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The Determination on Weight Hierarchies of q-Ary Linear Codes of Dimension 5 in Class IV*

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Abstract The weight hierarchy of a linear $[n; k; q]$ code C over $GF(q)$ is the sequence $(d_1, d_2, \cdots, d_k)$ where d_r is the smallest support of any r-dimensional subcode of C. "Determining all possible weight hierarchies of general linear codes" is a basic theoretical issue and has important scientific significance in communication system. However, it is impossible for q-ary linear codes of dimension k when q and k are slightly larger, then a reasonable formulation of the problem is modified as: "Determine almost all weight hierarchies of general q-ary linear codes of dimension k". In this paper, based on the finite projective geometry method, the authors study q-ary linear codes of dimension 5 in class IV, and find new necessary conditions of their weight hierarchies, and classify their weight hierarchies into 6 subclasses. The authors also develop and improve the method of the subspace set, thus determine almost all weight hierarchies of 5-dimensional linear codes in class IV. It opens the way to determine the weight hierarchies of the rest two of 5-dimensional codes (classes III and VI), and break through the difficulties. Furthermore, the new necessary conditions show that original necessary conditions of the weight hierarchies of k-dimensional codes were not enough (not most tight nor best), so, it is important to excogitate further new necessary conditions for attacking and solving the k-dimensional problem.

Keywords Difference sequence, q-ary linear code of dimension 5, weight hierarchy.

1 Introduction

Coding theory is an important part of information theory. Hamming, who is well-known coding scientist and one of the founders of coding theory, raised the concept of Hamming weight. The weight hierarchy of a linear code of dimension k is a sequence $(d_1, d_2, \cdots, d_k)$.

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These parameters were first introduced by Wei (see [1]). It is important in the analysis of the application of a linear code to the wiretap channel of type II (see [1]), the estimation of the trellis complexity of linear codes, and the analysis of linear codes for error detection on the local binomial channel. In short, the weight hierarchy is a sequence of basic important parameters and closely related to the design and security of communication systems. In [2] Klϕve etc proposed "determining all possible weight hierarchies of general linear codes", which is a basic theoretical issue of important scientific significance in communication system. The possible weight hierarchies of binary linear codes of dimension up to 4 were determined in [2, 3].

In 1996, Chen and Klϕve introduced the finite projective geometry method, that was first effectively used to study the weight hierarchies of q-ary linear codes of dimension 4 (see [4]). The weight hierarchies of linear codes of dimension 4 were split into 9 classes in [5], and there are a wealth of classified researches using the finite projective geometry method (see [5–7], etc.). However, the number of those unknown sequences increases sharply when q and k increase (see [8]). And we cannot determine whether those unknown sequences are weight hierarchies or not. So it is impossible to determine all weight hierarchies of q-ary linear codes of dimension k. A reasonable formulation of the problem is "determine almost all weight hierarchies of q-ary linear codes of dimension k", that was first introduced by Chen and Klϕve in 2003 (see [9]). This is a difficult problem. So far this problem was solved only for some small classes of k-dimensional codes (such as class I, see [9]), and 3-dimensional codes (see [8]) and 4-dimensional ones (see [10]).

There is little study about 5-dimensional codes. In [11], the weight hierarchies of binary linear codes of dimension 5 satisfying the chain condition were determined with the aid of computer. Thus, it is meaningful to further determine the weight hierarchies of 5-dimensional general linear codes. According to the method used in [5], we can classify higher dimensional linear codes. However, the number of classes obtained by this classification method is rapidly expanded with the increase of the dimension. There are 114 classes in 5-dimensional codes and it is difficult to determine one by one. Based on the necessary conditions in [3], The authors of this paper split 5-dimensional codes and their weight hierarchies into six classes (see [12]), greatly reducing the number of ones. Then we may improve research results of the 4-dimensional codes up to five-dimensional ones. "Determine almost all weight hierarchies of q-ary linear codes of dimension 5" is a challenging new topic, which is much more difficult and complicated than the corresponding problem of the 4-dimensional codes, and there is little study in this area. The authors of this paper developed the method of the subspace set which was first introduced in [9], and extended the fall method in [10], and determined the weight hierarchies of almost all linear codes of dimension 5 in class II using the finite projective geometry method (see [12]). Later, in [13] the authors of this paper determined the weight hierarchies of almost all linear codes in class V. Class IV studied in this paper face more difficulties. The necessary conditions of class IV in [12] were not most tight nor best, by which we can not determine almost all weight hierarchies of 5-dimensional linear codes in class IV. In order to solve this problem, in this paper we found four new necessary conditions by the finite projective geometry method, thus modified the necessary conditions of the weight hierarchies of 5-dimensional linear codes in class IV to be most tight and best. Further, we classified the weight hierarchies of 5-dimensional linear codes in class IV into 6 subclasses, improved the fall method, and completed the determination on almost all weight hierarchies of 5-dimensional linear codes in class IV. It opens the way to determine the weight hierarchies of the rest two of 5-dimensional codes (classes III and VI), and break through the difficulties. Furthermore, the new necessary conditions show that necessary conditions of the weight hierarchies of k-dimensional codes in [3] were not enough (not most tight nor best), so, it is important to excogitate further new necessary conditions for attacking and solving the k-dimensional problem.

2 Preliminaries

Throughout this paper, unless otherwise stated, C denotes a $[n,k;q]$ code, that is, a linear code of length n and dimension k over $GF(q)$. For any subcode D of C , the support of D is the set of positions where not all the codewords of D are zero, and we denote it by $\chi(D)$. Further, the support weight of D is the size of $\chi(D)$, and we denote it by $\omega_s(D)$.

