistency is necessary for filter optimality. Otherwise, wrong covariance will yield wrong gain.

For the general nonlinear systems (1), assume φ_k and ψ_k are the linearization errors at time-step k:

$$\begin{cases} \varphi_k = f(\widehat{X}_k) - f(X_k) - A_k^*(\widehat{X}_k - X_k), \\ \psi_{k+1} = g(\overline{X}_{k+1}) - g(X_{k+1}) - C_{k+1}^*(\overline{X}_{k+1} - X_{k+1}), \end{cases}$$

where $A_k^* = \frac{\partial f}{\partial X}|_{X_k}$, $C_{k+1}^* = \frac{\partial g}{\partial X}|_{X_{k+1}}$. Then for the EKF (2), the estimate error will be

$$\widehat{X}_{k+1} - X_{k+1} = (I - K_k C_{k+1}^*)(A_k^*(\widehat{X}_k - X_k) + \varphi_k(\cdot)) - K_k \psi_{k+1}(\cdot) + K_k v_{k+1}. \tag{3}$$

The mean square error of the estimation is

$$\begin{aligned} & E[(\widehat{X}_{k+1} - X_{k+1})(\widehat{X}_{k+1} - X_{k+1})^{\mathrm{T}}] \\ = {} & (I - K_k C_{k+1}) A_k E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{\mathrm{T}} (I - K_k C_{k+1})^{\mathrm{T}} + K_k E[v_{k+1} v_{k+1}^{\mathrm{T}}] K_k^{\mathrm{T}} \\ & + E[(I - K_k C_{k+1}^*)(A_k^*(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}} A_k^{*\mathrm{T}})(I - K_k C_{k+1}^*)^{\mathrm{T}}] \\ & - (I - K_k C_{k+1}) A_k E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{\mathrm{T}} (I - K_k C_{k+1})^{\mathrm{T}} \\ & + E[K_k \psi_{k+1}(\cdot) \psi_{k+1}^{\mathrm{T}}(\cdot) K_k^{\mathrm{T}}] + E[K_k v_{k+1} v_{k+1}^{\mathrm{T}} K_k^{\mathrm{T}}] - K_k E[v_{k+1} v_{k+1}^{\mathrm{T}}] K_k^{\mathrm{T}} \end{aligned}$$

$$
\begin{aligned}
&+E[(I-K_kC^*_{k+1})(A^*_k(\widehat{X}_k-X_k)\varphi_k^{\mathrm{T}}(\cdot)+\varphi_k(\cdot)(\widehat{X}_k-X_k)^{\mathrm{T}}A_k^{*\mathrm{T}}\\
&+\varphi_k(\cdot)\varphi_k^{\mathrm{T}}(\cdot))(I-K_kC^*_{k+1})^{\mathrm{T}}]-E[(I-K_kC^*_{k+1})(A^*_k(\widehat{X}_k-X_k)+\varphi_k(\cdot))\psi_{k+1}^{\mathrm{T}}(\cdot)K_k^{\mathrm{T}}]\\
&-E[K_k\psi_{k+1}(\cdot)(A^*_k(\widehat{X}_k-X_k)+\varphi_k(\cdot))^{\mathrm{T}}(I-K_kC^*_{k+1})^{\mathrm{T}}]\\
\triangleq\ & P_{k+1}+\Delta P_{k+1},
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
&\Delta P_{k+1}\\
=&E[(I-K_kC^*_{k+1})A^*_k(\widehat{X}_k-X_k)(\widehat{X}_k-X_k)^{\mathrm{T}}A_k^{*\mathrm{T}}(I-K_kC^*_{k+1})^{\mathrm{T}}]\\
&-(I-K_kC_{k+1})A_kE[(\widehat{X}_k-X_k)(\widehat{X}_k-X_k)^{\mathrm{T}}]A_k^{\mathrm{T}}(I-K_kC_{k+1})^{\mathrm{T}}\\
&+E[K_k\psi_{k+1}(\cdot)\psi_{k+1}^{\mathrm{T}}(\cdot)K_k^{\mathrm{T}}]+E[K_kv_{k+1}v_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]-K_kE[v_{k+1}v_{k+1}^{\mathrm{T}}]K_k^{\mathrm{T}}\\
&+E[(I-K_kC^*_{k+1})(A^*_k(\widehat{X}_k-X_k)\varphi_k^{\mathrm{T}}(\cdot)+\varphi_k(\cdot)(\widehat{X}_k-X_k)^{\mathrm{T}}A_k^{*\mathrm{T}}+\varphi_k(\cdot)\varphi_k^{\mathrm{T}}(\cdot))(I-K_kC^*_{k+1})^{\mathrm{T}}]\\
&-E[(I-K_kC^*_{k+1})(A^*_k(\widehat{X}_k-X_k)+\varphi_k(\cdot))\psi_{k+1}^{\mathrm{T}}(\cdot)K_k^{\mathrm{T}}]\\
&-E[K_k\psi_{k+1}(\cdot)(A^*_k(\widehat{X}_k-X_k)+\varphi_k(\cdot))^{\mathrm{T}}(I-K_kC^*_{k+1})^{\mathrm{T}}].
\end{aligned}
$$

If $\Delta P_{k+1}>0$, then the actual mean square errors do not equal to the calculated covariance P_{k+1}. Therefore, the EKF (2) is not consistent for the general nonlinear systems and the actual estimation errors, which are unknown in practice, cannot be evaluated by P_{k+1}. To make matters worse, the wrong covariance P_{k+1} may cause diverging.

