J Syst Sci Complex (2016) 29: 1505–1518

A Maximum Principle for General Backward Stochastic Differential Equation

WU Shuang · SHU Lan

DOI: 10.1007/s11424-016-5209-y
Received: 2 February 2015 / Revised: 1 June 2015

Abstract In this paper, the authors consider a stochastic control problem where the system is governed by a general backward stochastic differential equation. The control domain need not be convex, and the diffusion coefficient can contain a control variable. The authors obtain a stochastic maximum principle for the optimal control of this problem by virtue of the second-order duality method.

Keywords Adjoint equations, backward stochastic differential equation, maximum principle, variational inequality.

1 Introduction

This paper is concerned with the dynamic system of general backward differential equations (BSDEs). A BSDE is an Itô's stochastic differential equation (SDE) in which the terminal rather than the initial condition is given. The BSDEs were introduced by Bismut[1] in the linear case and by Pardoux and Peng[2] in the general case. Since their introduction, the BSDEs have received considerable research attention in a large range of domains, especially in mathematical finance (see, e.g., Cvitanic and Ma[3], El Karoui, Peng, and Quenez[4], Ma and Yong[5], Schroder and Skiadas[6], Yong and Zhou[7], etc.). In particular, the celebrated Black-Scholes option pricing formula can be derived from a class of linear BSDEs where the random terminal condition is just the option's payoff at the maturity. Since BSDEs are well-defined dynamic systems, it is very natural and appealing to consider the control problems of BSDEs. However, there exist only a few works along this line, including Peng[8], Xu[9], Wu[10], Lim and Zhou[11], Huang, Wang and Xiong[12], and Wang and Yu[13]. Our work distinguishes itself from

WU Shuang (Corresponding author)
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731*, China*; *Department of Applied Mathematics, China University of Petroleum, Qingdao* 266580*, China*.
Email: namozhuntipusa@126.com.
SHU Lan
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731*, China*. Email: shul@uestc.edu.cn.
◇ *This paper was recommended for publication by Editor YIN Gang George.*

the above ones in the following aspects: (i) The stochastic system is described by a general BSDE whose diffusion coefficient contains the state $y(t)$. (ii) The control domain need not be convex, and the diffusion coefficient contains the control variable. To our best knowledge, this system has not been found in existing works.

Our objective in this paper is to establish necessary optimality condition of the Pontryagin maximum principle type. In our problem, since the control domain is not necessarily convex, we must obtain the maximum principle in its global form. A classical way of treating such a problem is to use the "spike variation". Due to the appearance of the control variable in the diffusion coefficient and the control domain is not necessarily convex, the usual first-order expansion approach does not work. Hence, we must introduce a second-order expansion method to derive the variational inequality. This method was firstly used by Peng in [14] to derive general forward stochastic maximum principle. Based on this, we obtain the corresponding (second-order) variational inequality and adjoint equations that lead to the maximum principle.

The paper is organized as follows. In Section 2, we give the statement of the problem, our main assumptions and some preliminary results about BSDE. In Section 3, we consider the second-order expansion of the perturbed state variable $y^\varepsilon(t)$, $z^\varepsilon(t)$ and the perturbed cost function $J(u^\varepsilon(\cdot))$. We also treat the estimations of these terms. In Section 4, we introduce the adjoint equations. By means of the duality method, we derive the maximum principle. Finally, we end this paper with some concluding remarks.

2 Statement of the Problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathcal{F}_t$. Let $B(\cdot)$ be a 1-dimensional Brownian motion. We assume that $\mathcal{F}_t = \sigma\{B(s); 0 \le s \le t\}$. Throughout this paper, we use the following notations:

$$L^2(\mathcal{F}_T;\mathbb{R}) = \{\xi : \xi \text{ is } \mathbb{R}\text{-valued } \mathcal{F}_T\text{-measurable stochastic variable s.t. } \mathbb{E}|\xi|^2 < +\infty\};$$

$$L^2_{\mathcal{F}}(0,T;\mathbb{R}) = \Big\{\varphi(t) : \{\varphi(t), 0 \le t \le T\} \text{ is } \mathbb{R}\text{-valued } \mathcal{F}_t\text{-adapted stochastic process s.t. } \mathbb{E}\int_0^T |\varphi(t)|^2 dt < +\infty\Big\};$$

$$L^4_{\mathcal{F}}(0,T;\mathbb{R}) = \Big\{\psi(t) : \{\psi(t), 0 \le t \le T\} \text{ is } \mathbb{R}\text{-valued } \mathcal{F}_t\text{-adapted stochastic process s.t. } \mathbb{E}\int_0^T |\psi(t)|^4 dt < +\infty\Big\}.$$

Consider the following backward stochastic control system:

$$\begin{cases} dy(t) = b\left(y(t), z(t), v(t)\right) dt + [\sigma\left(y(t), v(t)\right) + \gamma(t) z(t)] dB(t), \\ y(T) = \xi, \end{cases} \tag{1}$$

where

$$b(y,z,v) : \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}, \quad \sigma(y,v) : \mathbb{R} \times U \longrightarrow \mathbb{R}$$

and $\xi \in L^2(\mathcal{F}_T; R)$. An admissible control $v(\cdot)$ is an $\mathcal{F}_t$-adapted process with value in U such that $\mathbb{E}\sup_{t\in[0,T]} |v_t|^2 < \infty$, where U is a nonempty subset of $\mathbb{R}$. We denote the set of all admissible controls by U_{ad}. Our problem is to minimize the following cost functional over U_{ad}:

$$\begin{aligned} &J\left(v(\cdot)\right) = \mathbb{E}\left[\int_0^T l\left(y(t), v(t)\right) dt + h\left(y(0)\right)\right], \\ &\inf\left\{J\left(v(\cdot)\right); v(\cdot) \in U_{ad}\right\}, \end{aligned} \tag{2}$$

where

$$l(y,v) : \mathbb{R} \times U \longrightarrow \mathbb{R}, \quad h(y) : \mathbb{R} \longrightarrow \mathbb{R}.$$

