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Decay Rates of the Hyperbolic Equation in an Exterior Domain with Half-Linear and Nonlinear Boundary Dissipations*

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Abstract This paper considers the energy decay of the wave equation with variable coefficients in an exterior domain. The damping is put on partly the boundary and partly on the interior of the domain. The energy decay results are established by Riemannian geometry method.

Keywords Energy decay, exterior domain, hyperbolic equation, Riemannian geometry method, variable coefficients.

1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^n (n \geq 3)$ such that $\mathbb{R}^n \backslash \Omega$ is compact, the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is smooth, where $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. We consider the initial-boundary value problem of the hyperbolic equation of the form:

$$\begin{cases} u_{tt} - \operatorname{div} A(x)\nabla u + \rho(x, u_t) = 0, & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ u_{\nu_A} = -f(u_t), & \text{on } \Gamma_1 \times (0, +\infty), \\ u(x, 0) = u_0(x), \ \ u_t(x, 0) = u_1(x), & \text{on } \Omega, \end{cases} \tag{1}$$

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where u_{tt} stands for $\partial^2 u/\partial t^2$, $A(x) = (a_{ij}(x))$ is a symmetric, positive, smooth $n \times n$ matrices, $\operatorname{div}(X)$ denotes the divergence of the vector field X in the Euclidean metric, $u_{\nu_A} = \langle A(x)\nabla u, \nu\rangle$, and $\nu(x)$ is the outside unit normal vector field at $x \in \Gamma$. $f(\upsilon)$ is a nonlinear term like $|\upsilon|^r \upsilon$, and $\rho(x, \upsilon)$ is a term like

$$\rho(x,\upsilon) = \begin{cases} a(x)|\upsilon|^p \upsilon, & \text{if} \quad |x| \le \dfrac{R}{2}, \\ a(x)\upsilon, & \text{if} \quad |x| \ge R, \end{cases}$$

where R is a large positive number.

The aim of this paper is to estimate the decay for the energy of the system (1) and some relations between the decay rate and the behavior of the nonlinearity ρ and f. The boundary (linear and nonlinear) stabilization problem of the wave equation has been studied by many authors, for example, see many results concerning the stability of such problems are available in the literatures, see, for instance [1–9] and references therein. They dealt with the constant coefficients case where $A(x)$ is the unit matrix.

In the case of variable coefficients where $A(x)$ depends on the space variable x, the classical analysis which was originally successful in dealing with the wave equation with constant coefficients is not enough. Here we use the Riemannian geometry method, which was introduced by [10] to deal with controllability of the wave equation with variable coefficients and has bee extended by [11–21], and many others. This method is a necessary tool for checking controllability/stabilization of the variable coefficient systems ([18, 22]). This is because without the sectional curvature information controllability/stabilization just holds true locally. Moreover, a computational technique (the Bochner technique) in Riemannian geometry provides great simplification in order to establish Carleman estimates.

The paper [6] studied the system (1) in the case of constant coefficients. The work [16] investigated the system (1) in the case of variable coefficients but the domain of the system was assumed to be the whole space $\mathbb{R}^n$ without boundary. In this paper, we study the system (1) in the case of variable coefficients where its the domain is assumed to an exterior region with two boundaries Γ_0 and Γ_1, by using the piecewise multiplier method, Riemannian geometry method, and an integral inequality introduced in [2].

Our paper is organized as follows. In Section 2, we state the main results. In Section 3, we give the estimates of L^2-norms. The proof of Theorem 2.1 will be presented in Section 4.

2 Statement of Results

For convenience, first we introduce some notation in Riemannian geometry (see [18, 23]). Let $A(x) = (a_{ij}(x))$ be a symmetric, positive, smooth $n \times n$ matrices for $x \in \mathbb{R}^n$. On $\mathbb{R}^n$, we introduce

$$g(x) = (g_{ij}(x)) = A^{-1}(x) \quad \text{for} \quad x \in \mathbb{R}^n,$$

as a Riemannian metric and regard $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, we define the inner product and the norm on the tangent space

$$g(X,Y) = \langle X,Y\rangle_g = \langle A^{-1}(x)X,Y\rangle, \quad |X|_g^2 = \langle X,X\rangle_g, \quad X,\ Y \in \mathbb{R}^n,$$

where $g = \langle\cdot,\cdot\rangle_g$ is the inner product of the metric g and $\langle\cdot,\cdot\rangle = \cdot$ is the Euclidean product of $\mathbb{R}^n$, respectively.

For any $u \in H^1(\mathbb{R}^n)$, we have

$$\nabla_g u = A(x)\nabla u, \quad |\nabla_g u|_g^2 = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j}, \quad x \in \mathbb{R}^n,$$

where ∇_g and ∇ are the gradients of the metric g and the Euclidean metric, respectively. Let $\Gamma = \partial\Omega$. Then (Γ, g) is a $(n-1)$-dimensional Riemannian manifold where g is the induced metric from $(\mathbb{R}^n, g)$. Denote by ∇_{Γ_g} the gradient of (Γ, g). Then for any $\upsilon \in H^1(\Omega)$,

$$\nabla_g \upsilon = \frac{\upsilon_{\nu_A}}{|\nu_A|_g^2}\nu_A + \nabla_{\Gamma_g}\upsilon \quad \text{and} \quad |\nabla_g \upsilon|_g^2 = \frac{\upsilon_{\nu_A}^2}{|\nu_A|_g^2} + |\nabla_{\Gamma_g}\upsilon|_g^2 \quad \text{for} \quad x \in \Gamma, \tag{2}$$

where $\langle\nabla_{\Gamma_g}\upsilon, \nu_A\rangle_g = 0$.

Let H be a vector field on $(\mathbb{R}^n, g)$. Denote by D the Levi-Civita connection of $\mathbb{R}^n$ in the Riemannian metric g. The covariant differential D of H determines a bilinear form on $\mathbb{R}^n \times \mathbb{R}^n$, for each $x \in \mathbb{R}^n$, by

$$DH(X,Y) = \langle D_X H, Y\rangle_g, \qquad \forall\ X,\ Y \in \mathbb{R}_x^n,$$

where $D_X H$ is the covariant derivative of vector field H with respect to X.

Now we state our main assumptions. Let $L > 0$ be a given constant. We set

$$B_L = \{x \in \mathbb{R}^n \mid |x| < L\}, \qquad \Omega_L = \Omega \cap B_L.$$

Assumption 1 There exists a vector field H on Riemannian manifold $(\mathbb{R}^n, g)$ such that

$$DH(X,X) \geq \sigma|X|_g^2, \quad \forall X \in \mathbb{R}_x^n, \quad x \in \overline{\Omega}_L, \tag{3}$$

for some constant $\sigma > 0$.

Assumption 2 Let $\rho(x,\upsilon)$ be given by (2) where $a(x)$ is a nonnegative bounded function on Ω satisfying: There exist a relatively open set $\omega \subset \overline{\Omega}$ and a positive number $L \gg 1$ such that

$$\Gamma_0' \subset \omega \quad \text{and} \quad a(x) \geq \varepsilon \quad \text{for} \quad x \in \omega \cup B_L^c, \tag{4}$$

where $\varepsilon > 0$ is a positive constant and

$$\Gamma_0' = \{x \in \Gamma_0 \mid \langle H, \nu\rangle > 0\}.$$

We further assume that ρ satisfies the following conditions:

$$\rho(x,\upsilon) = a(x)\upsilon, \quad \text{if} \quad (x,\upsilon) \in B_R^c \times \mathbb{R} \quad \text{for some} \ \ 0 < R < L.$$

If $(x, v) \in \Omega_R \times \mathbb{R}$, then

$$k_0 a(x)|v|^{p+2} \le \rho(x,v)v \le k_1 a(x)(|v|^{p+2} + |v|^2) \quad \text{if } |v| \le 1, \tag{5}$$

$$k_0 a(x)|v|^{q+2} \le \rho(x,v)v \le k_1 a(x)(|v|^{q+2} + |v|^2) \quad \text{if } |v| \ge 1, \tag{6}$$

where $k_0, k_1 > 0$, $-1 < p < +\infty$ and $-1 < q \le \frac{2}{n-2}$.

Assumption 3 $f(v)$ is strictly increasing and differentiable for $v \neq 0$, and satisfies

$$k_0|v|^{r+2} \le f(v)v \le k_1(|v|^{r+2} + |v|^2), \quad \text{if } |v| \le 1, \tag{7}$$

$$k_0|v|^{m+2} \le f(v)v \le k_1(|v|^{m+2} + |v|^2), \quad \text{if } |v| \ge 1, \tag{8}$$

where $k_0, k_1 > 0$, $-1 < r < +\infty$ and $-1 < m \le \frac{1}{n-2}$.

By Assumption 2, $\rho(x, u_t)$ is linear at u_t for large x and possibly nonlinear in a bounded region. It is said to be a 'half-linear' dissipation. The linearity for large x is essentially used to control the L^2-norm $\|u(t)\|_{L^2}$ of solutions.

Without loss of generality we may assume $\omega \subset B_R$ and $\omega \cap \Gamma_1 = \emptyset$. We define the energy of the system

$$E(t) = \frac{1}{2}\int_\Omega (|u_t|^2 + |\nabla_g u|_g^2)dx. \tag{9}$$

We note that the basic space $\widetilde{H}_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega \cup \Gamma_1)$ with respect to the $H^1(\Omega)$ norm. When $\Gamma_1 = \emptyset$, $\widetilde{H}_0^1(\Omega)$ coincides with the usual space $H_0^1(\Omega)$. To state the results we introduce the spaces (sets) $V_\infty \subset C_0^\infty(\overline{\Omega}) \times C_0^\infty(\overline{\Omega})$, $V_0 \subset \widetilde{H}_0^1(\Omega) \times L^2(\Omega)$ and $V_1 \subset H^2(\Omega) \cap \widetilde{H}_0^1(\Omega) \times H_0^1(\Omega)$ as follows:

$$V_\infty = \left\{(u_0, u_1) \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\overline{\Omega}) \mid u_0|_{\Gamma_0} = u_1|_{\Gamma_0} = 0, u_{0_{\nu_A}}|_{\Gamma_1} = -f(u_1)|_{\Gamma_1}\right\},$$
$$V_1 = V_\infty \ \text{ in } \ H^2(\Omega) \times H^1(\Omega) \ \text{ and } \ V_0 = \overline{V}_\infty \ \text{ in } \ H^1(\Omega) \times L^2(\Omega).$$

It is not difficult to see that $V_0 = \widetilde{H}_0^1(\Omega) \times L^2(\Omega)$. We note that under Assumptions 1, 2 and 3, the problem (1) admits a unique solution $u(t,x) \in C([0,\infty); \widetilde{H}_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ for each $(u_0, u_1) \in \widetilde{H}_0^1(\Omega) \times L^2(\Omega)$, and if $(u_0, u_1) \in V_1$ the solution belongs to

$$X_{loc}^2 = W_{loc}^{2,\infty}([0,\infty); L^2(\Omega)) \cap W_{loc}^{1,\infty}([0,\infty); \widetilde{H}_0^1(\Omega)) \cap L_{loc}^\infty([0,\infty); H^2(\Omega))$$

and satisfies

$$\begin{aligned}\|u_{tt}\| + \|\nabla_g u\| + \|\operatorname{div} A(x)\nabla u\| &\le C\big(\|\nabla_g u_1\| + \|\operatorname{div} A(x)\nabla u_0\| + \|\rho(x,u_1)\|\big) \\ &\equiv K < \infty, \quad 0 \le t < \infty,\end{aligned} \tag{10}$$

where C, K are positive constants and $\|\cdot\|$ denotes the usual L^2 norm on Ω.