For $1 \leq r \leq k$, the r-th minimum support weight (or Generalized Hamming weight) of C is defined by $d_r = d_r(C) = \min\{\omega_s(D)|D$ is a $[n,r;q]$ subcode of $C\}$. The sequence $(d_1, d_2, \cdots, d_k)$ is the weight hierarchy of C.

Without loss of generality, we may assume $n = d_k$. The difference sequence (DS) $(i_0, i_1, \cdots, i_k)$ of a $[n,k;q]$ code is defined by $i_r = d_{k-r} - d_{k-r-1}$ for $0 \leq r \leq k-1$, where $d_0 = 0$.

The difference sequence can easily be computed from the weight hierarchy and vice versa. Therefore, "determining the weight hierarchy" is equivalent to "determining the difference sequence".

Let G be a generator matrix for C. For any $x \in GF(q)^5$, $m(x)$ denotes the number of occurrences of x as a column in G. If y is a column in the generator matrix G, and $x = \alpha y$ for some nonzero $\alpha \in GF(q)$, then we may replace y by x without changing the support weight of any subcode. Therefore, we assume that all columns in G are non-zero. and we may describe the columns in G by points in the projective space $PG(4,q)$. Let V_4 be the projective space $PG(4,q)$. A value assignment is a function $m: \ V_4 \to N, \ N = \{0, 1, \cdots\}$.

For any point $p \in PG(4,q)$, we call $m(p)$ the value (or weight) of p. We use the following further notation: $m(S) = \sum_{p \in S} m(p)$ for $S \subset PG(4,q)$.

In [4], it was proved that the existence of a code with weight hierarchy $(d_1, d_2, d_3, d_4, d_5)$ is equivalent to the existence of a value assignment m such that:

$$\max\{m(U_r)|U_r \text{ is } r\text{-dimensional subspace of } V_4\} = \sum_{j=0}^{r} i_j, \quad 0 \leq r \leq 4. \tag{1}$$

Let p^*, l^*, P^*, V^* be the heaviest point, line, plane and body respectively, while the function take the maximum value of the right side of (1) as $r = 0, 1, 2, 3$. The core of the finite projective geometry method for determining almost all weight hierarchies of q-ary linear codes is that: First find the most tight and best necessary conditions of the difference sequences using the geometric method, and then construct the function m satisfying (1) as evenly as possible for almost all i_j satisfying this conditions.

3 Main Results

Definition 3.1 Let $N(i)$ be the number of difference sequences satisfying the sufficient condition of some class with $i_0 \leq i$, and $M(i)$ be the number of sequences satisfying the necessary condition of the same class with $i_0 \leq i$. If $\lim_{i\to\infty} \frac{N(i)}{M(i)} = 1$, we call the necessary condition almost sufficient.

In [12], The necessary conditions of the difference sequence of 5-dimensional linear codes were split into 6 classes, and there is no public sequence in arbitrary two classes. In this case, the difference sequences (the weight hierarchies) of 5-dimensional linear codes were split into 6 classes.

Definition 3.2 We call a linear code q-ary linear code of dimension 5 in class IV, if the necessary conditions for the difference sequence $(i_0, i_1, i_2, i_3, i_4)$ of the linear code are

$$\begin{aligned}&i_1 \leq qi_0, \quad qi_1 < i_2 \leq \frac{q^2}{q+1}(i_0+i_1), \quad i_3 \leq \frac{q^2}{q+1}(i_1+i_2),\\ &\max\left\{1, \frac{q}{q-1}(i_0-i_3)\right\} \leq i_4 \leq \min\{qi_3, (q^3+q^2+q)i_1 - i_2 - i_3\}.\end{aligned}$$

In this paper, we study the difference sequences of q-ary linear codes of dimension 5 in class IV. The necessary conditions of class IV in [12] were not most tight nor best. In Theorem 3.3 below, we add several new necessary conditions to the difference sequences in class IV, and show that the new necessary conditions are almost sufficient in class IV.

Theorem 3.3 *For q-ary linear codes of dimension* 5, *the necessary and almost sufficient conditions for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence of class IV are*

(i) $\frac{i_0}{q^2} < i_1 \leq qi_0$;

(ii) $qi_1 < i_2 \leq \min\left\{\frac{q^2}{q+1}(i_0+i_1), (q^2+q)i_1 - i_0\right\}$;

(iii) $1 \leq i_4 \leq qi_3$ $(if\ i_0 \leq i_3 \leq (q^2+q)i_1 - i_2\ )$;

(iv) $i_0 \leq i_4 \leq \min\{(q^3+q^2+q)i_1 - i_2 - i_3, (q^2+q)i_1 - i_2 + (q-1)i_3\}$ $\left(if\ (q^2+q)i_1 - i_2 < i_3 \leq \frac{q^2}{q+1}(i_1+i_2)\right)$.

Good sufficient conditions can determine almost all weight hierarchies of linear codes of dimension 5 in class IV.

4 New Necessary Conditions and Classification

We first deduce new key necessary conditions.

From [13], we have $p^* \notin P^*$ if $i_2 > qi_1$. Then p^* and P^* determine a body V, and we have $i_0 + i_0 + i_1 + i_2 = m(p^*) + m(P^*) \leq m(V) \leq i_0 + i_1 + i_2 + i_3$, so we get: $i_3 \geq i_0$.

If $p^* \in V^*$, we have $i_0 + i_1 + i_2 + i_3 = m(V^*) = m(p^*) + \sum_{p^*\in l\subset V^*}(m(l) - m(p^*)) \leq i_0 + (q^2+q+1)i_1$, and so we get: $i_3 \leq (q^2+q)i_1 - i_2$.

If $p^* \notin V^*$, we have $i_0+i_1+i_2+i_3+i_4 = m(V_4) \geq m(p^* \cup V^*) = i_0+i_0+i_1+i_2+i_3$, and so we get: $i_4 \geq i_0$.