In this paper, we only consider 1-dimensional stochastic system because the state of a backward system depends on two variables $(y(t), z(t))$, and $z(t)$ is hard to handle. Otherwise, there is an immediate difficult when we look for the second-order adjoint equation. In addition, we assume

$$\begin{cases} \text{(i)} & \sigma, l, h \text{ are twice continuously differentiable with respect} \\ & \text{to } y, \text{ and } b \text{ is twice continuously differentiable with respect to} \\ & (y,z). \text{ They and all their derivatives } b_y, b_{yy}, b_{yz}, b_z, b_{zz} \\ & \text{ are continuous in } (y,z,v); \\ \text{(ii)} & \sigma_y, \sigma_{yy}, l_y, l_{yy}, h_y, h_{yy} \text{ are continuous in } (y,v), \\ & b_y, b_{yy}, b_{yz}, b_z, b_{zz}, \sigma_y, \sigma_{yy}, l_{yy}, h_{yy} \text{ are bounded, and} \\ & b, \sigma, l_y, h_y \text{ are bounded by } \widetilde{C}\left(1+|y|+|z|+|v|\right), \text{ where } \widetilde{C} > 0; \\ \text{(iii)} & \sigma_y^{-1} \text{ is uniformly bounded with respect to } (t,\omega) \end{cases} \tag{3}$$

and

$$\begin{cases} \text{(i)} & \text{for all } v_1, v_2 \in U, \text{ there exists } \widetilde{c} > 0 \text{ such that} \\ & |b(y_1,z_1,v_1) - b(y_2,z_2,v_2)| + |\sigma(y_1,v) - \sigma(y_2,v)| \\ & \le \widetilde{c}(|y_1-y_2| + |z_1-z_2| + |v_1-v_2|); \\ \text{(ii)} & \gamma(\cdot) \text{ is } \mathcal{F}_t\text{-adapted, and for all } (t,\omega), \text{ there exists} \\ & \beta > 0 \text{ such that } |\gamma(t,\omega)| \ge \beta. \text{ Moreover, } \gamma(t)^{-1} \text{ is} \\ & \text{uniformly bounded with respect to } (t,\omega). \end{cases} \tag{4}$$

Remark 1 Equation (1) is a general BSDE whose diffusion coefficient contains $y(t)$ and $z(t)$, which is different from standard BSDEs whose diffusion terms only contain $z(t)$. When $\sigma \equiv 0$ and $\gamma \equiv 1$, (1) will regress to a standard BSDE.

The following theorem is the existence and uniqueness result, which comes from [2].

Theorem 1 *We suppose* (4) *holds. Then for any* $\xi \in L^2(\mathcal{F}_T; \mathbb{R})$, *there exists a unique pair* $(y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R})$ *which solves Equation* (1).

3 Variational Equation and Variational Inequality

Suppose that $(u(\cdot), y(\cdot), z(\cdot))$ is the solution to our optimal control problem. We introduce the spike variation as follows:

$$u^\varepsilon(t) = \begin{cases} v, & \text{if } t \in [\tau, \tau+\varepsilon], \\ u(t), & \text{otherwise}, \end{cases}$$

where $\varepsilon > 0$ is sufficiently small, $v \in U$ is an $\mathcal{F}_\tau$-measurable random variable such that $\sup_{\omega \in \Omega} |v(\omega)| < +\infty$.

Suppose that $(y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ is the trajectory of (1) corresponding to the control $u^\varepsilon(\cdot)$. We introduce the following first-order and second-order variational equations

$$\begin{cases} dy_1(t) = [b_y y_1(t) + b_z z_1(t) + b(u^\varepsilon(t)) - b(u(t))]dt \\ \qquad\qquad + [\sigma_y y_1(t) + \gamma(t) z_1(t) + \sigma(u^\varepsilon(t)) - \sigma(u(t))]dB(t), \\ y_1(T) = 0, \end{cases} \tag{5}$$

$$\begin{cases} dy_2(t) = \Big[b_y y_2(t) + b_z z_2(t) + \dfrac{1}{2} b_{yy} \cdot (y_1(t))^2 \\ \qquad\qquad + b_{yz} y_1(t) z_1(t) + (b_y(y(t), z(t), u^\varepsilon(t)) - b_y) y_1(t) \\ \qquad\qquad + (b_z(y(t), z(t), u^\varepsilon(t)) - b_z) z_1(t) + \dfrac{1}{2} b_{zz} \cdot (z_1(t))^2\Big] dt \\ \qquad + \Big[\sigma_y y_2(t) + \gamma(t) z_2(t) + \dfrac{1}{2}\sigma_{yy} \cdot (y_1(t))^2 \\ \qquad\qquad + (\sigma_y(y(t), u^\varepsilon(t)) - \sigma_y) y_1(t)\Big] dB(t), \\ y_2(T) = 0. \end{cases} \tag{6}$$

For convenience, we use the notations $g_y = g_y(y(t), z(t), u(t))$, $g(u^\varepsilon(t)) = g(y(t), z(t), u^\varepsilon(t))$, $g(u(t)) = g(y(t), z(t), u(t))$, $g_{yy} = g_{yy}(y(t), z(t), u(t))$, where $g = b, \sigma, h, l$. It is easy to know that (5) and (6) admit unique adapted solutions $(y_1(t), z_1(t))$ and $(y_2(t), z_2(t))$, respectively. We want to give the estimates of the variational state processes $(y_1(t), z_1(t))$ and $(y_2(t), z_2(t))$.

Lemma 1 *Under Assumptions* (3) *and* (4), *we have*

$$\mathbb{E}\left[\sup_{0\le t\le T} (y_1(t))^2\right] \le C\varepsilon, \quad \mathbb{E}\int_0^T (z_1(t))^2 dt \le C\varepsilon, \tag{7}$$

$$\mathbb{E}\left[\sup_{0\le t\le T} (y_1(t))^4\right] \le C\varepsilon^2, \quad \mathbb{E}\left(\int_0^T (z_1(t))^2 dt\right)^2 \le C\varepsilon^2, \tag{8}$$

$$\mathbb{E}\left[\sup_{0\le t\le T} (y_1(t))^8\right] \le C\varepsilon^4, \quad \mathbb{E}\left(\int_0^T (z_1(t))^2 dt\right)^4 \le C\varepsilon^4, \tag{9}$$

$$\mathbb{E}\left[\sup_{0\le t\le T} (y_2(t))^2\right] \le C\varepsilon^2, \quad \mathbb{E}\int_0^T (z_2(t))^2 dt \le C\varepsilon^2, \tag{10}$$

$$\mathbb{E}\left[\sup_{0\le t\le T} (y_2(t))^4\right] \le C\varepsilon^4, \quad \mathbb{E}\left(\int_0^T (z_2(t))^2 dt\right)^2 \le C\varepsilon^4. \tag{11}$$