Our main results on energy decay are as follows.

Theorem 2.1 *Let $u \in X_{loc}^2$ be a solution of* (1) *satisfying* (10). *Under Assumptions* 1, 2 *and* 3, *the following estimates hold:*

$$E(t) \le C(Q + \|u_0\|^2)(1+t)^{\vartheta - 1} \quad \textit{and} \quad \|u(t)\|^2 \le C(Q + \|u_0\|^2)(1+t)^{\vartheta}, \tag{11}$$

where C is a positive constant,

$$\vartheta = \max\left\{\frac{|q|(n-2)}{4-q(n-2)}, \frac{|p|}{p+2}, \frac{|m|(n-2)}{2-m(n-2)}, \frac{|r|}{r+2}\right\}, \tag{12}$$

and $Q \equiv E(0) + Q^1 + Q^2 + Q^3 + Q^4$ *is defined as follows:*

$$Q^1 = \begin{cases} E(0)^{\frac{2}{p+2}}, & \text{if } \ p \geq 0, \\ E(0)^{\frac{2(p+1)}{p+2}}, & \text{if } \ -1 < p \leq 0, \end{cases}$$

$$Q^2 = \begin{cases} (E(0)+K^2)^{(q+1)\theta_1} E(0)^{\frac{2(q+1)(1-\theta_1)}{q+2}}, & \text{if } \ 0 \leq q \leq \dfrac{2}{n-2}, \\ (E(0)+K^2)^{(1-\widetilde{\theta}_1)} E(0)^{\frac{4}{4-q(n-2)}}, & \text{if } \ -1 < q \leq 0, \end{cases}$$

$$Q^3 = \begin{cases} E(0)^{\frac{2}{r+2}}, & \text{if } r \geq 0, \\ E(0)^{\frac{2(r+1)}{r+2}}, & \text{if } \ -1 < r \leq 0, \end{cases}$$

$$Q^4 = \begin{cases} (E(0)+K^2)^{(m+1)\theta_2} E(0)^{\frac{2(m+1)(1-\theta_2)}{m+2}}, & \text{if } \ 0 \leq m \leq \dfrac{1}{n-2}, \\ (E(0)+K^2)^{\widetilde{\theta}_2} E(0)^{\frac{2}{2-m(n-2)}}, & \text{if} -1 < m \leq 0, \end{cases}$$

with

$$\theta_1 = \frac{nq}{(q+1)(4-q(n-2))}, \quad \theta_2 = \frac{m(n-1)}{(m+1)(2-m(n-2))},$$
$$\widetilde{\theta}_1 = \frac{2(q+2)}{4-q(n-2)}, \quad \textit{and} \quad \widetilde{\theta}_2 = \frac{-m(n-1)}{-m(n-2)+2}.$$

Remark 2.2 If $A(x) = I$, then assumption 1 hold automatically. The decay rates in Theorem 2.1 are the same as in [6].

Remark 2.3 Theorem 2.1 is also valid for $\Gamma_0 = \emptyset$.

Remark 2.4 If $\Gamma_1 = \emptyset$, then the decay rates in Theorem 2.1 are the same as [16].

Assumption 1 was introduced by [10] for the controllability of the wave equation with variable coefficients, which is also a useful condition for the controllability and the stabilization of the quasilinear wave equation (see [20, 21]). Existence of such a vector field depends on the sectional curvature of the Riemannian manifold $(\mathbb{R}^n, g)$. There are a number of methods and examples in [18] to find out a vector field H that satisfies Assumption 1.

If there is a vector field H such that

$$DH > 0 \quad \text{for} \quad \text{all } \ x \in \mathbb{R}^n,$$

then Assumption 1 holds for any bounded open set in $\mathbb{R}^n$.

Let h be a strictly convex function of the metric g on $\overline{\Omega}$. Then $H = \nabla_g h$ satisfies assumption 1. One of candidates for strictly convex functions is the distance function of the metric g from x_0 to $x \in \mathbb{R}^n$. If $A(x) = (\delta_{ij})$, then g is the standard metric of $\mathbb{R}^n$ and $\rho(x) = |x - x_0|$. For a general metric g, like (2), the structure of $\rho(x)$ is very complicated. For the properties of this function, see any Riemannian geometry book, for example, [23]. For more information on Assumption 1, we refer to [18].

3 Basic Inequalities and Estimates of L^2-Norms

With two metrics on $\mathbb{R}^n$ in mind, one is the Euclidean metric and the other is the Rimannian metric g, we have to deal with various notations carefully. Let us recall some basic relations between the two metric on $\mathbb{R}^n$.

Lemma 3.1 (see [18]) *Let $x = (x_1, x_2, \cdots, x_n)$ be the natural coordinate system on $\mathbb{R}^n$. Let H and X be vector field on $\mathbb{R}^n$, h_1, h_2 be C^1 functions. Then*

$$\begin{aligned}&\langle A(x)H(x), X(x)\rangle_g = \langle H(x), X(x)\rangle = H(x)\cdot X(x),\\&\nabla_g h_1 = A(x)\nabla h_1,\\&\langle \nabla_g h_1, \nabla_g h_2\rangle_g = \nabla_g h_1(h_2) = \langle A(x)\nabla h_1, \nabla h_2\rangle,\end{aligned}$$

for $x \in \mathbb{R}^n$, where $A(x)$ is the coefficient matrix in (2), and ∇_g, ∇ are the gradients of the metrics g and the Euclidean metric, respectively.

The following identity plays a crucial role in establishing multiplier identities in a version of the metric g.

Lemma 3.2 (see [18]) *Let h_1, h_2 be a function on $\mathbb{R}^n$ and let H be a vector field on $\mathbb{R}^n$. Then*

$$\begin{aligned}&\langle \nabla_g h_1, \nabla_g(H(h_2))\rangle_g + \langle \nabla_g h_2, \nabla_g(H(h_1))\rangle_g\\=\;& DH(\nabla_g h_1, \nabla_g h_2) + DH(\nabla_g h_2, \nabla_g h_1)\\&+ \operatorname{div}(\langle \nabla_g h_2, \nabla_g h_1\rangle_g H) - \langle \nabla_g h_2, \nabla_g h_1\rangle_g \operatorname{div} H,\end{aligned}\tag{13}$$

where $\operatorname{div} H$ is the divergence of the vector field H in the Euclidean metric.

In order to prove Theorem 2.1, first we present several multiplier identities.

Lemma 3.3 *Let H be a vector field on $\overline{\Omega}$ and $h(x)$ be a function on $\overline{\Omega}$. Then for a solution $u(t,x)$ to (1), we have the following identities:*

$$\frac{dE(t)}{dt} + \int_\Omega u_t \rho(x, u_t)dx + \int_{\Gamma_1} u_t f(u_t) d\Gamma = 0;\tag{14}$$

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_t H(u)dx + \int_\Omega \Big[\frac{1}{2}\operatorname{div} H(u_t^2 - |\nabla_g u|_g^2) + DH(\nabla_g u, \nabla_g u)\Big]dx + \int_\Omega \rho(x, u_t)H(u)dx\\&= \int_\Gamma \Big[\frac{1}{2}(u_t^2 - |\nabla_g u|_g^2)\langle H, \nu\rangle + H(u)\langle A(x)\nabla u, \nu\rangle\Big]d\Gamma;\end{aligned}\tag{15}$$

$$\begin{aligned}&\frac{d}{dt}\int_\Omega h u u_t dx + \int_\Omega h\big(|\nabla_g u|_g^2 - u_t^2 + u\rho(x, u_t)\big)dx - \frac{1}{2}\int_\Omega u^2 \operatorname{div} A(x)\nabla h dx\\&= \int_\Gamma \Big(h u u_{\nu_A} - \frac{1}{2}u^2 h_{\nu_A}\Big)d\Gamma.\end{aligned}\tag{16}$$

Proof We sketch an outline. We use a standard multiplier method. First, differentiating the energy (9) with respect to time t yields (14).

Using the above formulas in Lemmas 3.1 and 3.2, we multiply the equation in (1) with $H(u)$ and $h(x)u$, respectively. Then integrating by parts over Ω with respect to the variable x yields (15) and (16). ∎

Lemma 3.4 *Assume Assumptions* 1, 2 *and* 3 *hold. Let u be a solution of* (1) *in* X^2_{loc}. *Let $\zeta > 0$ be a sufficiently small constant and $k > 0$ be a constant such that*

$$k \geq \max\left\{\frac{2}{\varepsilon\zeta}(\sigma_1 + \sigma), \frac{3\sigma_3}{\sigma}\right\} \tag{17}$$

holds. Then it holds that

$$\begin{aligned}
&\frac{d}{dt}G(t) + \frac{2\sigma}{3}E(t) + \left(1 - \zeta - \frac{1}{16}\right)k\int_\Omega u_t\rho(x, u_t)dx + k\int_{\Gamma_1} u_t f(u_t)d\Gamma \\
&\leq C\int_{\Gamma_1}(u^2 + u_t^2 + |\nabla_g u|_g^2)d\Gamma + C\int_{\widehat{\widehat{\Omega}}_L} u^2dx + C\int_{\Omega_R}\left(a(x)u_t^2 + |\rho(x, u_t)|^2\right)dx + C\int_\omega |u_t|^2dx,
\end{aligned}$$

where

$$\begin{aligned}
G(t) = &\int_\Omega u_t\left(\varphi H(u) - C\widehat{H}(u)\right)dx + \int_\Omega\left(h_0 + \frac{\zeta k}{2}a(x)\phi + \frac{2\sigma}{3} + C\beta^2\right)uu_tdx \\
&+ \int_{\Omega_R^c}\left(\frac{h_0 a(x)}{2} + \frac{\sigma a(x)}{3} + \frac{\zeta k a^2(x)\phi}{4}\right)u^2dx + kE(t),
\end{aligned} \tag{18}$$