(3) and (4) show that for the general nonlinear system (1), it's difficult to exactly calculate the first two moments of the estimation errors. Next, a novel framework for designing a quasi-consistency EKF will be proposed. The definition of quasi-consistency is given as follows.

Definition 2.2 A state estimator (filter) is quasi-consistent, if its mean square error of the estimation equals or less than the one yielded by the filter.

3 The Quasi-Consistent Extended Kalman Filter

The QCEKF algorithm for the system (1) is provided as follows:

$$
\begin{cases}
\overline{X}_{k+1}=f(\widehat{X}_k),\\
\widetilde{\overline{P}}_{k+1}=A_k\widetilde{P}_kA_k^{\mathrm{T}}+\Delta Q_k,\\
K_k=\widetilde{\overline{P}}_{k+1}C_{k+1}^{\mathrm{T}}(C_{k+1}\widetilde{\overline{P}}_{k+1}C_{k+1}^{\mathrm{T}}+R_{k+1})^{-1},\\
\widehat{X}_{k+1}=\overline{X}_{k+1}+K_k(y_{k+1}-g(\overline{X}_{k+1})),\\
\widetilde{P}_{k+1}=(I-K_kC_{k+1})\widetilde{\overline{P}}_{k+1}(I-K_kC_{k+1})^{\mathrm{T}}+K_kR_{k+1}K_k^{\mathrm{T}}+\Delta R_{k+1},
\end{cases}
\tag{5}
$$

where $\widetilde{P}_0 = E[(\widehat{X}_0 - X_0)(\widehat{X}_0 - X_0)^{\mathrm{T}}]$, ΔQ_k and ΔR_{k+1} are symmetric matrices, satisfying the following conditions:

$$\begin{cases} \Delta Q_k \geq E[A_k^*(\widehat{X}_k - X_k)\varphi_k^{\mathrm{T}}] + E[\varphi_k(\widehat{X}_k - X_k)^{\mathrm{T}} A_k^{*\mathrm{T}}] + E[\varphi_k \varphi_k^{\mathrm{T}}] \\ \qquad + E[A_k^*(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}} A_k^{*\mathrm{T}}] - A_k E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{\mathrm{T}}, \\ \Delta R_{k+1} \geq - E[(I - K_k C_{k+1}^*)(\overline{X}_{k+1} - X_{k+1})\psi_{k+1}^{\mathrm{T}} K_k^{\mathrm{T}}] \\ \qquad - E[K_k \psi_{k+1}(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}(I - K_k C_{k+1}^*)^{\mathrm{T}}] \\ \qquad + E[K_k \psi_{k+1}\psi_{k+1}^{\mathrm{T}} K_k^{\mathrm{T}}] + E[K_k v_{k+1} v_{k+1}^{\mathrm{T}} K_k^{\mathrm{T}}] - K_k E[v_{k+1} v_{k+1}^{\mathrm{T}}] K_k^{\mathrm{T}} \\ \qquad - (I - K_k C_{k+1}) E[(\overline{X}_{k+1} - X_{k+1})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}](I - K_k C_{k+1})^{\mathrm{T}} \\ \qquad + E[(I - K_k C_{k+1}^*)(\overline{X}_{k+1} - X_{k+1})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}(I - K_k C_{k+1}^*)^{\mathrm{T}}], \\ k = 0, 1, \cdots. \end{cases} \tag{6}$$

Theorem 3.1 *The algorithm* (5)–(6) *is quasi-consistent, i.e.,*

$$E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] \leq \widetilde{P}_k, \quad k = 0, 1, \cdots.$$

Proof We use the mathematical induction to prove the result. Assume $E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] \leq \widetilde{P}_k$. Linearizing the process equation, the following equation holds

$$\overline{X}_{k+1} = f(\widehat{X}_k) = f(X_k) + A_k^*(\widehat{X}_k - X_k) + \varphi_k,$$

where φ_k is the linearization error. Then the predicted error at time-step $k+1$ is

$$\overline{X}_{k+1} - X_{k+1} = A_k^*(\widehat{X}_k - X_k) + \varphi_k.$$

Thus, the mean square error of the predicted value can be presented as follows:

$$E[(\overline{X}_{k+1} - X_{k+1})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}] = E[(A_k^*(\widehat{X}_k - X_k) + \varphi_k)(A_k^*(\widehat{X}_k - X_k) + \varphi_k)^{\mathrm{T}}].$$