Proof From the variational equation (5), we get

$$\begin{aligned}
&\mathbb{E}(y_1(t))^2+\beta^2\mathbb{E}\int_t^T(z_1(s))^2ds\\
\le&\mathbb{E}(y_1(t))^2+\mathbb{E}\int_t^T(\gamma(s)z_1(s))^2ds\\
=&\mathbb{E}\Big\{\int_t^T\left[b_yy_1(s)+b_zz_1(s)+b(u^\varepsilon(s))-b(u(s))\right]ds\\
&+\int_t^T\left[\sigma_yy_1(s)+\sigma(u^\varepsilon(s))-\sigma(u(s))\right]dB(s)\Big\}^2\\
\le&C_1\Big[T\mathbb{E}\int_t^T(y_1(s))^2ds+(T-t)\mathbb{E}\int_t^T(z_1(s))^2ds\Big]\\
&+7\mathbb{E}\Big(\int_t^T[b(u^\varepsilon(s))-b(u(s))]ds\Big)^2+5\int_t^T(\sigma(u^\varepsilon(s))-\sigma(u(s)))^2ds,
\end{aligned}$$

where C_1 is a constant. So for $t\in[T-\delta,T]$, $\delta=\frac{\beta^2}{2C_1}$, we have

$$\mathbb{E}(y_1(t))^2+\frac{\beta^2}{2}\mathbb{E}\int_t^T(z_1(s))^2ds\le C_2\mathbb{E}\int_t^T(y_1(s))^2ds+C\varepsilon,$$

where C_2 is a constant depending on C_1 and T, and C is a constant depending on Lipschitz constant. By the Gronwall inequality, we have

$$\mathbb{E}(y_1(t))^2\le C\varepsilon,\quad \mathbb{E}\int_t^T(z_1(s))^2ds\le C\varepsilon,\quad t\in[T-\delta,T].$$

Repeating this procedure, the above estimates hold for $t\in[T-2\delta,T-\delta]$. After a finite number of iterations, we obtain

$$\mathbb{E}(y_1(t))^2\le C\varepsilon,\quad \mathbb{E}\int_t^T(z_1(s))^2ds\le C\varepsilon,\quad t\in[0,T]. \tag{12}$$

On the other hand, by (5), we get

$$\begin{aligned}
\sup_{0\le t\le T}(y_1(t))^2\le&C_3\Big\{\int_0^T(y_1(t))^2dt+\int_0^T(z_1(t))^2dt+\int_0^T\left[b(u^\varepsilon(t))-b(u(t))\right]^2dt\Big\}\\
&+2\sup_{0\le t\le T}\Big\{\int_t^T\left[\sigma_yy_1(s)+\gamma(s)z_1(s)+\sigma(u^\varepsilon(s))-\sigma(u(s))\right]dB(s)\Big\}^2.
\end{aligned}$$

Taking expectation on both sides and by the Davis-Burkholder-Gundy inequality, we have

$$\begin{aligned}
\mathbb{E}[\sup_{0\le t\le T}(y_1(t))^2]\le&C_3\int_0^T\mathbb{E}\Big[\sup_{0\le s\le t}(y_1(s))^2\Big]dt+C_3\mathbb{E}\int_0^T(z_1(t))^2dt\\
&+C_4\mathbb{E}\int_0^T[(b(u^\varepsilon(t))-b(u(t)))^2+(\sigma(u^\varepsilon(t))-\sigma(u(t)))^2]dt,
\end{aligned}$$

where C_3, C_4 are constants. By the Gronwall inequality and (12), we obtain (7). And, by (5), we have

$$|y_1(s)| \le \mathbb{E}\Big[\int_t^T |b_y(y_1(s) + b_z z_1(s) + b(u^\varepsilon(s)) - b(u(s))|ds|\mathcal{F}_t\Big], \quad s \in [t, T].$$

The Doob martingale inequality gives

$$\mathbb{E}\Big[\sup_{t\le s\le T}(y_1(s))^4\Big] \le \overline{C}_1\mathbb{E}\Big[\int_t^T |b_y y_1(s) + b_z z_1(s) + b(u^\varepsilon(s)) - b(u(s))|ds\Big]^4,$$

where $\overline{C}_1$ is a constant. Then by the Davis-Burkholder-Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}(y_1(s))^4 + \beta^4\mathbb{E}\Big[\int_t^T (z_1(s))^2 ds\Big]^2\\
\le&\mathbb{E}\Big[\sup_{t\le s\le T}(y_1(s))^4\Big] + \overline{C}_2\mathbb{E}\Big[\sup_{t\le s\le T}\Big|\int_t^s \gamma(\tau)z_1(\tau)dB(\tau)\Big|\Big]^4\\
\le&\overline{C}_1\mathbb{E}\Big[\int_t^T |b_y y_1(s) + b_z z_1(s) + b(u^\varepsilon(s)) - b(u(s))|ds\Big]^4\\
&+\overline{C}_2\mathbb{E}\Big[\int_t^T |\sigma_y y_1(s) + \sigma(u^\varepsilon(s)) - \sigma(u(s))|^2 ds\Big]^2\\
\le&\overline{C}_3\mathbb{E}\int_t^T (y_1(s))^4 ds + \overline{C}_4(T-t)\mathbb{E}\Big[\int_t^T (z_1(s))^2 ds\Big]^2\\
&+\overline{C}_5\mathbb{E}\Big\{\Big[\int_t^T (b(u^\varepsilon(s)) - b(u(s)))^2 ds\Big]^2 + \Big[\int_t^T (\sigma(u^\varepsilon(s)) - \sigma(u(s)))^2 ds\Big]^2\Big\},
\end{aligned}$$

where $\overline{C}_3, \overline{C}_4, \overline{C}_5$ are constants. Using the same method as the proof of (7), we obtain (8). Similarly to (7) and (8), we can obtain (9). Note that