σ_1, σ_3, a_0 are positive constants that will be specified later on, φ, ϕ, h_0, β are functions and $\widehat{H}$ is a vector field that will be given later on, the vector field H and the constant $\sigma > 0$ are given in (3), *$\varepsilon > 0$ is given in* (4). *Here and in what follows, we use the constant $C > 0$ to denote some constants independent of functions involved, although it may have different value in different contexts.*

Proof Let H be a vector field such that Condition (3) holds. We take three bounded open sets such that

$$\overline{\Omega}_L \subset \widehat{\Omega}_L \subset \overline{\widehat{\Omega}}_L \subset \widehat{\widehat{\Omega}}_L. \tag{19}$$

Let $\varphi, \phi \in C_0^\infty(\overline{\Omega})$ be cut-off functions such that

$$0 \leq \varphi \leq 1, \quad 0 \leq \phi \leq 1, \quad \varphi = \begin{cases} 1, & x \in \Omega_L, \\ 0, & x \in \Omega\backslash\widehat{\Omega}_L, \end{cases} \qquad \phi = \begin{cases} 1, & x \in \widehat{\Omega}_L, \\ 0, & x \in \Omega\backslash\widehat{\widehat{\Omega}}_L. \end{cases} \tag{20}$$

We replace H with φH in the identity (15) and replace h with

$$h_0 = \frac{1}{2}\operatorname{div}\varphi H - \sigma\varphi \tag{21}$$

in the identity (16), respectively, where $\sigma > 0$ is given in (3). Then we add up the two identities and obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(u_t\varphi H(u) + h_0uu_t\right)dx + \int_\Omega\left[\sigma\varphi(u_t^2 - |\nabla_g u|_g^2) + D\varphi H(\nabla_g u, \nabla_g u)\right]dx \\
&-\frac{1}{2}\int_\Omega u^2\operatorname{div} A(x)\nabla h_0dx + \int_\Omega \rho(x, u_t)(\varphi H(u) + h_0u)dx \\
&= \frac{1}{2}\int_\Gamma\left[\frac{1}{2}(u_t^2 - |\nabla_g u|_g^2)\langle\varphi H, \nu\rangle + \varphi H(u)\langle A(x)\nabla u, \nu\rangle + h_0uu_{\nu_A} - \frac{1}{2}u^2h_{0_{\nu_A}}\right]d\Gamma.
\end{aligned}$$

Taking the Assumption 2, we have

$$\int_{\Omega} h_0 u\rho(x,u_t)dx = \int_{\Omega_R^c} h_0 a(x)uu_t dx + \int_{\Omega_R} h_0 u\rho(x,u_t)dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_{\Omega_R^c} h_0 a(x)u^2 dx + \int_{\Omega_R} h_0 u\rho(x,u_t)dx.$$

Then

$$\frac{d}{dt}\Big(\int_{\Omega}(u_t\varphi H(u) + h_0 uu_t)dx + \frac{1}{2}\int_{\Omega_R^c} h_0 a(x)u^2 dx\Big) + \int_{\Omega_R} h_0 u\rho(x,u_t)dx$$
$$+ \int_{\Omega}\rho(x,u_t)\varphi H(u)dx - \frac{1}{2}\int_{\Omega} u^2 \operatorname{div} A(x)\nabla h_0 dx$$
$$+ \int_{\Omega}\big[\sigma\varphi(u_t^2 - |\nabla_g u|_g^2) + D\varphi H(\nabla_g u,\nabla_g u)\big]dx$$
$$= \int_{\Gamma}\Big[\frac{1}{2}(u_t^2 - |\nabla_g u|_g^2)\langle\varphi H,\nu\rangle + \varphi H(u)\langle A(x)\nabla u,\nu\rangle + h_0 uu_{\nu_A} - \frac{1}{2}u^2 p_{0_{\nu_A}}\Big]d\Gamma.$$

Let $k>0$ be a constant. Multiplying the identity (14) by k and adding the obtained result to the identity (22) we have

$$\frac{d}{dt}\Big(\int_{\Omega}(u_t\varphi H(u) + h_0 uu_t)dx + \frac{1}{2}\int_{\Omega_R^c} h_0 a(x)u^2 dx + kE(t)\Big) + k\int_{\Omega} u_t\rho(x,u_t)dx$$
$$+ \int_{\Omega}\big[\sigma\varphi(u_t^2 - |\nabla_g u|_g^2) + D\varphi H(\nabla_g u,\nabla_g u)\big]dx - \frac{1}{2}\int_{\Omega} u^2 \operatorname{div} A(x)\nabla h_0 dx$$
$$+ \int_{\Omega}\rho(x,u_t)\varphi H(u)dx + \int_{\Omega_R} h_0 u\rho(x,u_t)dx + k\int_{\Gamma_1} u_t f(u_t)dx$$
$$= \int_{\Gamma}\Big[\frac{1}{2}[(u_t^2 - |\nabla_g u|_g^2)\langle\varphi H,\nu\rangle + \varphi H(u)\langle A(x)\nabla u,\nu\rangle + h_0 uu_{\nu_A} - \frac{1}{2}u^2 p_{0_{\nu_A}}\Big]d\Gamma. \tag{22}$$

We dispose:

$$k\int_{\Omega} u_t\rho(x,u_t)dx = [(1-\zeta)k + \zeta k]\int_{\Omega} u_t\rho(x,u_t)dx,$$

where $\zeta > 0$ is a suitable small constant that will be specified later on.

Let

$$Y(k,u) = \int_{\Omega}[\sigma\varphi(u_t^2 - |\nabla_g u|_g^2) + D\varphi H(\nabla_g u,\nabla_g u)]dx + \zeta k\int_{\Omega} u_t\rho(x,u_t)dx.$$

Since $a \geq a\phi \geq 0$, $\sigma \geq \sigma\varphi \geq 0$, taking Assumptions 1 and 2, from (23), we deduce

$$\int_{\Omega} D\varphi H(\nabla_g u,\nabla_g u)dx = \int_{\widehat{\Omega}_L\setminus\Omega_L} D\varphi H(\nabla_g u,\nabla_g u)dx + \int_{\Omega_L} D\varphi H(\nabla_g u,\nabla_g u)dx$$
$$\geq -\sigma_1\int_{\widehat{\Omega}_L\setminus\Omega_L} |\nabla_g u|_g^2 dx + \sigma\int_{\Omega_L} |\nabla_g u|_g^2 dx,$$

$$\begin{aligned}\int_\Omega u_t\rho(x,u_t)dx &= \int_{\Omega_R} u_t\rho(x,u_t)dx + \int_{\Omega_R^c} u_t\rho(x,u_t)dx \\ &= \int_{\Omega_R} u_t\rho(x,u_t)dx + \int_{\Omega_R^c} a(x)u_t^2dx \\ &\geq \int_\Omega a(x)u_t^2dx - \int_{\Omega_R} a(x)u_t^2dx \\ &\geq \frac{1}{2}\int_\Omega a(x)u_t^2dx + \frac{1}{2}\int_\Omega a(x)\phi u_t^2dx - \int_{\Omega_R} a(x)u_t^2dx,\end{aligned}$$

where

$$\sigma_1 = \sup_{X\in\mathbb{R}_x^n, |X|_g=1, x\in\widehat{\Omega}_L\setminus\Omega_L} |D\varphi H(X,X)|.$$

Then

$$\begin{aligned}Y(k,u) \geq &-(\sigma_1+\sigma)\int_{\widehat{\Omega}_L\setminus\Omega_L} |\nabla_g u|_g^2dx + \sigma\int_{\Omega_L} u_t^2dx + \frac{\zeta k}{2}\int_\Omega a(x)u_t^2dx \\ &+\frac{\zeta k}{2}\int_\Omega a(x)\phi u_t^2dx - \zeta k\int_{\Omega_R} a(x)u_t^2dx. \end{aligned}\tag{23}$$

Let $h = a(x)\phi$ in the identity (16). We have

$$\begin{aligned}\int_\Omega a(x)\phi u_t^2dx = &\frac{d}{dt}\int_\Omega a(x)\phi uu_tdx + \int_\Omega a(x)\phi|\nabla_g u|_g^2dx - \frac{1}{2}\int_\Omega u^2 \operatorname{div} A(x)\nabla(a(x)\phi)dx \\ &+\int_\Omega a(x)\phi u\rho(x,u_t)dx - \int_\Gamma \left(a(x)\phi uu_{\nu_A} - \frac{1}{2}u^2(a(x)\phi)_{\nu_A}\right)d\Gamma \end{aligned}\tag{24}$$

which, together with (23), leads to

$$\begin{aligned}Y(k,u) \geq &\int_{\widehat{\Omega}_L\setminus\Omega_L}\left(\frac{\zeta k}{2}a(x) - \sigma_1 - \sigma\right)|\nabla_g u|_g^2dx + \frac{\zeta k}{2}\int_\Omega a(x)u_t^2dx + \frac{\zeta k}{2}\frac{d}{dt}\int_\Omega a(x)\phi uu_tdx \\ &-\frac{\zeta k}{4}\int_\Omega u^2 \operatorname{div} A(x)\nabla(a(x)\phi)dx + \frac{\zeta k}{2}\int_\Omega a(x)\phi u\rho(x,u_t)dx \\ &+\sigma\int_{\Omega_L} u_t^2dx - \frac{\zeta k}{2}\int_\Gamma\left[a(x)\phi uu_{\nu_A} - \frac{1}{2}u^2(a(x)\phi)_{\nu_A}\right]d\Gamma - \zeta k\int_{\Omega_R} a(x)u_t^2dx. \end{aligned}\tag{25}$$

From the the inequality (17), we have

$$\frac{\zeta k}{2}\int_\Omega a(x)u_t^2dx \geq \frac{\zeta k}{2}\int_{\Omega_L^c} a(x)u_t^2dx \geq \frac{\zeta\varepsilon k}{2}\int_{\Omega_L^c} u_t^2dx \geq \sigma\int_{\Omega_L^c} u_t^2dx.$$

Then

$$\sigma\int_{\Omega_L} u_t^2dx + \frac{\zeta k}{2}\int_\Omega a(x)u_t^2dx \geq \sigma\int_\Omega u_t^2dx.$$