When ΔQ_k satisfies the inequality (6), we can get

$$\begin{aligned} & E(\overline{X}_{k+1} - X_{k+1})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}} \\ = {} & A_k E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{\mathrm{T}} + E[A_k^*(\widehat{X}_k - X_k)\varphi_k^{\mathrm{T}}] + E[\varphi_k(\widehat{X}_k - X_k)^{\mathrm{T}} A_k^{*\mathrm{T}}] \\ & + E(\varphi_k \varphi_k^{\mathrm{T}}) + E[A_k^*(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}} A_k^{*\mathrm{T}}] - A_k E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{\mathrm{T}} \\ \leq {} & \widetilde{\widetilde{P}}_{k+1}, \end{aligned} \tag{7}$$

where $\widetilde{\widetilde{P}}_{k+1} = A_k \widetilde{P}_k A_k^{\mathrm{T}} + \Delta Q_k$.

Linearizing the output function, the following equation holds

$$g(\overline{X}_{k+1}) = g(X_{k+1}) + C_{k+1}^*(\overline{X}_{k+1} - X_{k+1}) + \psi_{k+1},$$

where ψ_{k+1} is the linearization error. The filtering error at time-step $k+1$ is

$$\widehat{X}_{k+1} - X_{k+1} = (I - K_k C_{k+1}^*)(\overline{X}_{k+1} - X_{k+1}) - K_k \psi_{k+1} + K_k v_{k+1}.$$

Thus the mean square error of the filtering value can be presented as follows:

$$\begin{aligned}
&E[(\widehat{X}_{k+1}-X_{k+1})(\widehat{X}_{k+1}-X_{k+1})^{\mathrm{T}}]\\
=\,&(I-K_kC_{k+1})E[(\overline{X}_{k+1}-X_{k+1})(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}](I-K_kC_{k+1})^{\mathrm{T}}+K_kE[v_{k+1}v_{k+1}^{\mathrm{T}}]K_k^{\mathrm{T}}\\
&-E[(I-K_kC_{k+1}^*)(\overline{X}_{k+1}-X_{k+1})(K_k\psi_{k+1})^{\mathrm{T}}]\\
&-E[(K_k\psi_{k+1})(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}(I-K_kC_{k+1}^*)^{\mathrm{T}}]\\
&+E[K_k\psi_{k+1}\psi_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]+E[K_kv_{k+1}v_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]-K_kE[v_{k+1}v_{k+1}^{\mathrm{T}}]K_k^{\mathrm{T}}\\
&-(I-K_kC_{k+1})E[(\overline{X}_{k+1}-X_{k+1})(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}](I-K_kC_{k+1})^{\mathrm{T}}\\
&+E[(I-K_kC_{k+1}^*)(\overline{X}_{k+1}-X_{k+1})(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}(I-K_kC_{k+1}^*)^{\mathrm{T}}]\\
\le\,&\widetilde{P}_{k+1},
\end{aligned}\tag{8}$$

where $\widetilde{P}_{k+1}=(I-K_kC_{k+1})\widetilde{\overline{P}}_{k+1}(I-K_kC_{k+1})^{\mathrm{T}}+K_kR_{k+1}K_k^{\mathrm{T}}+\Delta R_{k+1}$, ΔR_{k+1} satisfies the inequality (6).

Remark 3.2 According to Theorem 3.1, the algorithm (5) is quasi-consistent and the filtering error can be evaluated by $\widetilde{P}_k$. Therefore, if $\widetilde{P}_k$ is bounded, the true filtering error is bounded.

Remark 3.3 Denote $\varPhi_k$ and $\varPsi_{k+1}$ as follows:

$$\begin{cases}
\varPhi_k=E[A_k^*(\widehat{X}_k-X_k)\varphi_k^{\mathrm{T}}]+E[\varphi_k(\widehat{X}_k-X_k)^{\mathrm{T}}A_k^{*\,\mathrm{T}}]+E[\varphi_k\varphi_k^{\mathrm{T}}]\\
\qquad+E[A_k^*(\widehat{X}_k-X_k)(\widehat{X}_k-X_k)^{\mathrm{T}}A_k^{*\,\mathrm{T}}]-A_kE[(\widehat{X}_k-X_k)(\widehat{X}_k-X_k)^{\mathrm{T}}]A_k^{\mathrm{T}},\\
\varPsi_{k+1}=-E[(I-K_kC_{k+1}^*)(\overline{X}_{k+1}-X_{k+1})\psi_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]\\
\qquad-E[K_k\psi_{k+1}(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}(I-K_kC_{k+1}^*)^{\mathrm{T}}]\\
\qquad+E[K_k\psi_{k+1}\psi_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]+E[K_kv_{k+1}v_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]-K_kE[v_{k+1}v_{k+1}^{\mathrm{T}}]K_k^{\mathrm{T}}\\
\qquad-(I-K_kC_{k+1})E[(\overline{X}_{k+1}-X_{k+1})(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}](I-K_kC_{k+1})^{\mathrm{T}}\\
\qquad+E[(I-K_kC_{k+1}^*)(\overline{X}_{k+1}-X_{k+1})(\overline{X}_{k+1}-X_{k+1})^{\mathrm{T}}(I-K_kC_{k+1}^*)^{\mathrm{T}}].
\end{cases}$$