If $p^* \notin V^*$, marking the body determined by p^*, P^* with V_1, then $m(V_1) \leq i_0+(q^2+q+1)i_1$, and $m(V_1)-m(P^*) \leq (q^2+q)i_1-i_2$. Because in V_4 there are $q+1$ bodies through P^*, so $m(P^*)+\frac{i_3+i_4-(m(V_1)-m(P^*))}{q} \leq m(V^*)$, and we get: $i_4 \leq (q^2+q)i_1-i_2+(q-1)i_3$.

Four new necessary conditions are got.

Because $qi_3-((q^3+q^2+q)i_1-i_2-i_3) = (q+1)i_3-((q^3+q^2+q)i_1-i_2)$, $(q^3+q^2+q)i_1-i_2-i_3-((q^2+q)i_1-i_2+(q-1)i_3) = q(q^2i_1-i_3)$ and $qi_3-((q^2+q)i_1-i_2+(q-1)i_3) = i_3-((q^2+q)i_1-i_2)$, and it is clear that $(q^2+q)i_1-i_2 < \frac{(q^3+q^2+q)i_1-i_2}{q+1} < q^2i_1 < \frac{q^2}{q+1}(i_1+i_2)$, so, let the upper bound of i_4 be qi_3 if $i_0 \leq i_3 \leq (q^2+q)i_1-i_2$, let the upper bound of i_4 be $(q^2+q)i_1-i_2+(q-1)i_3$ if $(q^2+q)i_1-i_2 < i_3 < q^2i_1$, let the upper bound of i_4 be $(q^3+q^2+q)i_1-i_2-i_3$ if $q^2i_1 \leq i_3 \leq \frac{q^2}{q+1}(i_1+i_2)$.

It was known in [13] that we have $i_2 \leq (q^2+q)i_1-i_0$ and $i_1 > \frac{i_0}{q^2}$ if $i_2 > qi_1$. Then let the upper bound of i_2 be $\frac{q^2}{q+1}(i_0+i_1)$ if $\frac{i_0}{q} < i_1 \leq qi_0$, and let the upper bound of i_2 be $(q^2+q)i_1-i_0$ if $\frac{i_0}{q^2} < i_1 \leq \frac{i_0}{q}$.

Hence, we can classify the necessary conditions of the difference sequences in class IV into 6 disjoint subclasses: $\mathrm{IV}_1, \mathrm{IV}_2, \mathrm{IV}_3, \mathrm{IV}_4, \mathrm{IV}_5, \mathrm{IV}_6$.

IV_1: $\frac{i_0}{q} < i_1 \leq qi_0,\ qi_1 < i_2 \leq \frac{q^2}{q+1}(i_0+i_1),\ i_0 \leq i_3 \leq (q^2+q)i_1-i_2,\ 1 \leq i_4 \leq qi_3$;

IV_2: $\frac{i_0}{q} < i_1 \leq qi_0,\ qi_1 < i_2 \leq \frac{q^2}{q+1}(i_0+i_1),\ (q^2+q)i_1-i_2 < i_3 < q^2i_1,\ i_0 \leq i_4 \leq (q^2+q)i_1-i_2+(q-1)i_3$;

IV_3: $\frac{i_0}{q} < i_1 \leq qi_0,\ qi_1 < i_2 \leq \frac{q^2}{q+1}(i_0+i_1),\ q^2i_1 \leq i_3 \leq \frac{q^2}{q+1}(i_1+i_2),\ i_0 \leq i_4 \leq (q^3+q^2+q)i_1-i_2-i_3$;

IV_4: $\frac{i_0}{q^2} < i_1 \leq \frac{i_0}{q},\ qi_1 < i_2 \leq (q^2+q)i_1-i_0,\ i_0 \leq i_3 \leq (q^2+q)i_1-i_2,\ 1 \leq i_4 \leq qi_3$;

IV_5: $\frac{i_0}{q^2} < i_1 \leq \frac{i_0}{q},\ qi_1 < i_2 \leq (q^2+q)i_1-i_0,\ (q^2+q)i_1-i_2 < i_3 < q^2i_1,\ i_0 \leq i_4 \leq (q^2+q)i_1-i_2+(q-1)i_3$;

IV_6: $\frac{i_0}{q^2} < i_1 \leq \frac{i_0}{q},\ qi_1 < i_2 \leq (q^2+q)i_1-i_0,\ q^2i_1 \leq i_3 \leq \frac{q^2}{q+1}(i_1+i_2),\ i_0 \leq i_4 \leq (q^3+q^2+q)i_1-i_2-i_3$.

As the necessary conditions of the difference sequences in class IV satisfying $\bigcup_{i=1}^{6} \mathrm{IV}_i = \mathrm{IV}$, the proof of Theorem 3.3 is transformed into proving that the necessary conditions of the difference sequences in classes IV_1–IV_6 are almost sufficient, that is, we find the sufficient conditions of the difference sequences in classes IV_1–IV_6, which are very close to the necessary conditions.

5 Sufficient Conditions of Class IV_1

In order to find the sufficient conditions of the difference sequences in class IV_1 (Theorem 5.4), firstly we construct an assignment function m satisfying (1) when i_j is the bound value. We call the construction as the bound construction.