Since $k \geq \frac{2(\sigma_1+\sigma)}{\varepsilon\zeta}$, we have

$$\begin{aligned}Y(k,u) \geq &\sigma\int_\Omega u_t^2dx + \frac{\zeta k}{2}\frac{d}{dt}\int_\Omega a(x)\phi uu_tdx - \frac{\zeta k}{4}\sigma_2\int_{\widehat{\widehat{\Omega}}_L} u^2dx + \frac{\zeta k}{2}\int_\Omega a(x)\phi u\rho(x,u_t)dx \\ &-\frac{\zeta k}{2}\int_\Gamma\left[a(x)\phi uu_{\nu_A} - \frac{1}{2}u^2(a(x)\phi)_{\nu_A}\right]d\Gamma - \zeta k\int_{\Omega_R} a(x)u_t^2dx, \end{aligned}\tag{26}$$

where

$$\sigma_2 = \sup_{x \in \hat{\hat{\Omega}}_L} |\operatorname{div} A(x)\nabla(a(x)\phi)|.$$

Applying $h = 1$ in (16) yields

$$\int_\Omega u_t^2 dx = \frac{d}{dt}\int_\Omega u u_t dx + \int_\Omega |\nabla_g u|_g^2 + \int_\Omega u\rho(x, u_t) dx - \int_\Gamma u u_{\nu_A} d\Gamma, \tag{27}$$

which, together with (26), leads to

$$\begin{aligned} Y(k,u) \geq{}& \frac{\sigma}{3}\int_\Omega u_t^2 dx + \frac{2\sigma}{3}\int_\Omega |\nabla_g u|_g^2 + \frac{2\sigma}{3}\int_\Omega u\rho(x,u_t)dx + \frac{d}{dt}\int_\Omega \left[\frac{\zeta k}{2}a(x)\phi + \frac{2\sigma}{3}\right] u u_t dx \\ & -\frac{\zeta k}{4}\sigma_2 \int_{\hat{\hat{\Omega}}_L} u^2 dx + \frac{\zeta k}{2}\int_\Omega a(x)\phi u \rho(x,u_t)dx - \zeta k \int_{\Omega_R} a(x) u_t^2 dx \\ & -\frac{\zeta k}{2}\int_\Gamma \left[a(x)\phi u u_{\nu_A} - \frac{1}{2}u^2 (a(x)\phi)_{\nu_A}\right] d\Gamma - \frac{2\sigma}{3}\int_\Gamma u u_{\nu_A} d\Gamma. \end{aligned} \tag{28}$$

Then

$$\begin{aligned} & \frac{d}{dt}\left[\int_\Omega \left(u_t \varphi H(u) + \left(h_0 + \frac{\zeta k}{2}a(x)\phi + \frac{2\sigma}{3}\right) u u_t\right) dx + \frac{1}{2}\int_{\Omega_R^c} h_0 a(x) u^2 dx + kE(t)\right] \\ & +(1-\zeta)k\int_\Omega u_t \rho(x,u_t)dx + \frac{\sigma}{3}\int_\Omega u_t^2 dx + \frac{2\sigma}{3}\int_\Omega |\nabla_g u|_g^2 + \frac{2\sigma}{3}\int_\Omega u\rho(x,u_t)dx \\ & +\frac{\zeta k}{2}\int_\Omega a(x)\phi u\rho(x,u_t)dx + \int_\Omega \rho(x,u_t)\varphi H(u)dx + k\int_{\Gamma_1} u_t f(u_t) dx \\ \leq{}& \frac{1}{2}\int_\Omega u^2 \operatorname{div} A(x)\nabla h_0 dx + \zeta k \int_{\Omega_R} a(x) u_t^2 dx + \frac{\zeta k}{4}\sigma_2 \int_{\hat{\hat{\Omega}}_L} u^2 dx - \int_{\Omega_R} h_0 u \rho(x,u_t) dx \\ & + \int_\Gamma \left[\frac{1}{2}(u_t^2 - |\nabla_g u|_g^2)\langle \varphi H, \nu\rangle + \varphi H(u)\langle A(x)\nabla u, \nu\rangle + h_0 u u_{\nu_A} - \frac{1}{2}u^2 p_{0_{\nu_A}}\right] d\Gamma \\ & + \frac{\zeta k}{2}\int_\Gamma \left[a(x)\phi u u_{\nu_A} + \frac{1}{2}u^2 (a(x)\phi)_{\nu_A}\right] d\Gamma + \frac{2\sigma}{3}\int_\Gamma u u_{\nu_A} d\Gamma. \end{aligned} \tag{29}$$

Since $u|_{\Gamma_0} = 0$, by (2) we have $H(u)u_{\nu_A} = |\nabla_g u|_g^2 \langle H, \nu\rangle = \frac{u_{\nu_A}^2}{|\nu_A|_g^2}\langle H, \nu\rangle$ on Γ_0. Noting $u_t|_{\Gamma_0} = 0$, then

$$\begin{aligned} & \int_\Gamma \left[\frac{1}{2}(u_t^2 - |\nabla_g u|_g^2)\langle \varphi H, \nu\rangle + \varphi H(u)\langle A(x)\nabla u, \nu\rangle + h_0 u u_{\nu_A} - \frac{1}{2}u^2 p_{0_{\nu_A}}\right] d\Gamma \\ & + \frac{\zeta k}{2}\int_\Gamma \left[a(x)\phi u u_{\nu_A} + \frac{1}{2}u^2 (a(x)\phi)_{\nu_A}\right] d\Gamma + \frac{2\sigma}{3}\int_\Gamma u u_{\nu_A} d\Gamma \\ \leq{}& C\int_{\Gamma_1} (u^2 + u_t^2 + |\nabla_g u|_g^2) d\Gamma + C\int_{\Gamma_0'} |u_{\nu_A}|^2 d\Gamma. \end{aligned}$$

Thus

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega\left(u_t\varphi H(u)+\left(h_0+\frac{\zeta k}{2}a(x)\phi+\frac{2\sigma}{3}\right)uu_t\right)dx+\frac{1}{2}\int_{\Omega_R^c}h_0a(x)u^2dx+kE(t)\right]\\
&+(1-\zeta)k\int_\Omega u_t\rho(x,u_t)dx+\frac{\sigma}{3}\int_\Omega u_t^2dx+\frac{2\sigma}{3}\int_\Omega|\nabla_g u|_g^2dx+\frac{2\sigma}{3}\int_\Omega u\rho(x,u_t)dx\\
&+\frac{\zeta k}{2}\int_\Omega a(x)\phi u\rho(x,u_t)dx+\int_\Omega\rho(x,u_t)\varphi H(u)dx+k\int_{\Gamma_1}u_tf(u_t)dx\\
\le&\frac{1}{2}\int_\Omega u^2\operatorname{div}A(x)\nabla h_0dx+\zeta k\int_{\Omega_R}a(x)u_t^2dx+\frac{\zeta k}{4}\sigma_2\int_{\hat{\hat{\Omega}}_L}u^2dx-\int_{\Omega_R}h_0u\rho(x,u_t)dx\\
&+C\int_{\Gamma_1}(u^2+u_t^2+|\nabla_g u|_g^2)d\Gamma+C\int_{\Gamma_0'}|u_{\nu_A}|^2d\Gamma. \qquad (30)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\int_\Omega\rho(x,u_t)\varphi H(u)dx\right|\\
\le&\left|\int_{\Omega_R}\rho(x,u_t)\varphi H(u)dx\right|+\left|\int_{\Omega_R^c}a(x)u_t\varphi H(u)dx\right|\\
\le&\frac{k}{16a_0}\int_{\Omega_R}|\rho(x,u_t)|^2dx+\frac{\sigma_3}{k}\int_\Omega|\nabla_g u|_g^2dx+\frac{k}{16}\int_{\Omega_R^c}a(x)u_t^2dx, \qquad (31)
\end{aligned}$$

where

$$\sigma_3=4a_0\sup_{x\in\hat{\Omega}_L}\varphi^2|H|_g^2,\qquad a_0=\sup_{x\in\overline{\Omega}}a(x).$$

We use (31) in (30) and (17) to obtain that

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega\left(u_t\varphi H(u)+\left(h_0+\frac{\zeta k}{2}a(x)\phi+\frac{2\sigma}{3}\right)uu_t\right)dx+\frac{1}{2}\int_{\Omega_R^c}h_0a(x)u^2dx+kE(t)\right]\\
&+(1-\zeta)k\int_\Omega u_t\rho(x,u_t)dx+\frac{\sigma}{3}\int_\Omega u_t^2dx+\frac{\sigma}{3}\int_\Omega|\nabla_g u|_g^2dx+\frac{2\sigma}{3}\int_\Omega u\rho(x,u_t)dx\\
&+\frac{\zeta k}{2}\int_\Omega a(x)\phi u\rho(x,u_t)dx-\frac{k}{16a_0}\int_{\Omega_R}|\rho(x,u_t)|^2dx-\frac{k}{16}\int_{\Omega_R^c}a(x)u_t^2dx+k\int_{\Gamma_1}u_tf(u_t)dx\\
\le&\left(\frac{\sigma_4}{2}+\frac{\zeta k}{4}\sigma_2\right)\int_{\hat{\hat{\Omega}}_L}u^2dx+\zeta k\int_{\Omega_R}a(x)u_t^2dx+C\int_{\Omega_R}|u||\rho(x,u_t)|dx\\
&+C\int_{\Gamma_1}(u^2+u_t^2+|\nabla_g u|_g^2)d\Gamma+C\int_{\Gamma_0'}|u_{\nu_A}|^2d\Gamma, \qquad (32)
\end{aligned}$$

where

$$\sigma_4=\sup_{x\in\hat{\Omega}_L}|\operatorname{div}A(x)\nabla h_0|.$$

Since

$$\begin{aligned}&\int_{\Omega}\left(\frac{2\sigma}{3}+\frac{\zeta k}{2}a(x)\phi\right)u\rho(x,u_t)dx\\=&\frac{d}{dt}\int_{\Omega_R^c}\left(\frac{\sigma}{3}+\frac{\zeta k}{4}a(x)\phi\right)a(x)u^2dx+\frac{2\sigma}{3}\int_{\Omega_R}u\rho(x,u_t)dx+\frac{\zeta k}{2}\int_{\Omega_R}a(x)\phi u\rho(x,u_t)dx,\\&\int_{\Omega_R^c}a(x)u_t^2dx=\int_{\Omega_R^c}u_t\rho(x,u_t)dx\le\int_{\Omega}u_t\rho(x,u_t)dx,\end{aligned}$$

then

$$\begin{aligned}&\frac{d}{dt}\left[X(t)+\int_{\Omega_R^c}\left(\frac{h_0a(x)}{2}+\frac{\sigma a(x)}{3}+\frac{\zeta ka^2(x)\phi}{4}\right)u^2dx\right]+\frac{2\sigma}{3}E(t)\\&+\left(1-\zeta-\frac{1}{16}\right)k\int_{\Omega}u_t\rho(x,u_t)dx+k\int_{\Gamma_1}u_tf(u_t)dx\\\le&\, C\int_{\Gamma_1}(u^2+u_t^2+|\nabla_gu|_g^2)d\Gamma+C\int_{\Gamma_0'}|u_{\nu_A}|^2d\Gamma\\&+C\int_{\hat{\hat{\Omega}}_L}u^2dx+\zeta k\int_{\Omega_R}a(x)u_t^2dx+C\int_{\Omega_R}|\rho(x,u_t)|^2dx,\end{aligned}\tag{33}$$

where

$$X(t)=\int_{\Omega}\left(u_t\varphi H(u)+\left(h_0+\frac{\zeta k}{2}a(x)\phi+\frac{2\sigma}{3}\right)uu_t\right)dx+kE(t)$$

and we choose $\zeta>0$ sufficiently small such that $(1-\zeta-\frac{1}{16})k>0$.