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T y_1(t)z_1(t)dt\Big]^2 &\le \mathbb{E}\Big[\int_0^T (y_1(t))^2 dt\int_0^T (z_1(t))^2 dt\Big]\\
&\le \Big(\mathbb{E}\Big[\int_0^T (y_1(t))^2 dt\Big]^2\Big)^{\frac12}\Big(\mathbb{E}\Big[\int_0^T (z_1(t))^2 dt\Big]^2\Big)^{\frac12}\\
&\le C\varepsilon^2,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\int_0^T [\sigma_y(y(t), u^\varepsilon(t)) - \sigma_y)]^2 (y_1(t))^2 dt\\
\le&\mathbb{E}\Big[\sup_{0\le t\le T}(y_1(t))^2\int_0^T [\sigma_y(y(t), u^\varepsilon(t)) - \sigma_y)]^2 dt\Big]\\
\le&\Big[\mathbb{E}\sup_{0\le t\le T}(y_1(t))^4\Big]^{\frac12}\Big(\mathbb{E}\Big(\int_0^T [\sigma_y(u^\varepsilon(t)) - \sigma_y]^2 dt\Big)^2\Big)^{\frac12} \le C\varepsilon^2,\\
&\mathbb{E}\Big[\int_0^T y_1(t)z_1(t)dt\Big]^4 \le C\varepsilon^4.
\end{aligned}$$

Using the same technique to deal with (6) as the proof of (5), we can get (10) and (11). We omit the details. ∎

The following lemma plays an important role in deriving the variational inequality. It gives the ε-order estimations of the differences between the perturbed state process $(y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ and the sum of the optimal state process and the variational processes $(y(\cdot)+y_1(\cdot)+y_2(\cdot), z(\cdot)+z_1(\cdot)+z_2(\cdot))$.

Lemma 2 *Under Assumptions* (3) *and* (4), *we have*

$$\sup_{0\le t\le T}\mathbb{E}|y^\varepsilon(t)-y(t)-y_1(t)-y_2(t)|^2=o(\varepsilon^2), \tag{13}$$

$$\mathbb{E}\int_0^T|z^\varepsilon(t)-z(t)-z_1(t)-z_2(t)|^2dt=o(\varepsilon^2). \tag{14}$$

Proof Set $\widetilde{y}=y_1+y_2, \widetilde{z}=z_1+z_2$. It follows from (1), (5) and (6) that (for simplification we omit the time subscript s)

$$\begin{aligned}&\int_t^T b(y+\widetilde{y},z+\widetilde{z},u^\varepsilon)ds+\int_t^T[\sigma(y+\widetilde{y},u^\varepsilon)+\gamma\cdot(z+\widetilde{z})]dB(s)\\ =&-y(t)-\widetilde{y}+\xi+\int_t^T C^\varepsilon(s)ds+\int_t^T D^\varepsilon(s)dB(s),\end{aligned}$$

where

$$\begin{aligned}C^\varepsilon(s)=&\frac{1}{2}b_{yy}(y,z,u)(y_2^2+2y_1y_2)\\&+\frac{1}{2}b_{zz}(y,z,u)(z_2^2+2z_1z_2)+2b_{yz}(y,z,u)(y_2z_2+y_1z_2+y_2z_1)\\&+(b_y(y,z,u^\varepsilon)-b_y(y,z,u))y_2+(b_z(y,z,u^\varepsilon)-b_z(y,z,u))z_2\\&+\int_0^1\int_0^1\lambda[b_{yy}(y+\lambda\mu\widetilde{y},z+\lambda\mu\widetilde{z},u^\varepsilon)-b_{yy}(y+\lambda\mu\widetilde{y},z+\lambda\mu\widetilde{z},u)]d\lambda d\mu\widetilde{y}^2\\&+2\int_0^1\int_0^1\lambda[b_{yz}(y+\lambda\mu\widetilde{y},z+\lambda\mu\widetilde{z},u^\varepsilon)-b_{yz}(y+\lambda\mu\widetilde{y},z+\lambda\mu\widetilde{z},u)]d\lambda d\mu\widetilde{y}\widetilde{z}\\&+\int_0^1\int_0^1\lambda[b_{zz}(y+\lambda\mu\widetilde{y},z+\lambda\mu\widetilde{z},u^\varepsilon)-b_{zz}(y+\lambda\mu\widetilde{y},z+\lambda\mu\widetilde{z},u)]d\lambda d\mu\widetilde{z}^2,\\ D^\varepsilon(s)=&\frac{1}{2}\sigma_{yy}(y,u)(y_2^2+2y_1y_2)+(\sigma_y(y,u^\varepsilon)-\sigma_y(y,u))y_2\\&+\int_0^1\int_0^1\lambda[\sigma_{yy}(y+\lambda\mu\widetilde{y},u^\varepsilon)-\sigma_{yy}(y+\lambda\mu\widetilde{y},u)]d\lambda d\mu\widetilde{y}^2.\end{aligned}$$

Then we have

$$\begin{aligned}y(t)+\widetilde{y}=&\xi-\int_t^T b(y+\widetilde{y},z+\widetilde{z},u^\varepsilon)ds-\int_t^T[\sigma(y+\widetilde{y},u^\varepsilon)+\gamma\cdot(z+\widetilde{z})]dB(s)\\&+\int_t^T C^\varepsilon(s)ds+\int_t^T D^\varepsilon(s)dB(s).\end{aligned}$$

Since

$$y^\varepsilon(t)=\xi-\int_t^T b(y^\varepsilon(s)),z^\varepsilon(s)),u^\varepsilon(s))ds-\int_t^T[\sigma(y^\varepsilon(s),u^\varepsilon(s))+\gamma\cdot z^\varepsilon(s)]dB(s),$$

it follows that

$$\begin{aligned}(y^\varepsilon - y - \widetilde{y})(t) =& \int_t^T [A_1^\varepsilon(s)(y^\varepsilon - y - \widetilde{y})(s) + A_2^\varepsilon(s)(z^\varepsilon - z - \widetilde{z})(s)]ds \\ &+ \int_t^T [B_1^\varepsilon(s)(y^\varepsilon - y - \widetilde{y})(s) + B_2^\varepsilon(s)(z^\varepsilon - z - \widetilde{z})(s)]dB(s) \\ &- \int_t^T C^\varepsilon(s)ds - \int_t^T D^\varepsilon(s)dB(s),\end{aligned}$$

where

$$|A_1^\varepsilon(s)| + |A_2^\varepsilon(s)| + |B_1^\varepsilon(s)| + |B_2^\varepsilon(s)| \le C, \quad \forall (s,\omega) \in [0,T] \times \Omega.$$

From Lemma 1, we can easily find

$$\sup_{0\le t\le T} \mathbb{E}\bigg\{\bigg[\int_t^T C^\varepsilon(s)ds\bigg]^2 + \bigg[\int_t^T D^\varepsilon(s)dB(s)\bigg]^2\bigg\} = o(\varepsilon^2).$$

Using the method once more as the proof of Lemma 1, we can obtain (13) and (14). The proof is completed. ∎

Now we can present the following variational inequality.