Lemma 5.1 *Let*

$$i_1 = qi_0 - (q+1), \tag{2}$$

$$i_2 = qi_1 + q, \tag{3}$$

$$i_3 = (q^2+q)i_1 - i_2, \tag{4}$$

$$i_4 = qi_3, \tag{5}$$

where i_1, i_3, i_4 are the upper bounds, i_2 is the lower bound (*when i_1 is the upper bound $qi_0-(q+1)$, the upper bound of i_2 is equal to its lower bound*). *Then the bound sequence $(i_0, i_1, i_2, i_3, i_4)$ is the difference sequence.*

Proof From the bounds, we can get:

$$\begin{aligned} &m(l^*) = i_0 + i_1 = (q+1)i_0 - (q+1), \\ &m(P^*) = i_0 + i_1 + i_2 = (q^2+q+1)i_0 - (q^2+q+1), \\ &m(V^*) = i_0 + i_1 + i_2 + i_3 = (q^3+q^2+q+1)i_0 - (q^3+2q^2+2q+1). \end{aligned}$$

Let $PG(4,q)$ be the 4-dimensional polyhedron of which five points e_1, e_2, e_3, e_4, e_5 not in a body are vertexes, shown in Figure 1(a). Let $\langle x_1, x_2, \cdots, x_t\rangle$ be the subspace of $PG(4,q)$ of dimension $t-1$ which is determined by the points $x_1, x_2, \cdots, x_t$.

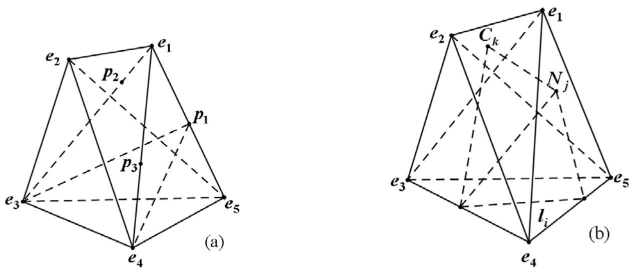
Let p_1 be a given point on $\langle e_1, e_5\rangle\backslash\{e_1, e_5\}$, p_2 be a given point on $\langle e_1, e_3\rangle\backslash\{e_1, e_3\}$, and p_3 be a given point on $\langle e_1, e_4\rangle\backslash\{e_1, e_4\}$.

We construct the function $m(x)$ as follows:

$$m(x) = \begin{cases} i_0, & x = e_1, \\ i_0 - 3, & x \in \langle e_2, e_3, p_1\rangle\backslash\langle e_3, p_1\rangle, \\ i_0 - 2, & x \in (\langle e_2, e_3, e_4, p_1\rangle \cup \langle e_3, p_3\rangle \cup \{p_2\}) \\ & \qquad \backslash(\langle e_3, e_4\rangle \cup (\langle e_2, e_3, p_1\rangle\backslash\langle e_3, p_1\rangle)), \\ i_0 - 1, & \text{others}. \end{cases}$$

Let e_1 be p^*, $\langle e_3, e_4\rangle$ be l^*, $\langle e_3, e_4, e_5\rangle$ be P^*, $\langle e_1, e_3, e_4, e_5\rangle$ be V^* (in fact, the plane through $\langle e_3, e_4, e_5\rangle$ is all V^*). It is easy to prove that $m(\cdot)$ satisfies the condition (1). ▌

In order to get general construction from the bound construction above, we will prove that i_1 can decrease to be near its lower bound with body sets, retaining that other i_j is still bound value (Lemma 5.2); and then we will prove that i_2 can increase to near its upper bound, retaining that i_3, i_4 are still bound values (Lemma 5.3); finally, we will decrease i_3, i_4 to near their lower bounds (Theorem 5.4).



(a) Graph for the bound construction (b) Graph used to decrease i_1

Figure 1 Graph for the bound construction (Figure(a)) of class IV_1 in $PG(4,q)$ and graph used to decrease i_1 (Figure(b)), where $\langle l_i, N_j, C_k\rangle$ is body in $PG(4,q)$

Lemma 5.2 *For all sequences* $(i_0, i_1, i_2, i_3, i_4)$ *satisfying* (3)–(5)*, if*

$$\frac{i_0}{q} + f_1(q) \le i_1 \le q i_0 - f_2(q), \tag{6}$$

where $f_1(q) = (q^7+q^3)(q^2-1)+2q-1$, $f_2(q) = (q^8+2q^7+q^5)(q^2-1)+q+1$, *then* $(i_0, i_1, i_2, i_3, i_4)$ *is the difference sequence.*

Proof Let l_i be the line in $\langle e_3, e_4, e_5\rangle$ except $\langle e_3, e_4\rangle$, $0 \le i < q^2+q$; $N_j \in \langle e_1, e_3, e_4, e_5\rangle \setminus (\langle e_3, e_4, e_5\rangle \cup \langle e_1, l_i\rangle)$, $0 \le j < q^3 - q^2$; $C_k \in V_4 \setminus \langle e_1, e_3, e_4, e_5\rangle$, $0 \le k < q^4$. All $\langle l_i, N_j, C_k\rangle$ form a body set (see Figure 1(b)).

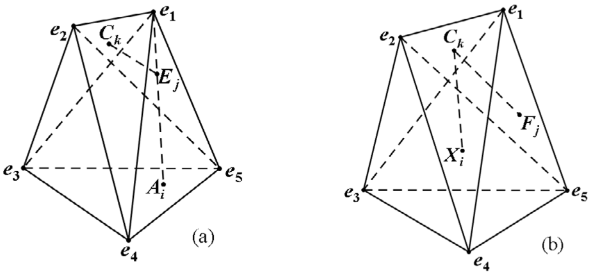
Based on $m(x)$, and modifying it slightly, we construct the function $m'(x)$ as follows:

$$m'(x) = \begin{cases} m(x) - 1, & x \in \langle l_i, N_j, C_k\rangle, \\ m(x), & \text{others.} \end{cases}$$

Each body in the body set and each line in V_4 at least intersect at one point, hence the value of each line decrease 1 (abbreviated as $\downarrow 1$) after modifying one time. $\langle e_1, e_2\rangle \downarrow 1$, it is still l^*. i_0 has no change, $i_1 \downarrow 1$. Similarly, body and plane at least intersect at one line, plane at least $\downarrow (q+1)$. $\langle e_1, e_3, e_4\rangle \downarrow (q+1)$, it is still P^*. Body and another body at least intersect at one plane. Body at least $\downarrow (q^2+q+1)$, $\langle e_1, e_3, e_4, e_5\rangle \downarrow (q^2+q+1)$, it is still V^*. $i_2, i_3, i_4 \downarrow q, q^2, q^3$ respectively, (1) still holds. The sequence after decreasing is still the difference sequence, and the bound formulas (3)–(5) still meet. This is the benefit of the method of subspace set like as body set.