In order to calculate the second term on the right-hand side of (33) we take a vector field $\widehat{H}$ satisfying $\widehat{H}\cdot\nu\ge 0$, $\widehat{H}=\nu$ on Γ_0' and $\widehat{H}=0$ on $\widetilde{\omega}^c$, where $\widetilde{\omega}$ is open in $\mathbb{R}^n$ such that $\Gamma_0'\subset\widetilde{\omega}\cap\overline{\Omega}\subset\omega$. Note that $\omega\cap\Gamma_1=\emptyset$. Then applying (15) with $H=\widehat{H}$ we have

$$\begin{aligned}&\frac{1}{2}\int_{\Gamma_0'}|u_{\nu_A}|^2d\Gamma\\\le&\frac{d}{dt}\int_{\Omega}u_t\widehat{H}(u)dx+\int_{\Omega}\left[\frac{1}{2}\operatorname{div}\widehat{H}(u_t^2-|\nabla_gu|_g^2)+D\widehat{H}(\nabla_gu,\nabla_gu)\right]dx+\int_{\Omega}\rho(x,u_t)\widehat{H}(u)dx\\\le&\frac{d}{dt}\int_{\Omega}u_t\widehat{H}(u)dx+C\int_{\omega}|\rho(x,u_t)|^2dx+C\int_{\widetilde{\omega}\cap\Omega}(|u_t|^2+|\nabla_gu|_g^2)dx.\end{aligned}\tag{34}$$

To estimate $|\nabla_gu|_g$ on $\widetilde{\omega}\cap\Omega$ of the right side of (34), we introduce the nonnegative function $\beta\in C^1(\overline{\Omega})$ such that $0\le\beta\le 1$, $|\nabla_gu|_g\in L^\infty(\Omega)$ and

$$\beta(x)=\begin{cases}1, & x\in\widetilde{\omega}\cap\Omega,\\0, & x\in\overline{\Omega}\cap\omega^c.\end{cases}$$

Applying (16) with $h=\beta^2$, we can show that

$$\begin{aligned}\int_{\widetilde{\omega}\cap\Omega}|\nabla_gu|_g^2dx&\le\int_{\Omega}\beta^2|\nabla_gu|_g^2dx\\&\le-\frac{d}{dt}\int_{\Omega}\beta^2uu_tdx+C\int_{\omega}(|u_t|^2+|u|^2+|\rho(x,u_t)|^2)dx.\end{aligned}\tag{35}$$

Form (33)–(35) we obtain (18). ▮

Lemma 3.5 *Suppose that Assumptions* 1, 2, *and* 3 *hold. Let* u *be a solution of* (1)*. Let* $\zeta > 0$ *be a sufficiently small constant and* $k > 0$ *be a constant such that*

$$k \geq \max\left\{\frac{2}{\varepsilon\zeta}(\sigma_1+\sigma), \frac{3\sigma_3}{\sigma}, \sup_{x\in\widehat{\Omega}_L}|H|_g + C\sup_{x\in\widetilde{\omega}}|\widehat{H}|_g + \frac{3\sigma_5^2}{\sigma\varepsilon}\right\} \tag{36}$$

holds, where $\varepsilon, \sigma, \sigma_1, \sigma_3$ *are positive constants and* $H, \widehat{H}$ *are vector fields which have been given,* σ_5 *is a positive constant which is specified later on. There exist* $C_1,\ C_2 > 0$ *such that for any* $t \geq 0$,

$$C_1(E(t)+\|u(t)\|^2) - C_1\int_{\widehat{\Omega}_L}|u|^2dx \leq G(t) \leq C_2(E(t)+\|u(t)\|^2), \tag{37}$$

where $G(t)$ *is given in* (18)*.*

Proof Using the assumption of $a(x)$ in (4), we exploit the Cauchy's and Hölder's inequlities in (18) to get

$$\begin{aligned}
G(t) &= \int_\Omega u_t\varphi H(u)dx - C\int_\Omega u_t\widehat{H}(u)dx + \int_\Omega\Big(h_0+\frac{\zeta k}{2}a(x)\phi+\frac{2\sigma}{3}+C\beta^2\Big)uu_tdx \\
&\quad+\int_{\Omega_R^c}\Big(\frac{h_0a(x)}{2}+\frac{\sigma a(x)}{3}+\frac{\zeta ka^2(x)\phi}{4}\Big)u^2dx + kE(t) \\
&\geq kE(t) + \frac{\sigma}{3}\int_{\Omega_R^c}a(x)u^2dx - \Big(\sup_{x\in\widehat{\Omega}_L}\varphi|H|_g + C\sup_{x\in\Omega\cap\widetilde{\omega}}|\widehat{H}|_g\Big)\|u_t\|\|\nabla_g u\| \\
&\quad-\sup_{x\in\widehat{\widehat{\Omega}}_L}\Big|h_0+\frac{\zeta k}{2}a(x)\phi+\frac{2\sigma}{3}+C\beta^2\Big|\|u_t\|\|u\| - \sup_{x\in\widehat{\Omega}_L}\Big|\frac{a(x)h_0}{2}\Big|\int_{\widehat{\Omega}_L}|u|^2dx \\
&\geq \frac{\sigma\varepsilon}{6}\|u\|^2 + \Big(k-\max\Big\{\sup_{x\in\widehat{\Omega}_L}\varphi|H|_g + C\sup_{x\in\Omega\cap\widetilde{\omega}}|\widehat{H}|_g, \frac{3\sigma_5^2}{\sigma\varepsilon}\Big\}\Big)E(t) \\
&\quad-\Big(\sup_{x\in\widehat{\Omega}_L}\Big|\frac{a(x)h_0}{2}\Big| + \frac{\sigma\varepsilon}{3}\Big)\int_{\widehat{\Omega}_L}|u|^2dx,
\end{aligned}\tag{38}$$

where

$$\sigma_5 = \sup_{x\in\widehat{\widehat{\Omega}}_L}\Big|h_0+\frac{\zeta k}{2}a(x)\phi+\frac{2\sigma}{3}+C\beta^2\Big|.$$

Thanks to the choice of k, we obtain that there exists a constant $C > 0$ such that

$$G(t) \geq C(E(t)+\|u(t)\|^2) - C\int_{\widehat{\Omega}_L}|u|^2dx. \tag{39}$$

On the other hand,

$$G(t) \leq \Big(k+\sup_{x\in\widehat{\Omega}_L}\varphi|H|_g + C\sup_{x\in\Omega\cap\widetilde{\omega}}|\widehat{H}|_g + \sigma_5\Big)E(t) + \Big(\frac{\sigma_5}{2}+\sigma_6\Big)\|u(t)\|^2,$$

where

$$\sigma_6 = \sup_{x\in\widehat{\widehat{\Omega}}_L}\Big|\frac{h_0a(x)}{2}+\frac{\sigma a(x)}{3}+\frac{\zeta ka^2(x)\phi}{4}\Big|.$$

Therefore, there exists a constant $C > 0$ such that $G(t) \leq C(E(t) + \|u(t)\|^2)$ which, together with (39), leads to the result. ∎

Lemma 3.6 *Let $u(t,x)$ be a solution in Theorem 2.1. Then there exists $T_0 > 0$, independent of u such that if $T \geq T_0$, the solution u satisfies*

$$\begin{aligned}&\int_t^{t+T}\int_{\hat{\hat{\Omega}}_L}|u|^2dxds+\int_t^{t+T}\int_{\Gamma_1}|u|^2d\Gamma ds\\ &\leq C_\eta\Big[\int_t^{t+T}\int_{\Omega_L}|\rho(x,u_t)|^2dxds+\int_t^{t+T}\int_{\Gamma_1}(|u_{\nu_A}|^2+|u_t|^2)d\Gamma ds\\ &\quad+\int_t^{t+T}\int_{\Omega_L^c\cup\omega}|u_t|^2dxds\Big]+\eta\int_t^{t+T}E(s)ds \qquad (40)\end{aligned}$$

for any $\eta > 0$, where C_η is a constant independent of u.

Proof As usual, we prove the lemma by contradiction. We assume that for some $\eta_0 > 0$, the number $C_{\eta_0} > 0$ does not exist. Then for any $n \geq 1$, there are solutions $\{u_n\}$ and $\{t_n\} > 0$ such that

$$\begin{aligned}&\int_{t_n}^{t_n+T}\int_{\hat{\hat{\Omega}}_L}|u_n|^2dxds+\int_{t_n}^{t_n+T}\int_{\Gamma_1}|u_n|^2d\Gamma ds\\ &\geq n\Big[\int_{t_n}^{t_n+T}\int_{\Omega_L}|\rho(x,u_{nt})|^2dxds+\int_{t_n}^{t_n+T}\int_{\Gamma_1}(|u_{n\nu_A}|^2+|u_{nt}|^2)d\Gamma ds\\ &\quad+\int_{t_n}^{t_n+T}\int_{\Omega_L^c\cup\omega}|u_{nt}|^2dxds\Big]+\eta_0\int_{t_n}^{t_n+T}E_{u_n}(s)ds, \qquad (41)\end{aligned}$$

where $E_{u_n}(t)$ are the $E(t)$ with u replaced by u_n.