Lemma 3 (*variational inequality*) *Under Assumptions* (3) *and* (4), *we have*

$$\begin{aligned}&\mathbb{E}\int_0^T \bigg[l_y(y(s),u(s))(y_1(s)+y_2(s)) + \frac{1}{2}l_{yy}(y(s),u(s))(y_1(s))^2\bigg]ds \\ &+\mathbb{E}\int_0^T [l(y(s),u^\varepsilon(s)) - l(y(s),u(s))]ds \\ &+\mathbb{E}\bigg[h_y(y(0))(y_1(0)+y_2(0)) + \frac{1}{2}h_{yy}(y(0))(y_1(0))^2\bigg] \ge o(\varepsilon). \qquad (15)\end{aligned}$$

Proof Since $(y(\cdot), z(\cdot), u(\cdot))$ is optimal, we have

$$\mathbb{E}\int_0^T l(y(s),u^\varepsilon(s))ds + \mathbb{E}h_y(y^\varepsilon(0)) - \mathbb{E}\int_0^T l(y(s),u(s))ds - \mathbb{E}h(y(0)) \ge 0.$$

It follows from Lemma 2 that

$$\begin{aligned}&\mathbb{E}\int_0^T l(y(s),u^\varepsilon(s))ds + \mathbb{E}h_y(y^\varepsilon(0)) - \mathbb{E}\int_0^T l(y(s),u(s))ds - \mathbb{E}h(y(0)) \\ =&\mathbb{E}\int_0^T [l(y+y_1+y_2,u^\varepsilon) - l(y,u)]ds + \mathbb{E}[h(y+y_1+y_2)(0) - h(y(0))] + o(\varepsilon) \\ =&\mathbb{E}\int_0^T [l(y+y_1+y_2,u) - l(y,u)]ds + \mathbb{E}\int_0^T [l(y+y_1+y_2,u^\varepsilon) \\ &- l(y+y_1+y_2,u)]ds + \mathbb{E}[h(y+y_1+y_2)(0) - h(y(0))] + o(\varepsilon) \\ =&\mathbb{E}\int_0^T \bigg[l_y(y,u)(y_1+y_2) + \frac{1}{2}l_{yy}(y,u)(y_1+y_2)^2\bigg]ds\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^T[l(y,u^\varepsilon)-l(y,u)]ds+\mathbb{E}\int_0^T[l_y(y,u^\varepsilon)-l_y(y,u)](y_1+y_2)ds\\
&+\frac{1}{2}\mathbb{E}\int_0^T[l_{yy}(y,u^\varepsilon)-l_{yy}(y,u)](y_1)^2]ds+\mathbb{E}[h_y(y(0))(y_1(0)+y_2(0))]\\
&+\frac{1}{2}\mathbb{E}[h_{yy}(y(0))(y_1(0))^2]+o(\varepsilon)\geq 0.
\end{aligned}
$$

Then by Lemma 1, the desired variational inequality (15) can be obtained. ∎

4 Adjoint Equation and Maximum Principle

In this section, we introduce the first-order and second-order adjoint equations, then we use the duality method to obtain the necessary condition of optimality. Let us consider the following forward SDEs (for simplification we omit the time subscript t in some places)

$$
\begin{cases}
dp(t)=\Big(\Big(-b_y+\sigma_y\dfrac{b_z}{\gamma}\Big)p(t)-ly\Big)dt-\dfrac{b_z}{\gamma}p(t)dB(t),\\
p(0)=-h_y(y(0)),
\end{cases}
\tag{16}
$$

and

$$
\begin{cases}
dP(t)=\Big[\Big(-\sigma_y^2-2b_y+2\sigma_y\Big(1+\dfrac{b_z}{\sigma_y\gamma}\Big)+\Big(\dfrac{b_z}{\gamma}-\sigma_y\Big)\Big)P(t)\\
\qquad\qquad+\Big(-2\dfrac{b_{yz}}{\gamma}+b_{yy}-\dfrac{b_z}{\gamma}\sigma_{yy}\Big)p(t)+l_{yy}\Big]dt\\
\qquad\qquad+\Big[\Big(-\Big(1+\dfrac{b_z}{\sigma_y\gamma}\Big)-\dfrac{1}{2\sigma_y}\Big(\dfrac{b_z}{\gamma}-\sigma_y\Big)\Big)P(t)+\dfrac{b_{yz}}{\sigma_y\gamma}p(t)\Big]dB(t),\\
P(0)=h_{yy}(y(0)).
\end{cases}
\tag{17}
$$

Because the solution of BSDE (5) contains process $z_1(\cdot)$, which will bring us more difficulties to deal with, especially when we use the duality technique to derive the maximum principle for backward stochastic systems, it is essentially necessary to look for more explicit estimates of $z_1(\cdot)$. We have the following lemma.