There are $q^7(q^2-1)$ bodies like as $\langle l_i, N_j, C_k\rangle$. Using the $q^7(q^2-1)$ bodies one by one and function $m'(x)$ after iteratively modified $q^7(q^2-1)$ times (say i_1 decreases by one cycle), $i_1 \downarrow q^7(q^2-1)$. The value of every point on $\langle e_3, e_4\rangle \downarrow q(q^3-q^2)q^4 = q^7(q-1)$, The value of every point on $\langle e_3, e_4, e_5\rangle \setminus \langle e_3, e_4\rangle \downarrow (q+1)(q^3-q^2)q^4 = q^6(q^2-1)$ (for each line through this point, because $\langle l_i, N_j\rangle$ is not through point e_1, so there are q^3-q^2 points N_j that can be selected). The value of every point on $\langle e_1, e_3, e_4, e_5\rangle \setminus (\langle e_3, e_4, e_5\rangle \cup \langle e_1, e_3, e_4\rangle)$ (altogether q^3-q^2 points) $\downarrow (q^2+q-q-1)q^4q^2 = q^6(q^2-1)$. The value of every point on $\langle e_1, e_3, e_4\rangle \setminus (\{e_1\} \cup \langle e_3, e_4\rangle)$ (altogether q^2-1 points) $\downarrow q^2q^4q^2 = q^8$. The value of every point out of $\langle e_1, e_3, e_4, e_5\rangle$ (altogether q^4 points) $\downarrow (q^2+q)(q^3-q^2)q^3 = q^6(q^2-1)$.

In order to make the values of all points larger than 0, computing the cycle times ω_1 for i_1 decreasing: $(i_0-3)-\omega_1 q^8 \geq 0$, we get $\omega_1 \leq \frac{i_0-3}{q^8}$. Let $\omega_1=\lfloor\frac{i_0-3}{q^8}\rfloor$, then i_1 can decrease to $qi_0-(q+1)-\omega_1 q^7(q^2-1) < qi_0-(q+1)-\big(\frac{i_0-3}{q^8}-1\big)q^7(q^2-1)=\frac{i_0}{q}+\frac{3(q^2-1)}{q}+q^7(q^2-1)-(q+1)$. The proof is completed. ∎



(a) Graph used to increase i_2 (b) Graph used to decrease i_3

Figure 2 Graph used to increase i_2 (Figure(a)) and graph used to decrease i_3 (Figure(b)) in class IV_1

Lemma 5.3 *For all sequences* (i_0,i_1,i_2,i_3,i_4) *satisfying* (4)–(6), *if*

$$qi_1+f_3(q)\leq i_2\leq \frac{q^2}{q+1}(i_0+i_1)-f_4(q), \tag{7}$$

where $f_3(q)=q$,$f_4(q)=(q^9+2q^8+2q^6)(q-1)$, *then* (i_0,i_1,i_2,i_3,i_4) *is the difference sequence.*

Proof Let $A_i\in\langle e_3,e_4,e_5\rangle\setminus\langle e_3,e_4\rangle,\ 0\leq i<q^2$; $E_j\in\langle e_1,A_i\rangle\setminus\{e_1,A_i\},\ 1\leq j\leq q-1$; $C_k\in V_4\setminus\langle e_1,e_3,e_4,e_5\rangle,\ 0\leq k<q^4$ (see Figure 2(a)).

We construct the function $m''(x)$ as follows:

$$m''(x)=\begin{cases} m'(x)+1, & x=A_i,\\ m'(x)-1, & x\in\langle E_j,C_k\rangle,\\ m'(x), & \text{others},\end{cases}$$

where $m'(x)$ is the corresponding assignment function (after iterating repeatedly) after i_1 taking some value in (6).

There are $q^2(q-1)q^4=q^6(q-1)$ groups of points like as A_i,E_j,C_k. After one cycle, $A_i\uparrow q^4(q-1)$. The value of every point on $\langle e_1,e_3,e_4,e_5\rangle\setminus(\langle e_3,e_4,e_5\rangle\cup\langle e_1,e_3,e_4\rangle)$ (altogether q^3-q^2 points) $\downarrow q^4$. The value of every point out of $\langle e_1,e_3,e_4,e_5\rangle$ (altogether q^4 points) $\downarrow (q^3-q^2)q=q^3(q-1)$.

Suppose $i_2\uparrow\omega_2$ cycles, let $\omega_2=\lfloor\frac{qi_0-(q+1)-i_1-(q^8+2q^7+q^5)(q^2-1)}{q^5(q^2-1)}\rfloor$, then i_2 can increase to $qi_1+q+q^6(q-1)\omega_2>\frac{q^2}{q+1}(i_0+i_1)-(q^9+2q^8+2q^6)(q-1)=\frac{q^2}{q+1}(i_0+i_1)-f_4(q)$, that is, i_2 can increase to near its upper bound. we can verify that:

1) The value of point on $\langle e_1,e_3,e_4,e_5\rangle\setminus(\langle e_3,e_4,e_5\rangle\cup\langle e_1,e_3,e_4\rangle)\geq i_0-2-\lfloor\frac{qi_0-(q+1)-i_1}{q^7(q^2-1)}\rfloor q^6(q^2-1)-q^6(q^2-1)-q^4\omega_2-q^4\geq 0$;