In the right hand of $\varPhi_k$, the first three items are caused by the linearization error φ_k and the last two items are caused by the randomness of A_k. In the right hand of $\varPsi_{k+1}$, the first three items are caused by the linearization error ψ_{k+1}, the error $E[K_kv_{k+1}v_{k+1}^{\mathrm{T}}K_k^{\mathrm{T}}]-K_kE[v_{k+1}v_{k+1}^{\mathrm{T}}]K_k^{\mathrm{T}}$ is caused by the randomness of K_k and the last two items are caused by the randomness of C_k and K_k. Hence, when the error introduced during linearization can not be ignored, the estimation error of EKF may be inconsistent. In QCEKF, the quasi-consistency is assured by ΔQ_k and ΔR_{k+1}.

Theorem 3.1 can be extended to the systems with determined disturbance.

Considering the system

$$\begin{cases}X_{k+1}=f(X_k)+d_{w,k},\\ y_k=g(X_k)+v_k+d_{v,k},\end{cases}\qquad k=0,1,\cdots,\tag{9}$$

where X_k, y_k, v_k, f and g are the same as those in (1). $d_{w,k}$ and $d_{v,k}$ are the determined disturbances, with known upper boundary $\overline{d}_{w,k}, \overline{d}_{v,k}$ respectively. The initial state X_0 has an unbiased estimator $\widehat{X}_0$, which is independent of v_k, and the covariance is known:

$$E[\widehat{X}_0] = X_0, \quad E[(\widehat{X}_0 - X_0)(\widehat{X}_0 - X_0)^{\mathrm{T}}] = \widetilde{P}_0.$$

For the system (9), the parameter turning law (6) should be adjusted as follows:

$$\begin{cases} \Delta Q_k \geq E[A_k^*(\widehat{X}_k - X_k)(\varphi_k + \overline{d}_{w,k})^{\mathrm{T}}] + E[(\varphi_k + \overline{d}_{w,k})(\widehat{X}_k - X_k)^{\mathrm{T}} A_k^{*\mathrm{T}}] \\ \qquad + E[(\varphi_k + \overline{d}_{w,k})(\varphi_k + \overline{d}_{w,k})^{\mathrm{T}}] + A_k^* E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{*\mathrm{T}} \\ \qquad - A_k E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] A_k^{\mathrm{T}}, \\ \Delta R_{k+1} \geq - E[(I - K_k C_{k+1}^*)(\overline{X}_{k+1} - X_{k+1})(\psi_{k+1} + \overline{d}_{v,k})^{\mathrm{T}} K_k^{\mathrm{T}}] \\ \qquad - E[K_k(\psi_{k+1} + \overline{d}_{v,k})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}(I - K_k C_{k+1}^*)^{\mathrm{T}}] \\ \qquad + E[K_k(\psi_{k+1} + \overline{d}_{v,k})(\psi_{k+1} + \overline{d}_{v,k})^{\mathrm{T}} K_k^{\mathrm{T}}] + E[K_k v_{k+1} v_{k+1}^{\mathrm{T}} K_k^{\mathrm{T}}] \\ \qquad - K_k E[v_{k+1} v_{k+1}^{\mathrm{T}}] K_k^{\mathrm{T}} \\ \qquad - (I - K_k C_{k+1}) E[(\overline{X}_{k+1} - X_{k+1})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}](I - K_k C_{k+1})^{\mathrm{T}} \\ \qquad + E[(I - K_k C_{k+1}^*)(\overline{X}_{k+1} - X_{k+1})(\overline{X}_{k+1} - X_{k+1})^{\mathrm{T}}(I - K_k C_{k+1}^*)^{\mathrm{T}}], \\ k = 0, 1, \cdots. \end{cases} \tag{10}$$

Theorem 3.4 *For the system* (9)*, if the parameters* ΔQ_k*,* ΔR_{k+1} *satisfy the inequality* (10)*, then the algorithm* (5), (10) *is quasi-consistent, i.e.,*

$$E[(\widehat{X}_k - X_k)(\widehat{X}_k - X_k)^{\mathrm{T}}] \leq \widetilde{P}_k, \quad k = 0, 1, \cdots.$$

The proof of Theorem 3.4 can be easily obtained by the method similar to that in the proof of Theorem 3.1.

Remark 3.5 For the system with continuous dynamic, we can transform the continuous equation into discrete equation, viewing the discretization error as the deterministic disturbance, then the original system can be transformed into the discrete system (9).