Setting

$$\lambda_n^2=\int_{t_n}^{t_n+T}\int_{\hat{\hat{\Omega}}_L}|u_n|^2dxds+\int_{t_n}^{t_n+T}\int_{\Gamma_1}|u_n|^2d\Gamma ds,$$

and

$$\upsilon_n(t)\equiv\frac{u_n(t+t_n)}{\lambda_n},$$

then we have

$$\begin{aligned}1\geq n\Big[&\int_0^T\int_{\Omega_L}\frac{1}{\lambda_n^2}|\rho(x,u_{nt}(s+t_n))|^2dxds+\int_0^T\int_{\Gamma_1}(|\upsilon_{n\nu_A}|^2+|\upsilon_{nt}|^2)d\Gamma ds\\ &+\int_0^T\int_{\Omega_L^c\cup\omega}|\upsilon_{nt}|^2dxds\Big]+\eta_0\int_0^T E_{\upsilon_n}(s)ds.\end{aligned}$$

Consequently,

$$\int_0^T\int_{\Omega_L}\frac{1}{\lambda_n^2}|\rho(x,u_{nt}(s+t_n))|^2dxds\to 0,\quad \text{as}\quad n\to\infty$$

and

$$\int_0^T \int_{\Gamma_1} (|\upsilon_{n\nu_A}|^2 + |\upsilon_{nt}|^2) d\Gamma ds \to 0, \quad \text{as} \quad n \to \infty,$$

where $E_{\upsilon_n}(t)$ are the $E(t)$ with u replaced by υ_n.

Note that

$$\int_0^T \int_{\widehat{\widehat{\Omega}}_L} |\upsilon_n|^2 dx ds + \int_0^T \int_{\Gamma_1} |\upsilon_n|^2 d\Gamma ds = 1, \tag{42}$$

and

$$\int_0^T \int_{\Omega} \big(|\upsilon_n|^2 + |\nabla_g \upsilon_n|_g^2\big) dx ds = 2 \int_0^T E_{\upsilon_n}(s) ds \le 2\eta_0^{-1} < \infty. \tag{43}$$

Similar to [12, 19], taking (42) and (43) into account, we may assume that there is a subsequence of $\{\upsilon_n\}$ which are still denoted by $\{\upsilon_n\}$ and a υ such that

$$\begin{cases} \upsilon_{nt} \rightharpoonup \upsilon_t, & \text{in } L^2([0,T] \times \Omega) \text{ weakly}, \\ \nabla_g \upsilon_n \rightharpoonup \nabla_g \upsilon, & \text{in } L^2([0,T] \times \Omega) \text{ weakly}, \\ \upsilon_n \to \upsilon, & \text{in } L^2([0,T] \times \widehat{\widehat{\Omega}}_L) \cup L^2([0,T]; \Gamma_1) \text{ strongly}, \end{cases}$$

and the limit function υ satisfies

$$\begin{cases} \upsilon_{tt} - \operatorname{div} A(x) \nabla \upsilon = 0, & \text{in} \quad [0,T] \times \Omega; \\ \upsilon = 0, & \text{on} \quad [0,T] \times \Gamma_0; \\ \upsilon_t = \upsilon_{\nu_A} = 0, & \text{on} \quad [0,T] \times \Gamma_1; \end{cases} \tag{44}$$

$$\int_0^T \int_{\widehat{\widehat{\Omega}}_L} |\upsilon|^2 dx ds + \int_0^T \int_{\Gamma_1} |\upsilon|^2 d\Gamma ds = 1; \tag{45}$$

$$\int_0^T \int_{\Omega_L^c \cup \omega} |\upsilon_t|^2 dx ds = 0. \tag{46}$$

The identity (46) means

$$\upsilon_t(t,x) = 0 \quad \text{in} \quad [0,T] \times (\Omega_L^c \cup \omega).$$

By the unique continuation of the wave equation (see [16]), there exists $T_0 > 0$ such that if $T > T_0$, then

$$\upsilon_t(t,x) = 0 \quad \text{in} \quad [0,T] \times \Omega,$$

which, together with (44), yields $\upsilon(t,x) = \upsilon(x)$ and $\operatorname{div} A(x) \nabla \upsilon = 0$ in $[0,T] \times \Omega$.

From the identity (44) we know $\upsilon_{\nu_A}\big|_{\Gamma_1} = 0$. Since $\nabla_g \upsilon \in L^2(\Omega)$ and $\upsilon\big|_{\Gamma_0} = 0$, we can easily prove that

$$\upsilon(t,x) = 0 \quad \text{in} \quad [0,T] \times \Omega,$$

which contradicts the relation (45). The proof is completed. ∎

When $t > T_0$, take T such that $T_0 < T < 2T_0$. Let m be the natural number with $mT \le t < (m+1)T$. We can divide the interval $[0,t]$ as $\bigcup_{i=0}^{m-1}[iT,(i+1)T] \cup [t-T,t]$. Then Lemma 3.6 immediately implies the following result which plays an essential role in our argument.

Lemma 3.7 *Let u be a solution of* (1) *in Theorem* 2.1. *For $t > T_0$, we have*

$$\begin{aligned}&\int_0^t \int_{\widehat{\widehat{\Omega}}_L} |u|^2 dx ds + \int_0^t \int_{\Gamma_1} |u|^2 d\Gamma ds \\ &\le C_\eta \Big[\int_0^t \int_{\Omega_L} |\rho(x,u_t)|^2 dx ds + \int_0^t \int_{\Gamma_1} (|u_{\nu_A}|^2 + |u_t|^2) d\Gamma ds \\ &\quad + \int_0^t \int_{\Omega_L^c \cup \omega} |u_t|^2 dx ds \Big] + \eta \int_0^t E(s) ds \end{aligned} \tag{47}$$

for any $0 < \eta \ll 1$, where C_η is a positive constant depending on η but independent of u.

Finally, we shall consider the tangential derivatives in the right-hand sides of (18). For this we rely on a result in [24]. To apply a result in [24] we reduce our problem to the one in a bounded domain. We take a cut-off function $\varphi(x) \in C^1(\Omega)$ such that $\varphi(x) = 1$ in a neighbourhood of Γ_1 and $\varphi(x) = 0$ for $|x| \ge L$ and set $\phi(x,t) = \varphi(x)u$. Then

$$\begin{aligned}\phi_{tt} - \operatorname{div} A(x)\nabla\phi &= -\varphi\rho(x,u_t) - \langle \nabla_g \varphi(x), \nabla_g u \rangle_g - u \operatorname{div} A(x) \nabla\varphi \\ &\equiv \widetilde{f}(x,t) \quad \text{in} \quad \Omega_L \times [0,\infty)\end{aligned}$$

with $\phi\big|_{\Gamma_0} = \phi\big|_{\partial B_L} = 0$. Applying a result in [24] (see also [5]) we have

Proposition 3.8 (see [24]) *We fix $T > 0$. Let $t \ge T > 0$ and let u be a solution in Theorem* 2.1. *Then for any ε_0 constant with $0 < \varepsilon_0 < 1/2$ the following trace estimate holds*

$$\begin{aligned}&\int_{t-3T/4}^{t-T/4} \int_{\Gamma_1} |\nabla_{\Gamma_g} u|_g^2 d\Gamma ds \\ &\le C \int_{t-T}^t \left[\int_{\Gamma_1} \left(|u_{\nu_A}|^2 + |u_t|^2\right) d\Gamma + \|\widetilde{f}(s)\|^2_{H^{-\frac{1}{2}+\varepsilon_0}(\Omega_L)} \right] ds + C_{\varepsilon_0} \|\phi\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,T,t})},\end{aligned} \tag{48}$$

where we set $Q_{L,T,t} = \Omega_L \times [t-T,t]$. The constants C, C_{ε_0} in the above are independent of t and u.

By interpolation, it is easy to see that

$$\int_{t-T}^t \|\widetilde{f}(s)\|^2_{H^{-\frac{1}{2}+\varepsilon_0}(\Omega_L)} ds \le C \int_{t-T}^t \|\rho(x,u_t)\|^2_{L^2(\Omega_L)} ds + C_{\varepsilon_0} \|u\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,T,t})} \tag{49}$$

and

$$\|\phi\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,T,t})} \le C \|u\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,T,t})}. \tag{50}$$

Hence, from (48)–(50) we obtain

$$\begin{aligned}&\int_{t-3T/4}^{t-T/4} \int_{\Gamma_1} |\nabla_{\Gamma_g} u|_g^2 d\Gamma ds \\ &\le C \int_{t-T}^t \left[\int_{\Gamma_1} \left(|u_{\nu_A}|^2 + |u_t|^2\right) d\Gamma + \|\rho(x,u_t)\|^2_{L^2(\Omega_L)} \right] ds + C_{\varepsilon_0} \|u\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,T,t})}.\end{aligned}$$

Applying the above to $t = iT/2, i = 2, 3, \cdots, 2n$ with $nT \le t < (n+1)T$ and summing up the resulted inequalities we have for $t \ge T$,

$$\int_{T/4}^{t-T/4} \int_{\Gamma_1} |\nabla_{\Gamma_g} u|_g^2 d\Gamma ds$$
$$\le C \int_0^t \left[\int_{\Gamma_1} \big(|u_{\nu_A}|^2 + |u_t|^2 \big) d\Gamma + \|\rho(x, u_t)\|^2_{L^2(\Omega_L)} \right] ds + C_{\varepsilon_0} \|u\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,t})}, \tag{51}$$

where C is a constant independent of t, u and we set $Q_{L,t} = \Omega_L \times [0, t]$.

Using an interpolation theory, Lemma 3.7 and the boundary condition on Γ_1 we have

$$\begin{aligned} C\|u\|^2_{H^{\frac{1}{2}+\varepsilon_0}(Q_{L,t})} &\le \frac{\varepsilon_1}{4} \int_0^t (\|\nabla_g u\|^2 + \|u_t\|^2) ds + C \int_0^t \int_{\Omega_L} |u|^2 dx ds \\ &\le C \left[\int_0^t \int_{\Omega_L} |\rho(x, u_t)|^2 dx ds + \int_0^t \int_{\Gamma_1} \big(|u_{\nu_A}|^2 + |u_t|^2 \big) d\Gamma ds \right] \\ &\quad + C \int_0^t \int_{\Omega_L^c \cup \omega} |u_t|^2 dx ds + \frac{\varepsilon_1}{2} \int_0^t E(s) ds. \end{aligned} \tag{52}$$

Combining the estimates (51) and (52) we have the following lemma.

Lemma 3.9 *We fix $T > 0$. Let $t \ge T > 0$ and let u be a solution in Theorem 2.1. Then for any ε_0 constant with $0 < \varepsilon_0 < 1/2$ the following trace estimate holds*

$$\begin{aligned} \int_{T/4}^{t-T/4} \int_{\Gamma_1} |\nabla_{\Gamma_g} u|_g^2 d\Gamma ds &\le C \left[\int_0^t \int_{\Omega_L} |\rho(x, u_t)|^2 dx ds + \int_0^t \int_{\Gamma_1} \big(|u_{\nu_A}|^2 + |u_t|^2 \big) d\Gamma ds \right] \\ &\quad + C \int_0^t \int_{\Omega_L^c \cup \omega} |u_t|^2 dx ds + \frac{\varepsilon_1}{2} \int_0^t E(s) ds, \end{aligned} \tag{53}$$

where the constants C, C_{ε_0} in the above are independent of t and u.