2) The value of point on $\langle e_3,e_4,e_5\rangle\setminus\langle e_3,e_4\rangle\leq i_0-1-\lfloor\frac{qi_0-(q+1)-i_1}{q^7(q^2-1)}\rfloor q^6(q^2-1)+q^4(q-1)\omega_2+q^4(q-1)<i_0$;

3) For line l in $\langle e_3, e_4, e_5\rangle$ (not $\langle e_3, e_4\rangle$), we have $m''(l) \le (q+1)i_0-(q+1)-\lfloor\frac{qi_0-(q+1)-i_1}{q^7(q^2-1)}\rfloor q^6 (q^2-1)q-\lfloor\frac{qi_0-(q+1)-i_1}{q^7(q^2-1)}\rfloor q^7(q-1)+q^4(q-1)q\omega_2+q^4(q-1)q \le m''(l^*)=i_0+i_1$.

$\langle e_3, e_4, e_5\rangle$ is still P^* because the value of it increases most; each body is at most not decreased, hence $\langle e_1, e_3, e_4, e_5\rangle$ is still V^*.

Furthermore, from $\omega_2 \ge 0$, we get: $qi_0 - i_1 \ge (q^8+2q^7+q^5)(q^2-1)+(q+1) = f_2(q)$, that is, after the value of i_1 is away from its upper bound more than $(q^8+2q^7+q^5)(q^2-1)$, i_2 can begin to increase.

Using construction $m''(x)$, i_2 can increase to near its upper bound, retaining that i_3, i_4 are still bound values and sequence is still the difference sequence. ∎

Theorem 5.4 *For q-ary linear codes of dimension* 5*, the sufficient conditions for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence in class* IV_1 *are that*

(i) $f_0(q) \le i_0$;

(ii) $\frac{i_0}{q} + f_1(q) \le i_1 \le qi_0 - f_2(q)$;

(iii) $qi_1 + f_3(q) \le i_2 \le \frac{q^2}{q+1}(i_0+i_1) - f_4(q)$;

(iv) $i_0 + f_5(q) \le i_3 \le (q^2+q)i_1 - i_2$;

(v) $1 \le i_4 \le qi_3$

where $f_0(q) = q^9+3q^8+q^6+q^4+4,\ f_1(q) = (q^7+q^3)(q^2-1)+2q-1,\ f_2(q) = (q^8+2q^7+q^5)(q^2-1)+q+1,\ f_3(q) = q,\ f_4(q) = (q^9+2q^8+2q^6)(q-1),\ f_5(q) = (q^9+2q^4+3)(q^2-1)+3q^6(q-1)+q^3-3q^2-q$.

Proof Let $X_i \in \langle e_1, e_3, e_4\rangle \backslash (\{e_1\} \cup \langle e_3, e_4\rangle), 0 \le i < q^2-1$; $F_j \in \langle e_1, e_3, e_4, e_5\rangle \backslash (\langle e_3, e_4, e_5\rangle \cup \langle e_1, e_3, e_4\rangle), 0 \le j < q^3-q^2$; $C_k \in V_4 \setminus \langle e_1, e_3, e_4, e_5\rangle,\ 0 \le k < q^4$. (see Figure 2(b)).

We construct the function $m_1'''(x), m_2'''(x)$ as follows:

$$m_1'''(x) = \begin{cases} m''(x)-1, & x \in \langle F_j, C_k\rangle, \\ m''(x), & \text{others}, \end{cases}$$

$$m_2'''(x) = \begin{cases} m_1'''(x)-1, & x \in \langle X_i, C_k\rangle, \\ m_1'''(x), & \text{others}, \end{cases}$$

where $m''(x)$ is the corresponding assignment function (after iterating repeatedly) after i_2 taking some value in (7), and $m_1'''(x)$ is the corresponding assignment function (after iterating repeatedly by the function $m_1'''(x)$) after i_3 taking some value.

There are $q^6(q-1)$ lines like as $\langle F_j, C_k\rangle$. After one cycle, $F_j \downarrow q^4, C_k \downarrow (q^3-q^2)q = q^3(q-1)$.

There are $q^4(q^2-1)$ lines like as $\langle X_i, C_k\rangle$. After one cycle, $X_i \downarrow q^4, C_k \downarrow (q^2-1)q$.

Suppose i_3 decrease ω_3 cycles by lines like as $\langle F_j, C_k\rangle$, in order to keep value of $F_j \ge i_0 - 2 - \lfloor\frac{qi_0-(q+1)-i_1}{q^7(q^2-1)}\rfloor q^6(q^2-1) - q^6(q^2-1) - \lfloor\frac{i_2-qi_1-q}{q^6(q-1)}\rfloor q^4 - q^4 - q^4\omega_3 - q^4 \ge 0$, we have $\omega_3 \le \frac{q^2i_1-i_2-q^8(q-1)(q^2-1)-2q^6(q-1)-q^3+2q^2}{q^6(q-1)}$.

Suppose i_3 decrease ω_4 cycles by lines like as $\langle X_i, C_k\rangle$, in order to keep value of $X_i \ge i_0 - 2 - \lfloor\frac{qi_0-(q+1)-i_1}{q^7(q^2-1)}\rfloor q^8 - q^8 - q^4\omega_4 - q^4 \ge 0$, we have $\omega_4 \le \frac{qi_1-i_0-(q^8+q^4+3)(q^2-1)+q(q+1)}{q^4(q^2-1)}$.