Lemma 4 *Let $p(\cdot)$ and $P(\cdot)$ are solutions of* (16) *and* (17), *respectively. Then we have*

$$
\mathbb{E}\int_0^T\Big(\frac{b_z}{\gamma}-\sigma_y\Big)P\gamma y_1z_1dt=\mathbb{E}\int_0^T[P\gamma^2z_1^2+P\gamma z_1(\sigma(u^\varepsilon)-\sigma(u))]dt+o(\varepsilon),
\tag{18}
$$

$$
\mathbb{E}\int_0^T P\gamma z_1(\sigma(u^\varepsilon)-\sigma(u))dt=-\mathbb{E}\int_0^T P(\sigma(u^\varepsilon)-\sigma(u))^2dt+o(\varepsilon),
\tag{19}
$$

$$
\mathbb{E}\int_0^T p(b_z(u^\varepsilon)-b_z(u))z_1dt=-\mathbb{E}\int_0^T p\gamma^{-1}(b(u^\varepsilon)-b(u))(\sigma(u^\varepsilon)-\sigma(u))dt+o(\varepsilon).
\tag{20}
$$

Proof In what follows, we take several steps to prove the lemma (for simplification we omit the time subscript t in some places). Firstly, we introduce a process $\varphi(t)$ such that

$$
\varphi(t)=\int_0^t P_0^{-1}P\gamma z_1dB(s),
$$

where $P_0(\cdot)$ satisfies

$$\begin{cases} dP_0(t) = -\Big(b_y - \dfrac{b_z}{\gamma}\sigma_y\Big)P_0(t)dt - \dfrac{b_z}{\gamma}P_0(t)dB(t), \\ P_0(0) = 1, \end{cases}$$

and it is obvious that $P_0^{-1}(\cdot)$ exists and $P_0(\cdot), P_0^{-1}(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$. Note that $\varphi(0) = 0$ and $y_1(T) = 0$. Using Itô formula to $\varphi(t)y_1(t)$, we have

$$\begin{aligned} d\varphi y_1 =&[\varphi b_y y_1 + \varphi b_z z_1 + \varphi(b(u^\varepsilon) - b(u)) \\ &+ \sigma_y P_0^{-1}P\gamma y_1 z_1 + P_0^{-1}P\gamma^2 z_1^2 + P_0^{-1}P\gamma z_1(\sigma(u^\varepsilon) - \sigma(u))]dt \\ &+ [\varphi\sigma_y y_1 + \varphi\gamma z_1 + \varphi(\sigma(u^\varepsilon) - \sigma(u)) + P_0^{-1}P\gamma y_1 z_1]dB(t). \end{aligned} \tag{21}$$

Set $Y(t) = \varphi(t)y_1(t)$ and $Z(t) = \varphi(t)z_1(t)$. Then (21) can be rewritten as

$$\begin{aligned} dY =&[b_y Y + b_z Z + \varphi(b(u^\varepsilon) - b(u)) + \sigma_y P_0^{-1}P\gamma y_1 z_1 \\ &+ \gamma P_0^{-1}P\gamma z_1^2 + P_0^{-1}P\gamma z_1(\sigma(u^\varepsilon) - \sigma(u))]dt \\ &+ [\sigma_y Y + \gamma Z + \varphi(\sigma(u^\varepsilon) - \sigma(u)) + P_0^{-1}P\gamma y_1 z_1]dB(t). \end{aligned}$$

Using Itô formula to $P_0(t)Y(t)$ and taking expectation, we get

$$\begin{aligned} &\mathbb{E}\int_0^T [P_0 b_y Y + P_0 b_z Z + P_0\varphi(b(u^\varepsilon) - b(u))]dt \\ &+ \mathbb{E}\int_0^T [P\sigma_y\gamma y_1 z_1 + P\gamma^2 z_1^2 + P\gamma z_1(\sigma(u^\varepsilon) - \sigma(u))]dt \\ &+ \mathbb{E}\int_0^T -\Big(b_y - \frac{b_z}{\gamma}\sigma_y\Big)P_0 Y dt \\ &+ \mathbb{E}\int_0^T -\Big[\frac{b_z}{\gamma}P_0\sigma_y Y + \frac{b_z}{\gamma}P_0\gamma Z + \frac{b_z}{\gamma}P_0\varphi(\sigma(u^\varepsilon) - \sigma(u))\Big]dt \\ &+ \mathbb{E}\int_0^T -\frac{b_z}{\gamma}P\gamma y_1 z_1 dt = 0. \end{aligned} \tag{22}$$

Note that

$$\begin{aligned} &\mathbb{E}\int_0^T \frac{b_z}{\gamma}P_0\varphi(\sigma(u^\varepsilon) - \sigma(u))dt \\ =&\mathbb{E}\int_0^T \frac{b_z}{\gamma}P_0\Big(\int_0^t P_0^{-1}P\gamma z_1 dB(s)\Big)(\sigma(u^\varepsilon) - \sigma(u))dt \\ \le&\mathbb{E}\Big[\sup_{0\le t\le T}\int_0^t P_0^{-1}P\gamma z_1 dB(s)\int_0^T \frac{b_z}{\gamma}P_0(\sigma(u^\varepsilon) - \sigma(u))dt\Big] \\ =&o(\varepsilon), \\ &\mathbb{E}\int_0^T P_0\varphi(b(u^\varepsilon) - b(u))dt = o(\varepsilon). \end{aligned}$$

Then by (22), we can obtain (18). Secondly, we introduce the following process

$$\phi(t) = \int_0^t P_0^{-1} P(\sigma(u^\varepsilon) - \sigma(u)) dB(s).$$

Using Itô formula to $\phi(t)y_1(t)$, we have

$$\begin{aligned} d\phi y_1 =& [\phi b_y y_1 + \phi b_z z_1 + \phi(b(u^\varepsilon) - b(u)) \\ &+ P_0^{-1} P \sigma_y y_1 (\sigma(u^\varepsilon) - \sigma(u)) + P_0^{-1} P \gamma z_1 (\sigma(u^\varepsilon) - \sigma(u)) \\ &+ P_0^{-1} P (\sigma(u^\varepsilon) - \sigma(u))^2] dt + [\phi \sigma_y y_1 + \phi \gamma z_1 + \phi(\sigma(u^\varepsilon) - \sigma(u)) \\ &+ P_0^{-1} P y_1 (\sigma(u^\varepsilon) - \sigma(u))] dB(t). \end{aligned} \tag{23}$$

Set $\widetilde{Y}(t) = \phi(t)y_1(t)$ and $\widetilde{Z}(t) = \phi(t)z_1(t)$. Then (23) can be rewritten as

$$\begin{aligned} d\widetilde{Y} =& [b_y \widetilde{Y} + b_z \widetilde{Z} + \phi(b(u^\varepsilon) - b(u)) + \sigma_y P_0^{-1} P y_1 (\sigma(u^\varepsilon) - \sigma(u)) \\ &+ P_0^{-1} P \gamma z_1 (\sigma(u^\varepsilon) - \sigma(u)) + P_0^{-1} P (\sigma(u^\varepsilon) - \sigma(u))^2] dt \\ &+ [\sigma_y \widetilde{Y} + \gamma \widetilde{Z} + \phi(\sigma(u^\varepsilon) - \sigma(u)) + P_0^{-1} P y_1 (\sigma(u^\varepsilon) - \sigma(u))] dB(t). \end{aligned}$$