Lemma 3.10 *Let u be a solution of (1) in Theorem 2.1. There exists a constant $T_0 > 0$ such that if $t > T_0 > 0$, then*

$$E(t) + \|u(t)\|^2 + \varepsilon \int_0^t E(s) ds$$
$$\le C \left[\int_0^t \int_{\Gamma_1} (u_t^2 + u_{\nu_A}^2) d\Gamma ds + \int_0^t \int_{\Omega_R} a(x) u_t^2 dx ds + \int_0^t \int_{\Omega_L} |\rho(x, u_t)|^2 dx ds \right]$$
$$+ C \int_0^t \int_\omega |u_t|^2 dx ds + C\big(E(0) + \|u_0\|^2\big) \tag{54}$$

with some positive constants ε, C.

Proof Integrating (18) over $[\frac{T}{4}, t - \frac{T}{4}]$ with respect to time t, and combining the estimates (47) and (53), we have

$$
\begin{aligned}
& G\left(t-\frac{T}{4}\right)-G\left(\frac{T}{4}\right)+\left(\frac{2\sigma}{3}-C\eta-C\varepsilon_1\right)\int_0^t E(s)ds \\
\le{} & C\left[\int_0^t\int_{\Gamma_1}(u_t^2+u_{\nu_A}^2)d\Gamma ds+\int_0^t\int_{\Omega_R}a(x)u_t^2dxds+\int_0^t\int_{\Omega_L}|\rho(x,u_t)|^2dxds\right] \\
& +C\int_0^t\int_\omega |u_t|^2dxds+CE(0).
\end{aligned} \tag{55}
$$

Using the estimate (37) to the above inequality, and the fact

$$
\|u(t)\|^2\le \mathrm{e}^{\frac{T}{4}}\left(\int_{t-\frac{T}{4}}^t\|u_t(s)\|^2ds+\left\|u\left(t-\frac{T}{4}\right)\right\|^2\right),
$$

we see

$$
\begin{aligned}
G(t)\le C\big(E(t)+\|u(t)\|^2\big) &\le C\left(E\left(t-\frac{T}{4}\right)+\int_{t-\frac{T}{4}}^t\|u_t(s)\|^2ds+\left\|u\left(t-\frac{T}{4}\right)\right\|^2\right) \\
&\le C\left(G\left(t-\frac{T}{4}\right)+\|u(t)\|^2_{\widehat{\Omega}_L}\right),
\end{aligned}
$$

for $t\ge\frac{T}{4}$.

Noting the following estimate

$$
\|u(t)\|^2_{\widehat{\Omega}_L}\le\|u(0)\|^2_{\widehat{\Omega}_L}+\delta\int_0^t\int_{\widehat{\Omega}_L}|u_t|^2dxds+C_\delta\int_0^t\int_{\widehat{\Omega}_L}|u|^2dxds,
$$

we obtain from (55)

$$
\begin{aligned}
& E(t)+\|u(t)\|^2+\left(\frac{2\sigma}{3}-C\eta-C\varepsilon_1-C\delta\right)\int_0^t E(s)ds \\
\le{} & C\left[\int_0^t\int_{\Gamma_1}(u_t^2+u_{\nu_A}^2)d\Gamma ds+\int_0^t\int_{\Omega_R}a(x)u_t^2dxds+\int_0^t\int_{\Omega_L}|\rho(x,u_t)|^2dxds\right] \\
& +C\int_0^t\int_\omega|u_t|^2dxds+C\big(E(0)+\|u(0)\|^2\big),
\end{aligned}
$$

where we could find suitable constants $\delta,\ \eta,\ \varepsilon_1$ such that $\varepsilon=\frac{2\sigma}{3}-C\eta-C\varepsilon_1-C\delta>0$. ∎

4 Proof of Theorem 2.1

To prove Theorem 2.1 we must estimate the following terms appearing in the right side of (54):

$$
\begin{aligned}
I_1(t)&=\int_0^t\int_\omega|u_t|^2dxds, \\
I_2(t)&=\int_0^t\int_{\Omega_L}|\rho(x,u_t)|^2dxd,
\end{aligned}
$$

$$I_3(t) = \int_0^t \int_{\Omega_R} a(x) u_t^2 dxds,$$
$$I_4(t) = \int_0^t \int_{\Gamma_1} (u_t^2 + u_{\nu_A}^2) d\Gamma ds.$$

We devices similar to those in [6]. We set

$$\Omega_R^1(t) = \{x \in \Omega_R \mid |u_t(x,t)| \le 1\} \quad \text{and} \quad \Omega_R^2(t) = \{x \in \Omega_R \mid |u_t(x,t)| \ge 1\}.$$

Similarly, we set

$$\Gamma_1^1(t) = \{x \in \Gamma_1 \mid |u_t(x,t)| \le 1\} \quad \text{and} \quad \Gamma_1^2(t) = \{x \in \Gamma_1 \mid |u_t(x,t)| \ge 1\}.$$

From the energy identity, we see for $t \ge 0$,

$$k_0 \int_0^t \left(\int_{\Omega_R^1} a(x)|u_t|^{p+2} dx + \int_{\Omega_R^2} a(x)|u_t|^{q+2} dx \right) ds \le \int_0^t \int_\Omega \rho(x, u_t) u_t dxds \le CE(0),$$
$$k_0 \int_0^t \left(\int_{\Gamma_1^1} a(x)|u_t|^{r+2} dx + \int_{\Gamma_1^2} a(x)|u_t|^{m+2} dx \right) ds \le \int_0^t \int_\Omega f(u_t) u_t dxds \le CE(0).$$

First, we shall estimate $I_1(t)$, $I_2(t)$ and $I_3(t)$ separately in the different cases.

(i) The case $0 \le p < \infty$ and $0 \le q \le \frac{2}{n-2}$:

$$\begin{aligned} &I_1(t) \\ &\le C \left(\int_0^t \int_{\Omega_R^1} a(x)|u_t|^{p+2} dxds \right)^{\frac{2}{p+2}} \left(\int_0^t \int_{\Omega_R^1} dxds \right)^{\frac{p}{p+2}} + C \int_0^t \int_{\Omega_R^2} a(x)|u_t|^{q+2} dxds \\ &\le C \left(E(0)^{\frac{2}{p+2}} (1+t)^{\frac{p}{p+2}} + E(0) \right), \end{aligned} \tag{56}$$

$$\begin{aligned} &I_2(t) \\ &\le C \int_0^t \left(\int_{\Omega_R^1} a(x)|u_t|^2 dx + \int_{\Omega_R^2} a(x)|u_t|^{2(q+1)} dx \right) ds + \int_0^t \int_{\Omega_L \cap \Omega_R^c} a(x) u_t \rho(x, u_t) dxds \\ &\le C \left(E(0) + E(0)^{\frac{2}{p+2}} (1+t)^{\frac{p}{p+2}} \right) + C \int_0^t \left(\int_{\Omega_R^2} a(x)|u_t|^{q+2} dx \right)^{\frac{2(q+1)(1-\theta_1)}{q+2}} \|u_t\|_{L^{\frac{2n}{n-2}}}^{2(q+1)\theta_1} ds \\ &\le C \left(E(0) + E(0)^{\frac{2}{p+2}} (1+t)^{\frac{p}{p+2}} \right) \\ &\quad + C(E(0) + K^2)^{(q+1)\theta_1} \left(\int_0^t \int_{\Omega_R^2} a(x)|u_t|^{q+2} dxds \right)^{\frac{2(q+1)(1-\theta_1)}{q+2}} \left(\int_0^t \int_{\Omega_R^2} dxds \right)^{\frac{2(q+1)(1-\theta_1)}{q+2}} \\ &\le C \left(E(0) + E(0)^{\frac{2}{p+2}} (1+t)^{\frac{p}{p+2}} + (E(0) + K^2)^{(q+1)\theta_1} E(0)^{\frac{2(q+1)(1-\theta_1)}{q+2}} (1+t)^{\frac{q(n-2)}{4-q(n-2)}} \right) \end{aligned} \tag{57}$$

with $\theta_1 = \frac{nq}{(q+1)(4-q(n-2))}$, and

$$\begin{aligned} I_3(t) &\le C \left(\int_0^t \int_{\Omega_R^1} a(x)|u_t|^{p+2} dxds \right)^{\frac{2}{p+2}} \left(\int_0^t \int_{\Omega_R^1} dxds \right)^{\frac{p}{p+2}} + \int_0^t \int_{\Omega_R^2} a(x)|u_t|^{q+2} dxds \\ &\le C \left(E(0)^{\frac{2}{p+2}} (1+t)^{\frac{p}{p+2}} + E(0) \right). \end{aligned} \tag{58}$$

(ii) The case $-1 < p \le 0$ and $-1 < q \le 0$:

We have

$$I_1(t) \le C\int_0^t \int_{\Omega_R^1} a(x)|u_t|^{p+2}dxds + C\int_0^t \int_{\Omega_R^2} a(x)|u_t|^{2\widetilde{\theta}_1}|u_t|^{2(1-\widetilde{\theta}_1)}dxds$$
$$\le CE(0) + C\int_0^t \Big(\int_{\Omega_R^2} a(x)|u_t|^{q+2}dx\Big)^{\frac{2\widetilde{\theta}_1}{q+2}} \Big(\int_{\Omega_R^2} |u_t|^{\frac{2n}{n-2}}dx\Big)^{1-\frac{2\widetilde{\theta}_1}{q+2}}$$
$$\le CE(0) + C(E(0)+K^2)^{(1-\widetilde{\theta}_1)}E(0)^{\frac{4}{4-q(n-2)}}(1+t)^{\frac{-q(n-2)}{4-q(n-2)}},$$

with $\widetilde{\theta}_1 = \frac{2(q+2)}{4-q(n-2)}$.