Let

$$\omega_3 = \lfloor \frac{q^2 i_1 - i_2 - q^8(q-1)(q^2-1) - 2q^6(q-1) - q^3 + 2q^2}{q^6(q-1)} \rfloor,$$
$$\omega_4 = \lfloor \frac{q i_1 - i_0 - (q^8+q^4+3)(q^2-1) + q(q+1)}{q^4(q^2-1)} \rfloor.$$

Then i_3 can decrease to $(q^2+q)i_1 - i_2 - q^6(q-1)\omega_3 - q^4(q^2-1)\omega_4 < i_0 + (q^9+2q^4+3)(q^2-1) + 3q^6(q-1) + q^3 - 3q^2 - q = i_0 + f_5(q)$. That is, i_3 can decrease to near its lower bound, furthermore no values of point, line, plane and body exceeds the ones of p^*, l^*, P^* and V^* respectively. In this process, e_1 is p^*, $\langle e_3, e_4\rangle$ is l^*, $\langle e_3, e_4, e_5\rangle$ is P^*, $\langle e_1, e_3, e_4, e_5\rangle$ is V^*. i_4 is still the bound value, and the sequence is still the difference sequence.

i_4 can directly decrease to its lower bound.

Furthermore, from $\omega_4 \geq 0$, we get $i_1 \geq \frac{i_0}{q} + \frac{(q^8+q^4+3)(q^2-1)-q(q+1)}{q}$. We can let $f_1(q) = (q^7+q^3)(q^2-1)+2q-1$. And from $\frac{i_0}{q}+f_1(q) \leq i_1 \leq qi_0 - f_2(q)$, we get $i_0 \geq q^9+3q^8+q^6+q^4+4 = f_0(q)$. In summary, the theorem is proved. ▌

Let $N_1(i)$ be the number of difference sequences satisfying the sufficient condition in Class IV_1 with $i_0 \leq i$, and $M_1(i)$ be the number of sequences satisfying the necessary condition in class IV_1 with $i_0 \leq i$. From Theorem 5.4, on computer we can get $\lim_{i\to\infty} \frac{N_1(i)}{M_1(i)} = 1$.

6 Sufficient Conditions of Class IV_2

With the similar method in Section 5, in this section we only give the result and the construction used to prove the result. When $\frac{i_0}{q} < i_1 \leq qi_0$, $qi_1 < i_2 \leq \frac{q^2}{q+1}(i_0+i_1)$, $(q^2+q)i_1 - i_2 < i_3 < q^2 i_1$, $i_0 \leq i_4 \leq (q^2+q)i_1 - i_2 + (q-1)i_3$, we first make bound construction.

Lemma 6.1 *Let*

$$i_1 = qi_0 - (q+1), \tag{8}$$
$$i_2 = qi_1 + q, \tag{9}$$
$$i_3 = (q^2+q)i_1 - i_2 + 1, \tag{10}$$
$$i_4 = (q^2+q)i_1 - i_2 + (q-1)i_3 - (q-1), \tag{11}$$

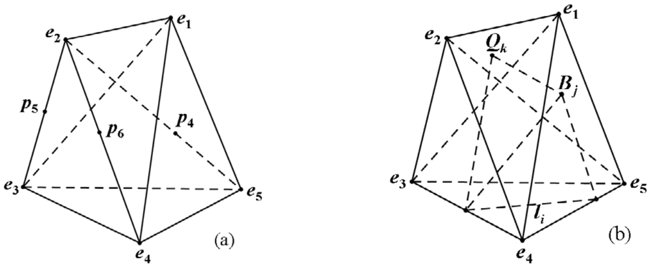
where i_1, i_4 are the upper bounds, i_2, i_3 are the lower bounds, then the bound sequence $(i_0, i_1, i_2, i_3, i_4)$ is the difference sequence.

Proof Let p_4 be a given point on $\langle e_2, e_5\rangle \backslash \{e_2, e_5\}$, p_5 be a given point on $\langle e_2, e_3\rangle \backslash \{e_2, e_3\}$, and p_6 be a given point on $\langle e_2, e_4\rangle \backslash \{e_2, e_4\}$ (see Figure 3(a)).

We construct the function $m(x)$ as follows:

$$m(x) = \begin{cases} i_0, & x = e_2, \\ i_0 - 2, & x \in (\langle e_1, e_3, e_4, p_4\rangle \cup \langle e_1, e_4, p_5\rangle \cup \langle e_1, e_5\rangle \cup \{p_6\}) \\ & \qquad \backslash(\langle e_3, e_4\rangle \cup \langle e_1, e_4\rangle \cup \{e_5\}), \\ i_0 - 1, & \text{others,} \end{cases}$$

where e_2 is p^*, $\langle e_3, e_4\rangle$ is l^*, $\langle e_3, e_4, e_5\rangle$ is P^*, and $\langle e_1, e_3, e_4, e_5\rangle$ is V^*. ∎



(a) Graph for the bound construction (b) Graph used to decrease i_1

Figure 3 Graph for the bound construction (Figure(a)) of class IV$_2$ in $PG(4,q)$ and graph used to decrease i_1 (Figure(b)), where $\langle l_i, B_j, Q_k\rangle$ is body in $PG(4,q)$

Lemma 6.2 *For all sequences* $(i_0, i_1, i_2, i_3, i_4)$ *satisfying* (9)–(11), *if*

$$\frac{i_0}{q} + g_1(q) \le i_1 \le qi_0 - g_2(q), \tag{12}$$

where $g_1(q) = 2q^9 - 2q^7 + q,\ g_2(q) = (q^9 + 2q^8 + 2q^7 + 3q^4)(q^2 - 1) + 2q^3 + 2q^2 + q + 1$, *then* $(i_0, i_1, i_2, i_3, i_4)$ *is the difference sequence.*

Proof Let l_i be line in $\langle e_3, e_4, e_5\rangle$ except $\langle e_3, e_4\rangle$, $0 \le i < q^2 + q$; $B_j \in \langle e_1, e_3, e_4, e_5\rangle \setminus \langle e_3, e_4, e_5\rangle$, $0 \le j < q^3$; $Q_k \in V_4 \setminus (\langle e_1, e_3, e_4, e_5\rangle \cup \langle l_i, B_j, e_2\rangle)$, $0 \le k < q^4 - q^3$. $\langle l_i, B_j, Q_k\rangle$ form a body set (see Figure 3(b)).