Using Itô formula to $P_0(t)\widetilde{Y}(t)$ and taking expectation, we have

$$\begin{aligned} &\mathbb{E}\int_0^T [P_0 b_y \widetilde{Y} + P_0 b_z \widetilde{Z} + P_0 \phi(b(u^\varepsilon) - b(u))] dt \\ &+ \mathbb{E}\int_0^T [P \sigma_y y_1 (\sigma(u^\varepsilon) - \sigma(u)) + P \gamma z_1 (\sigma(u^\varepsilon) - \sigma(u)) \\ &+ P(\sigma(u^\varepsilon) - \sigma(u))^2] dt + \mathbb{E}\int_0^T -\left(b_y - \frac{b_z}{\gamma}\sigma_y\right) P_0 \widetilde{Y} dt \\ &+ \mathbb{E}\int_0^T -\left[\frac{b_z}{\gamma} P_0 \sigma_y \widetilde{Y} + \frac{b_z}{\gamma} P_0 \gamma \widetilde{Z} + \frac{b_z}{\gamma} P_0 \phi(\sigma(u^\varepsilon) - \sigma(u))\right] dt \\ &+ \mathbb{E}\int_0^T -\frac{b_z}{\gamma} P_0^{-1} P y_1 (\sigma(u^\varepsilon) - \sigma(u)) dt = 0. \end{aligned} \tag{24}$$

Note that

$$\mathbb{E}\int_0^T P_0 \phi(b(u^\varepsilon) - b(u)) dt = o(\varepsilon), \quad \mathbb{E}\int_0^T \frac{b_z}{\gamma} P_0 \phi(\sigma(u^\varepsilon) - \sigma(u)) dt = o(\varepsilon),$$
$$\mathbb{E}\int_0^T -\frac{b_z}{\gamma} P_0^{-1} P y_1 (\sigma(u^\varepsilon) - \sigma(u)) dt = o(\varepsilon), \quad \mathbb{E}\int_0^T P \sigma_y y_1 (\sigma(u^\varepsilon) - \sigma(u)) dt = o(\varepsilon).$$

From (24), we obtain (19). Finally, we introduce the process

$$\psi(t) = \int_0^t P_0^{-1} p \gamma^{-1} (b(u^\varepsilon) - b(u)) dB(s).$$

Then using the same method as the proof of (18) and (19), we get

$$\mathbb{E}\int_0^T p(b_z(u^\varepsilon) - b_z(u)) z_1 dt = -\mathbb{E}\int_0^T p\gamma^{-1} (b(u^\varepsilon) - b(u))(\sigma(u^\varepsilon) - \sigma(u)) dt + o(\varepsilon),$$

which is (20). We omit the details. ∎

Now we define the Hamilton function as

$$H(y,z,v,p)=l(y,v)+p\bigg(b(y,z,v)-\frac{b_z(y,z,v)}{\gamma}\sigma(y,v)\bigg)$$

and state the main result of this paper.

Theorem 2 (*stochastic maximum principle*) *Let* (3) *and* (4) *hold. Suppose that* $(y(\cdot),z(\cdot),u(\cdot))$ *is a solution of the optimal control problem, then we have*

$$\begin{aligned}&H(y(\tau),z(\tau),v,p(\tau))-H(y(\tau),z(\tau),u(\tau),p(\tau))\\&-\frac{1}{2}p(\tau)\gamma(\tau)^{-1}[b(y(\tau),z(\tau),v)-b(y(\tau),z(\tau),u(\tau))]\\&\times[\sigma(y(\tau),v)-\sigma(y(\tau),u(\tau))]\\&+\frac{1}{2}P(\tau)(\sigma(y(\tau),v)-\sigma(y(\tau),u(\tau)))^2\geq 0,\quad\forall v\in U,\quad\text{a.e., a.s.,}\end{aligned}$$

where $p(\cdot)$ *and* $P(\cdot)$ *are solutions of* (16) *and* (17), *respectively.*

Proof Applying Itô formula to $p(t)y_1(t)$ and $p(t)y_2(t)$, taking expectation and by Lemma 1, we have

$$\begin{aligned}&\mathbb{E}h_y(y(0))y_1(0)+\mathbb{E}\int_0^T l_y y_1 dt\\=&\mathbb{E}\int_0^T p\bigg[(b(u^\varepsilon)-b(u))-\frac{b_z}{\gamma}(\sigma(u^\varepsilon)-\sigma(u))\bigg]dt,\end{aligned}\tag{25}$$

$$\begin{aligned}&\mathbb{E}h_y(y(0))y_2(0)+\mathbb{E}\int_0^T l_y y_2 dt\\=&\mathbb{E}\int_0^T p\bigg[\frac{1}{2}b_{yy}(y_1(t))^2+b_{yz}y_1(t)z_1(t)\\&\qquad+(b_z(u^\varepsilon)-b_z(u))z_1(t)\bigg]dt-\mathbb{E}\int_0^T\frac{b_z}{2\gamma}p\sigma_{yy}(y_1(t))^2dt+o(\varepsilon).\end{aligned}\tag{26}$$

Note that

$$\begin{aligned}dy_1^2=&[(\sigma_y^2+2b_y)y_1^2+\gamma^2z_1^2+(2\sigma_y\gamma+2b_z)y_1z_1\\&+2\sigma_y(b(u^\varepsilon)-b(u))y_1+2\gamma(\sigma(u^\varepsilon)-\sigma(u))z_1+(\sigma(u^\varepsilon)-\sigma(u))^2]dt\\&+[2\sigma_y y_1^2+2\sigma_y\gamma y_1z_1+2(\sigma(u^\varepsilon)-\sigma(u))y_1]dB(t).\end{aligned}$$