Further,

$$I_2(t) \le C\int_0^t \Big(\int_{\Omega_R^1} a(x)|u_t|^{2(p+1)}dx + \int_{\Omega_R^2} a(x)|u_t|^2dx\Big)ds + \int_0^t \int_{\Omega_L\cap\Omega_R^c} u_t\rho(x,u_t)dxds$$
$$\le C\Big(E(0) + (E(0)+K^2)^{(1-\widetilde{\theta}_1)}E(0)^{\frac{4}{4-q(n-2)}}(1+t)^{\frac{-q(n-2)}{4-q(n-2)}}\Big)$$
$$+C\Big(\int_0^t \int_{\Omega_R^1} a(x)|u_t|^{p+2}dxds\Big)^{\frac{2(p+1)}{p+2}} \Big(\int_0^t \int_{\Omega_R^1} dxds\Big)^{\frac{-p}{p+2}}$$
$$\le C\Big(E(0) + (E(0)+K^2)^{(1-\widetilde{\theta}_1)}E(0)^{\frac{4}{4-q(n-2)}}(1+t)^{\frac{-q(n-2)}{4-q(n-2)}} + E(0)^{\frac{2(p+1)}{p+2}}(1+t)^{\frac{-p}{p+2}}\Big),$$
$$I_3(t) \le C\int_0^t \int_{\Omega_R^1} a(x)|u_t|^{p+2}dxds + C\int_0^t \int_{\Omega_R^2} a(x)|u_t|^2dxds$$
$$\le C\Big(E(0) + (E(0)+K^2)^{(1-\widetilde{\theta}_1)}E(0)^{\frac{4}{4-q(n-2)}}(1+t)^{\frac{-q(n-2)}{4-q(n-2)}}\Big).$$

(iii) Other cases $-1 < p < 0$, $0 \le q \le \frac{2}{n-2}$ and $0 \le p < \infty$, $-1 < q < 0$ are treated by combining the above estimations.

Next, we consider the boundary integral $I_4(t)$. Again we separate the cases.

The case $0 \le r < \infty$ and $0 \le m \le \frac{1}{n-2}$:

From the boundary condition and assumption on $f(u_t)$, we have

$$I_4(t) \le C\int_0^t \int_{\Gamma_1^1} |u_t|^2 d\Gamma ds + C\int_0^t \int_{\Gamma_1^1} |u_t|^{2(m+1)}d\Gamma ds.$$

Here,

$$\int_0^t \int_{\Gamma_1^1} |u_t|^2 d\Gamma ds \le \Big(\int_0^t \int_{\Gamma_1^1} |u_t|^{r+2}d\Gamma ds\Big)^{\frac{2}{r+2}} \Big(\int_0^t \int_{\Gamma_1} d\Gamma ds\Big)^{\frac{r}{r+2}}$$
$$\le CE(0)^{\frac{2}{r+2}}(1+t)^{\frac{r}{r+2}}.$$

Also, by the Gagliardo-Nirenberg inequality and trace theorem, we see that

$$\|u_t(t)\|_{L^{2(m+1)}(\Gamma_1^2)} \le C\|u_t(t)\|^{1-\theta_2}_{L^{m+2}(\Gamma_1^2)}\|u_t(t)\|^{\theta_2}_{H^{\frac{1}{2}}(\Gamma_1^2)}$$
$$\le C\|u_t(t)\|^{1-\theta_2}_{L^{m+2}(\Gamma_1^2)}\|u_t(t)\|^{\theta_2}_{H^1(\Omega_R)}, \qquad (59)$$

where $\theta_2 = \frac{m(n-1)}{(m+1)(2-m(n-2))}$.

Hence,

$$\int_0^t \int_{\Gamma_1^2} |u_t|^{2(m+1)} d\Gamma ds \le C \int_0^t \left(\int_{\Gamma_1^2} |u_t|^{m+2} d\Gamma \right)^{\frac{2(m+1)(1-\theta_2)}{m+2}} \|u_t\|_{H^1(\Omega_R)}^{2(m+1)\theta_2} ds$$
$$\le C(E(0)+K^2)^{(m+1)\theta_2} E(0)^{\frac{2(m+1)(1-\theta_2)}{m+2}} (1+t)^{\frac{-m+2\theta_2(m+1)}{m+2}}. \quad (60)$$

Note that

$$\frac{-m+2\theta_2(m+1)}{m+2} = \frac{m(n-2)}{2-m(n-2)}.$$

Thus, we have from (59) and (60),

$$I_4(t) \le CE(0)^{\frac{2}{r+2}} (1+t)^{\frac{r}{r+2}} + C(E(0)+K^2)^{(m+1)\theta_2} E(0)^{\frac{2(m+1)(1-\theta_2)}{m+2}} (1+t)^{\frac{m(n-2)}{2-m(n-2)}}. \quad (61)$$

The case $-1 < r \le 0$, $-1 < m \le 0$:

In this case we see that

$$I_4(t) \le C \int_0^t \int_{\Gamma_1^1} |u_t|^{2(r+1)} d\Gamma ds + C \int_0^t \int_{\Gamma_1^1} |u_t|^2 d\Gamma ds$$

and, instead of (59),

$$\int_0^t \int_{\Gamma_1^1} |u_t|^{2(r+1)} d\Gamma ds \le \left(\int_0^t \int_{\Gamma_1^1} |u_t|^{r+2} d\Gamma ds \right)^{\frac{2(r+1)}{r+2}} \left(\int_0^t \int_{\Gamma_1} d\Gamma ds \right)^{\frac{-r}{r+2}}$$
$$\le CE(0)^{\frac{2(r+1)}{r+2}} (1+t)^{\frac{-r}{r+2}}.$$

Also,

$$\int_0^t \int_{\Gamma_1^2} |u_t|^2 d\Gamma ds \le C \int_0^t \|u_t\|_{L^{m+2}(\Gamma_1^2)}^{2(1-\widetilde{\theta}_2)} \|u_t\|_{H^{\frac{1}{2}}(\Gamma_1^2)}^{2\widetilde{\theta}_2} ds$$
$$\le C(E(0)+K^2)^{\widetilde{\theta}_2} \left(\int_0^t \int_{\Gamma_1^2} |u_t|^{m+2} d\Gamma ds \right)^{\frac{2(1-\widetilde{\theta}_2)}{m+2}} \left(\int_0^t ds \right)^{1-\frac{2(1-\widetilde{\theta}_2)}{m+2}}$$
$$\le C(E(0)+K^2)^{\widetilde{\theta}_2} E(0)^{\frac{2(1-\widetilde{\theta}_2)}{m+2}} (1+t)^{1-\frac{2(1-\widetilde{\theta}_2)}{m+2}},$$

where $\widetilde{\theta}_2 = \frac{-m(n-1)}{-m(n-2)+2}$ and hence,

$$1 - \frac{2(1-\widetilde{\theta}_2)}{m+2} = \frac{-m(n-2)}{2-m(n-2)} < 1.$$

Therefore,

$$I_4(t) \le C(E(0)+K^2)^{\widetilde{\theta}_2} E(0)^{\frac{2}{2-m(n-2)}} (1+t)^{\frac{-m(n-2)}{2-m(n-2)}} + CE(0)^{\frac{2(r+1)}{r+2}} (1+t)^{\frac{-r}{r+2}}. \quad (62)$$

Other cases $-1 < r \le 0$, $0 \le m \le \frac{2}{n-2}$ and $0 \le r < \infty$, $-1 < m \le 0$ can be treated similarly.

For the proof of Theorem 2.1 it will be sufficient to consider the cases:
1) $0 \le p < \infty$, $0 \le q \le \frac{2}{n-2}$, $0 \le r < \infty$ and $0 \le m \le \frac{1}{n-2}$,
2) $-1 < p, q, r, m \le 0$, since other cases can be treated similarly.
The case 1) $0 \le p < \infty$, $0 \le q \le \frac{2}{n-2}$, $0 \le r < \infty$ and $0 \le m \le \frac{2}{n-2}$.
We have from (56), (57), (58) and (61),

$$\begin{aligned} &E(t) + \|u(t)\|^2 + \varepsilon \int_0^t E(s)ds \\ &\le CE(0) + C\|u_0\|^2 + CE(0)^{\frac{2}{p+2}}(1+t)^{\frac{p}{p+2}} + CE(0)^{\frac{2}{r+2}}(1+t)^{\frac{r}{r+2}} \\ &\quad + C(E(0)+K^2)^{(m+1)\theta_2} E(0)^{\frac{2(m+1)(1-\theta_2)}{m+2}} (1+t)^{\frac{m(n-2)}{2-m(n-2)}} \\ &\quad + C(E(0)+K^2)^{(q+1)\theta_1} E(0)^{\frac{2(q+1)(1-\theta_1)}{q+2}} (1+t)^{\frac{q(n-2)}{4-q(n-2)}} \\ &\le C(Q_0 + \|u_0\|^2)(1+t)^{\vartheta}, \end{aligned}$$

where we set

$$\begin{aligned} Q_0 = {} & E(0) + E(0)^{\frac{2}{p+2}} + E(0)^{\frac{2}{r+2}} + (E(0)+K^2)^{(q+1)\theta_1} E(0)^{\frac{2(q+1)(1-\theta_1)}{q+2}} \\ & + (E(0)+K^2)^{(m+1)\theta_2} E(0)^{\frac{2(m+1)(1-\theta_2)}{m+2}} \end{aligned}$$

and

$$\vartheta = \max\left\{ \frac{p}{p+2}, \frac{r}{r+2}, \frac{m(n-2)}{2-m(n-2)}, \frac{q(n-2)}{4-q(n-2)} \right\}.$$

Then the above inequality implies

$$\|u(t)\|^2 \le C(Q_0 + \|u_0\|^2)(1+t)^{\vartheta}.$$

By the inequality (14), we have

$$\frac{d}{dt}\big[(1+t)E(t)\big] = E(t) + (1+t)\frac{d}{dt}E(t) \le E(t). \tag{63}$$

Integrating (63) over $(0,t)$, we have

$$(1+t)E(t) \le E(0) + \int_0^t E(s)ds \le C(Q_0 + \|u_0\|^2)(1+t)^{\vartheta}$$

and so we can conclude

$$E(t) \le C(Q_0 + \|u_0\|^2)(1+t)^{\vartheta-1}.$$

The case 2) $-1 < p, q, r, m \le 0$, since other cases can be treated similarly.
In this case we have, instead of (63),

$$E(t) + \|u(t)\|^2 + \varepsilon \int_0^t E(s)ds \le C(Q_0 + \|u_0\|^2)(1+t)^{\vartheta}$$

with

$$Q_0 = E(0) + (E(0) + K^2)^{(1-\widetilde{\theta}_1)} E(0)^{\frac{4}{4-q(n-2)}} + E(0)^{\frac{2(p+1)}{p+2}} \\ + (E(0) + K^2)^{\widetilde{\theta}_2} E(0)^{\frac{2}{2-m(n-2)}} + E(0)^{\frac{2(r+1)}{r+2}}$$

and

$$\vartheta = \max\left\{\frac{-q(n-2)}{4-q(n-2)}, \frac{-p}{p+2}, \frac{-m(n-2)}{2-m(n-2)}, \frac{-r}{r+2}\right\}.$$

The inequality again gives the desired estimates for $\|u(t)\|^2$ and $E(t)$. This completes the proof of Theorem 2.1.

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