Remark 3.6 According to the equations (7) and (8), the more conservative ΔQ_k and ΔR_{k+1} are, the more conservative $\widetilde{P}_{k+1}$ will be. If the parameters ΔQ_k and ΔR_{k+1} are too conservative, the true filtering error may bounded while $\widetilde{P}_k$ divergence. This feature can be illustrated by the following Example.

Example 3.7 Consider the system

$$\begin{cases} x_{1,k+1} = x_{1,k} + x_{2,k}, \\ x_{2,k+1} = x_{2,k}^2, \end{cases}$$
$$y_k = x_{1,k} + v_k, \quad k = 0, 1, \cdots, \tag{11}$$

where $X_k = [x_{1,k}, x_{2,k}]^{\mathrm{T}}$ is a 2-dimension state vector, y_k is a 1-dimension measurement vector. v_k is a zero-mean white noise processes with covariances being $R_k = 1$. The initial value

$x_{2,0} \in [-0.5, 0.5]$. The initial estimation $\widehat{X}_0$ is independent of v_k, and the covariance of the initial estimation are known as follows:

$$E[(\widehat{X}_0 - X_0)(\widehat{X}_0 - X_0)^{\mathrm{T}}] = \widetilde{P}_0 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

When the initial estimation $\widehat{X}_0 = [1, 0]^{\mathrm{T}}$, according to the QCEKF algorithm (5), $\overline{X}_1 = [1, 0]^{\mathrm{T}}$ and $A_0 = [\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}]$. We choose $\Delta Q_0 = \left[\begin{smallmatrix} 2\widetilde{P}_0(1,1)+2\widetilde{P}_0(2,2) & 0 \\ 0 & 1 \end{smallmatrix}\right]$. For the system (11), it can be verified that ΔQ_0 satisfies the inequality (6). Substituting ΔQ_0 into the algorithm (5), we have

$$\tilde{\tilde{P}}_1 = A_0 \widetilde{P}_0 {A_0}^{\mathrm{T}} + \Delta Q_0 = \begin{bmatrix} 3\widetilde{P}_0(1,1) + 3\widetilde{P}_0(2,2) & 0 \\ 0 & \Delta Q_0(2,2) \end{bmatrix}, \quad K_0 = \begin{bmatrix} \frac{\tilde{\tilde{P}}_1(1,1)}{\tilde{\tilde{P}}_1(1,1)+R_1} \\ 0 \end{bmatrix}.$$

Since K_0 is independent of $\{y_k\}$, we can choose $\Delta R_1 = 0$. Substituting $\Delta R_1 = 0$ into the algorithm (5), we have

$$\widetilde{P}_1 = \begin{bmatrix} \frac{\tilde{\tilde{P}}_1(1,1) R_1}{\tilde{\tilde{P}}_1(1,1)+R_1} & 0 \\ 0 & \tilde{\tilde{P}}_1(2,2) \end{bmatrix}. \tag{12}$$

Similar to $\widetilde{P}_0$, $\widetilde{P}_1$ is also a diagram matrix, and it is easy to know that:

$$\widetilde{P}_1(1,1) \le R_1, \quad \widetilde{P}_1(2,2) = \Delta Q_0(2,2), \quad \widehat{x}_{2,1} = 0.$$

Moreover, $\widetilde{P}_1$ and $\widehat{x}_{2,1}$ are independent of $\{y_k\}$.

Similarly, we can choose the parameters ΔQ_k and ΔR_{k+1} as follows:

$$\Delta Q_k = \begin{bmatrix} 2\widetilde{P}_k(1,1) + 2\widetilde{P}_k(2,2) & 0 \\ 0 & k \end{bmatrix}, \quad \Delta R_{k+1} = 0. \tag{13}$$

Then $\widetilde{P}_k(1,1) \le R_k$, $\widetilde{P}_k(2,2) = \Delta Q_{k-1}(2,2) = k$, $\widehat{x}_{2,k} = 0$. Obviously, $\widetilde{P}_k$ is divergent.

However, since QCEKF algorithm is quasi-consistent, we have

$$\begin{cases} E[(\widehat{x}_{1,k} - x_{1,k})^2] \le \widetilde{P}_k(1,1) \le R_k, \\ \widehat{x}_{2,k} - x_{2,k} = -x_{2,k} \to 0. \end{cases}$$

Since $x_{2,0} \in [-0.5, 0.5]$, $x_{2,k} \to 0$. Hence, the mean square of filter error is bounded.

In this example, the mean square error of filter is bounded while $\widetilde{P}_k$ is divergent. The reason is the parameter ΔQ_k is too conservative.

This example shows that the selection of the parameters ΔQ_k, ΔR_{k+1} are the key issue of QCEKF.

Theorems 3.1 and 3.4 provide the rules of how to determine the parameters ΔQ_k, ΔR_{k+1}. Next some specific methods for choosing the parameters ΔQ_k, ΔR_{k+1} will be discussed for the orbit determination problem.