Applying Itô formula to $P(t)(y_1(t))^2$ and taking expectation, we get

$$\begin{aligned}&\mathbb{E}h_{yy}(y(0))y_1^2+\mathbb{E}\int_0^T[P\gamma^2z_1^2+2P\gamma(\sigma(u^\varepsilon)-\sigma(u))z_1]dt\\&-\mathbb{E}\int_0^T\bigg(-\sigma_y+\frac{b_z}{\gamma}\bigg)P\gamma y_1z_1dt+\mathbb{E}\int_0^T 2pb_{yz}y_1z_1dt\\&+\mathbb{E}\int_0^T\bigg(pb_{yy}-p\frac{b_z}{\gamma}\sigma_{yy}+l_{yy}\bigg)y_1^2dt+\mathbb{E}\int_0^T P(\sigma(u^\varepsilon)-\sigma(u))^2dt=0.\end{aligned}\tag{27}$$

By (25), (26), (27) and Lemma 4, the variational inequality (15) can be rewritten as

$$\mathbb{E}\int_0^T p\bigg[(b(u^\varepsilon)-b(u))-\frac{b_z}{\gamma}(\sigma(u^\varepsilon)-\sigma(u))\bigg]dt$$
$$-\frac{1}{2}\mathbb{E}\int_0^T p\gamma^{-1}(b(u^\varepsilon)-b(u))(\sigma(u^\varepsilon)-\sigma(u))dt$$
$$+\frac{1}{2}\mathbb{E}\int_0^T P(\sigma(u^\varepsilon)-\sigma(u))^2dt+\mathbb{E}\int_0^T(l(u^\varepsilon)-l(u))dt\geq o(\varepsilon).$$

This gives the maximum principle immediately. ∎

We discuss a special case: $\sigma\equiv 0$, $\gamma\equiv 1$. Now (1) is reduced to the following standard BSDE

$$\begin{cases} dy(t)=b\left(y(t),z(t),v(t)\right)dt+z(t)dB(t), \\ y(T)=\xi, \end{cases} \tag{28}$$

and the cost functional is given by (2). In this special case, In this special case, the second-order adjoint equation (17) is not necessary. By virtue of first-order adjoint equation (16), we can easily obtain the following result.

Proposition 1 *Suppose $u(\cdot)$ is an optimal control subject to* (28) *and* (2)*, then we have*

$$H(y(\tau),z(\tau),v,p(\tau))-H(y(\tau),z(\tau),u(\tau),p(\tau))\geq 0,\quad \forall v\in U,\quad a.e.,\ a.s.,$$

where

$$H(y,z,v,p)=l(y,v)+pb(y,z,v)$$

and $p(\cdot)$ satisfies

$$\begin{cases} dp(t)=(-b_yp(t)-ly)dt-b_zp(t)dB(t), \\ p(0)=-h_y(y(0)). \end{cases}$$

5 Conclusions

In this paper, we have discussed the stochastic control problem for a kind of general BSDE. A necessary condition for optimality called maximum principle is obtained based on the variation and duality method. Compared with the forward maximum principle, the backward maximum principle contains a covariant term $-1/2p(\tau)\gamma(\tau)^{-1}[b(y(\tau),z(\tau),v)-b(y(\tau),z(\tau),u(\tau))]\cdot[\sigma(y(\tau),v)-\sigma(y(\tau),u(\tau))]$ that cannot be found in the forward maximum principle. Moreover, if we replace the diffusion coefficient $\sigma\left(y(t),v(t)\right)+\gamma(t)z(t)$ of (1) with a more general term $\sigma(y(t),z(t),v(t))$, we will meet some trouble because the second-order adjoint equation (6) in this case will contain $1/2\sigma_{zz}z_1^2dB(t)$, and we are not sure whether $1/2\sigma_{zz}z_1^2dB(t)$ is well-defined because we only know $z_1(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ other than $L^4_{\mathcal{F}}(0,T;\mathbb{R})$. This is the main reason that we only consider the case that diffusion coefficient is linear with respect to $z(t)$. It is remarkable that Lemma 4 plays an important role on our maximum principle, and we can extend this result to multidimensional systems whenever we impose some heavy assumptions on the derivatives of the coefficients. We hope this paper will serve as a stimulus to the study of backward stochastic control systems.

References

[1] Bismut J M, An introductory approach to duality in optimal stochastic control, *SIAM Rev.*, 1978, **20**: 62–78.

[2] Pardoux E and Peng S, Adapted solution of backward stochastic differential equation, *Systems Control Lett.*, 1990, **14**: 55–61.

[3] Cvitanic J and Ma J, Hedging options for a large investor and froward-backward SDEs, *Ann. Appl. Probab.*, 1996, **6**: 370–398.

[4] EI Karoui N, Peng S, and Quenez M C, Backward stochastic differential equations in finance, *Math. Finance*, 1997, **7**(1): 1–71.

[5] Ma J and Yong J, Forward-backward stochastic differential equations and their applications, *Lecture Notes in Math.*, Vol. 1702, New York, Springer-Verlag, 1999.

[6] Schroder M and Skiadas C, Lifetime consumption-portfolio choice under trading constraints, recursive preferences, and nontradable income, *Stochastic Process. Appl.*, 2005, **115**: 1–30.

[7] Yong J and Zhou X, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, New York, Springer-Verlag, 1999.

[8] Peng S, Backward stochastic differential equations and applications to optimal control, *Appl. Math. Optim.*, 1993, **27**: 125–144.

[9] Xu W, Stochastic maximum principle for optimal control problem of forward and backward system, *J. Aust. Math. Soc. Ser. B,* 1995, **37**: 172–185.

[10] Wu Z, Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems, *J. Systems Sci. Math. Sci.*, 1998, **11**(3): 249–259 (in Chinese).

[11] Lim A E B and Zhou X, Linear-quadratic control of backward stochastic differential equations, *SIAM J. Control Optim.*, 2001, **40**: 450–474.

[12] Huang J, Wang G, and Xiong J, A maximum principle for partial information backward stochastic control problem with applications, *SIAM J. Control Optim.*, 2009, **48**(4): 2106–2117.

[13] Wang G and Yu Z, A Pontryagin's maximum principle for non-zero sum differential game of backward stochastic differential equations with applications, *IEEE Trans. Automat. Control*, 2010, **55**: 1742–1747.

[14] Peng S, A general stochastic maximum principle for optimal control problems, *SIAM J. Control Optim.*, 1990, **28**(4): 966–979.