4 QCEKF Based Orbit Determination Algorithm

For the flight during free flight phase, denote the position vector as $\left[\,{}^{r_x, r_y, r_z}\,\right]^{\mathrm{T}}$, the velocity vector as $\left[\,{}^{v_x, v_y, v_z}\,\right]^{\mathrm{T}}$. The dynamic equation of the flight can be presented as follows[22]:

$$\begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_x \\ v_x \\ -\frac{u_e}{q^3} r_x + 2\omega v_y + \omega^2 r_x \\ -\frac{u_e}{q^3} r_y - 2\omega v_x + \omega^2 r_y \\ -\frac{u_e}{q^3} r_z \end{bmatrix} \triangleq F(X), \quad X = \begin{bmatrix} r_x \\ r_y \\ r_z \\ v_x \\ v_y \\ v_z \end{bmatrix}, \tag{14}$$

where $u_e = 3.986005 \times 10^{14}$ is the geocentric gravitational constant, $\omega = 7.292115 \times 10^{-5}$ is the angular speed of rotation. $q = \sqrt{r_x^2 + r_y^2 + r_z^2}$ is the distance from the earth 's center to the vehicle.

The measurement of the target is obtained in the polar coordinates[23]:

$$y_k = h\left(\begin{bmatrix} r_{x,k} \\ r_{y,k} \\ r_{z,k} \end{bmatrix} - S_k \right) + v_k, \tag{15}$$

where S_k is the position vector from the earth's center to the sensor. The measurement noise v_k valued uncorrelated zero-mean Gaussian noise processes with covariance being $R_k = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_b^2 & 0 \\ 0 & 0 & \sigma_e^2 \end{bmatrix}$.

The nonlinear function $h\left(\begin{bmatrix} z_x \\ z_y \\ z_z \end{bmatrix}\right)$ is presented as follows:

$$h\left(\begin{bmatrix} z_x \\ z_y \\ z_z \end{bmatrix}\right) = \begin{bmatrix} \sqrt{z_x^2 + z_y^2 + z_z^2} \\ \arctan(\frac{z_y}{z_x}) \\ \arctan(\frac{z_z}{\sqrt{z_x^2 + z_y^2}}) \end{bmatrix}. \tag{16}$$

In this example, since the randomness of the gain matrix K_k can be neglected, the parameters ΔQ_k, ΔR_{k+1} can be chosen as follows:

$$\begin{cases} \Delta Q_k = ((1+\alpha_{1,k})(1+\alpha_{2,k}) - 1) A_k \widetilde{P}_k {A_k}^{\mathrm{T}} \\ \qquad + (1+\alpha_{1,k})\left(1 + \dfrac{1}{\alpha_{2,k}}\right) M_{Q,k} + \left(1 + \dfrac{1}{\alpha_{1,k}}\right) \Delta Q_{\varphi,k}, \\ \Delta R_{k+1} = ((1+\beta_{1,k})(1+\beta_{2,k}) - 1)(I - K_k C_{k+1}) \widetilde{\widetilde{P}}_{k+1} (I - K_k C_{k+1})^{\mathrm{T}} \\ \qquad + (1+\beta_{1,k})\left(1 + \dfrac{1}{\beta_{2,k}}\right) K_k M_{R,k} K_k^{\mathrm{T}} + \left(1 + \dfrac{1}{\beta_{1,k}}\right) K_k \Delta R_{\varphi,k} K_k^{\mathrm{T}}, \end{cases} \tag{17}$$

where

$$\alpha_{1,k}=\sqrt{\frac{\|\Delta Q_{\varphi,k}\|}{\|A_k\widetilde{P}_kA_k{}^{\mathrm{T}}\|}},\quad \alpha_{2,k}=\sqrt{\frac{\|M_{Q,k}\|}{\|A_k\widetilde{P}_kA_k{}^{\mathrm{T}}\|}},$$

$$\Delta Q_{\varphi,k}=6\times\mathrm{diag}\Big([(\varphi_{k,1}+d_{k,1})^2,(\varphi_{k,2}+d_{k,2})^2,\cdots,(\varphi_{k,6}+d_{k,6})^2]\Big),$$

$$\varphi_{k,i}^2=\frac{9}{4}\mathrm{trace}\Big(\sqrt{\widetilde{P}_k}\frac{\partial^2 f_i}{\partial X^2}|_{\widehat{X}_k}\widetilde{P}_k\frac{\partial^2 f_i}{\partial X^2}|_{\widehat{X}_k}\sqrt{\widetilde{P}_k}^{\mathrm{T}}\Big),\quad i=1,2,\cdots,6,\quad \widetilde{P}_k=\sqrt{\widetilde{P}_k}^{\mathrm{T}}\sqrt{\widetilde{P}_k},$$

$$[d_{k,1},d_{k,2},d_{k,3}]^{\mathrm{T}}=\frac{T^2}{2}[f_{4,k},f_{5,k},f_{6,k}]^{\mathrm{T}},$$

$$[d_{k,4},d_{k,5},d_{k,6}]^{\mathrm{T}}=\frac{T^2}{2}\Big(|A_{22,k}[f_{4,k},f_{5,k},f_{6,k}]^{\mathrm{T}}|+|A_{21,k}[8000,8000,8000]^{\mathrm{T}}|\Big),$$

$$A_{21,k}=[0,I_3]A_k[I_3,0]^{\mathrm{T}},\quad A_{22,k}=[0,I_3]A_k[0,I_3]^{\mathrm{T}},$$

$$M_{Q,k}=4\times 6\mathrm{diag}\Big(\Big[\varphi_{k,1}^2,\varphi_{k,2}^2,\cdots,\varphi_{k,6}^2\Big]\Big),$$

$$\beta_{1,k}=\max\left\{\frac{\psi_{k,2}}{\sigma_b},\frac{\psi_{k,3}}{\sigma_e}\right\},\quad \beta_{2,k}=\max\left\{\frac{M_{R,k}(2,2)}{\sigma_b},\frac{M_{R,k}(3,3)}{\sigma_e}\right\},$$

$$\Delta R_{\varphi,k}=3\times\mathrm{diag}\Big([\psi_{k,1}{}^2,\psi_{k,2}{}^2,,\psi_{k,3}{}^2]\Big),$$

$$\psi_{k,i}^2=\frac{9}{4}\mathrm{trace}\left[\sqrt{\widetilde{\widetilde{P}}_k}\frac{\partial^2 h_i}{(\partial X)^2}|_{\overline{X}_{k+1}}\widetilde{\widetilde{P}}_k\frac{\partial^2 h_i}{(\partial X)^2}|_{\overline{X}_{k+1}}\sqrt{\widetilde{\widetilde{P}}_k}^{\mathrm{T}}\right],\quad i=1,2,3,\quad \widetilde{\widetilde{P}}_k=\sqrt{\widetilde{\widetilde{P}}_k}^{\mathrm{T}}\sqrt{\widetilde{\widetilde{P}}_k},$$

$$M_{R,k}=4\times 3\mathrm{diag}\big([\psi_{k,1}{}^2,\psi_{k,2}{}^2,\psi_{k,3}{}^2]\big).$$

Remark 4.1 For the system (14), the simulations show that the randomness of the gain matrix K_k is very weak.

In the simulation, the initial state is $X_0=[S_k,0,0,0]^{\mathrm{T}}+[1\times 10^6\mathrm{m},2\times 10^6\mathrm{m},2\times 10^6\mathrm{m},4\times 10^3\mathrm{m/s},5\times 10^3\mathrm{m/s},1\times 10^3\mathrm{m/s}]^{\mathrm{T}}$, where $S_k=[-1.007499,5.48943,3.077004]^{\mathrm{T}}\times 10^6\mathrm{m}$. The standard deviation of the initial filtering error is $\sigma_{r,x}=\sigma_{r,y}=\sigma_{r,z}=1\times 10^4\mathrm{m}$, $\sigma_{v,x}=\sigma_{v,y}=\sigma_{v,z}=1\times 10^2\mathrm{m/s}$. The sampling step is $T=1\mathrm{s}$. The standard deviation of the measurement noise is $\sigma_r=20\mathrm{m},\sigma_b=5\times 10^{-3}\mathrm{rad},\sigma_e=5\times 10^{-3}\mathrm{rad}$.

Figures 1–4 are the simulation results of the traditional EKF and QCEKF algorithms. The red lines stand for the mean square error (MSE) of filtering results, which is calculated by 100 times experiments, and the blue lines stand for the corresponding diagonal elements of $\widetilde{P}_k$. In Figures 1–4, the diagonal elements of P_k are smaller than the mean square error. Hence, the traditional EKF is not quasi-consistent. In Figures 1–4, the diagonal elements of $\widetilde{P}_k$ are the upper boundary of the corresponding mean square error, i.e., the QCEKF algorithm is quasi-consistent. Hence, we can evaluate the filtering error based on $\widetilde{P}_k$ in real time.

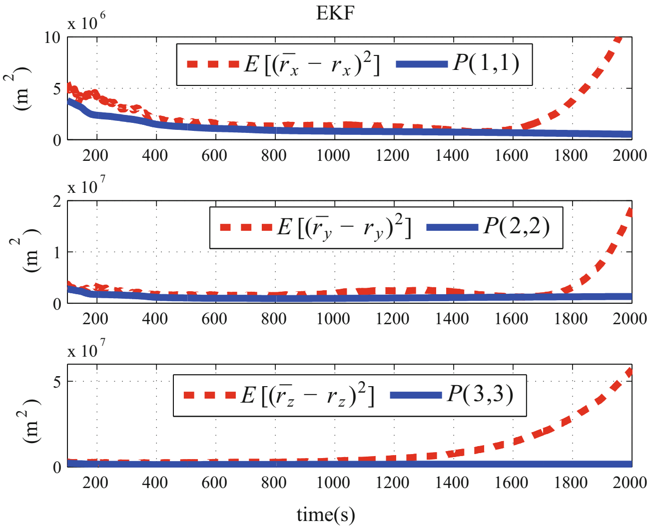

Figure 1 Numerical simulations of the position vector by the traditional EKF algorithm: The mean square error (MSE) of the true filtering error (dotted line) and the corresponding diagonal element in P_k (solid line) in real time

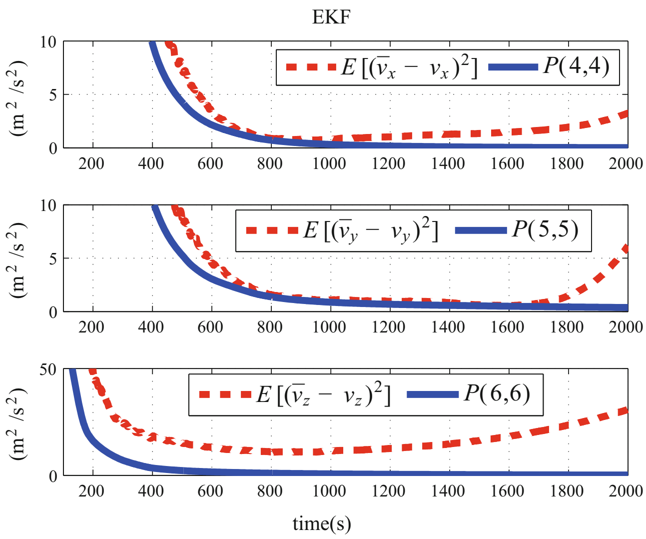

Figure 2 Numerical simulations of the velocity vector by the traditional EKF algorithm: The mean square error (MSE) of the true filtering error (dotted line) and the corresponding diagonal element in P_k (solid line) in real time

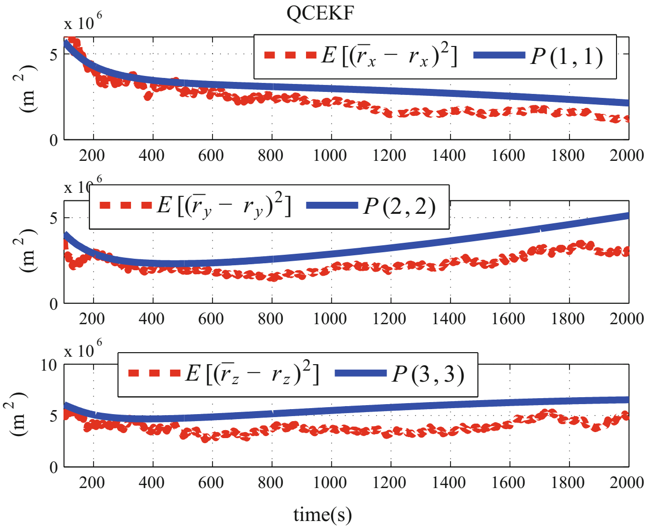

Figure 3 Numerical simulations of the position vector by the QCEKF algorithm: The mean square error (MSE) of the true filtering error (dotted line) and the corresponding diagonal element in $\widetilde{P}_k$ (solid line) in real time

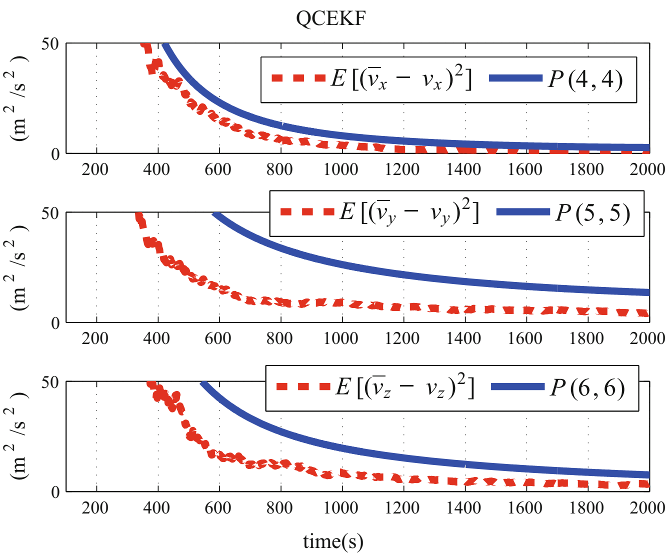

Figure 4 Numerical simulations of the velocity vector by the QCEKF algorithm: The mean square error (MSE) of the true filtering error (dotted line) and the corresponding diagonal element in $\widetilde{P}_k$ (solid line) in real time

5 Conclusion

This paper proposed a QCEKF algorithm for a kind of nonlinear systems, and the quasi-consistency of the algorithm is proved. Then the feasibility of QCEKF algorithm is illustrated by the numerical simulation for an orbit determination example.

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