Based on $m(x)$, and modifying it slightly, we construct the function $m'(x)$ as follows:

$$m'(x) = \begin{cases} m(x) - 1, & x \in \langle l_i, B_j, Q_k\rangle, \\ m(x), & \text{others.} \end{cases}$$

The proof is completed. ∎

Lemma 6.3 *For all sequences* $(i_0, i_1, i_2, i_3, i_4)$ *satisfying* (10)–(12), *if*

$$qi_1 + g_3(q) \le i_2 \le \frac{q^2}{q+1}(i_0 + i_1) - g_4(q), \tag{13}$$

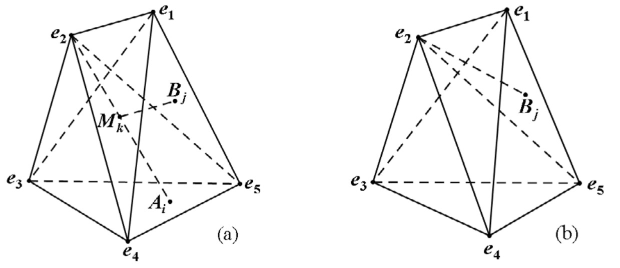
where $g_3(q) = q^{11} - q^9 + q^6 - q^5 + 2q^3 + q,\ g_4(q) = (q^9 + 2q^8 + 2q^5)(q - 1)$, *then* $(i_0, i_1, i_2, i_3, i_4)$ *is the difference sequence.*

Proof Let $A_i \in \langle e_3, e_4, e_5\rangle \setminus \langle e_3, e_4\rangle$, $0 \le i < q^2$; $B_j \in \langle e_1, e_3, e_4, e_5\rangle \setminus \langle e_3, e_4, e_5\rangle$, $0 \le j < q^3$; $M_k \in \langle e_2, A_i\rangle \setminus \{e_2, A_i\}$, $1 \le k \le q - 1$ (see Figure 4(a)).

We construct the function $m''(x)$ as follows:

$$m''(x) = \begin{cases} m'(x) + 1, \ x = A_i, \\ m'(x) - 1, \ x \in \langle B_j, M_k\rangle, \\ m'(x), \qquad \text{others,} \end{cases}$$

where $m'(x)$ is the corresponding assignment function (after iterating repeatedly) after i_1 taking some value in (12). ∎



(a) Graph used to increase i_2 (b) Graph used to increase i_3

Figure 4 Graph used to increase i_2 (Figure(a)) and graph used to increase i_3 (Figure(b)) in class IV_2

Theorem 6.4 *For q-ary linear codes of dimension* 5, *the sufficient conditions for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence of in class* IV_2 *are that*

(i) $g_0(q) \le i_0$;

(ii) $\frac{i_0}{q} + g_1(q) \le i_1 \le qi_0 - g_2(q)$;

(iii) $qi_1 + g_3(q) \le i_2 \le \frac{q^2}{q+1}(i_0 + i_1) - g_4(q)$;

(iv) $(q^2 + q)i_1 - i_2 + g_5(q) \le i_3 \le q^2 i_1 - g_6(q)$;

(v) $i_0 \le i_4 \le (q^2 + q)i_1 - i_2 + (q - 1)i_3$,

where $g_0(q) = q^{10} + 2q^9 + 4q^8 + 3q^5 + 2q^2 + 2q + 8,\ g_1(q) = 2q^9 - 2q^7 + q,\ g_2(q) = (q^9 + 2q^8 + 2q^7 + 3q^4)(q^2 - 1) + 2q^3 + 2q^2 + q + 1,\ g_3(q) = q^{11} - q^9 + q^6 - q^5 + 2q^3 + q,\ g_4(q) = (q^9 + 2q^8 + 2q^5)(q - 1),\ g_5(q) = 1,\ g_6(q) = q^9(q^2 - 1) + q^5(q - 1) + 2q^3 + q - 1.$

Proof Let $B_j \in \langle e_1, e_3, e_4, e_5\rangle \setminus \langle e_3, e_4, e_5\rangle,\ 0 \le j < q^3$ (see Figure 4(b)).

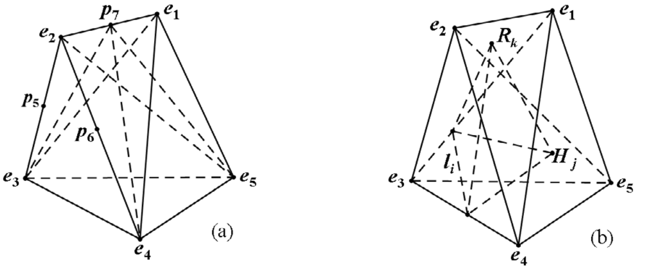
We construct the function $m'''(x)$ as follows:

$$m'''(x) = \begin{cases} m''(x) + 1, & x \in \langle e_2, B_j\rangle \setminus \{e_2\}, \\ m''(x), & \text{others}, \end{cases}$$

where $m''(x)$ is the corresponding assignment function (after iterating repeatedly) after i_2 taking some value in (13). ▮

7 Sufficient Conditions of Class IV_3

In this section we only give the result and the construction used to prove the result. When $\frac{i_0}{q} < i_1 \le qi_0,\ qi_1 < i_2 \le \frac{q^2}{q+1}(i_0+i_1),\ q^2 i_1 \le i_3 \le \frac{q^2}{q+1}(i_1+i_2),\ i_0 \le i_4 \le (q^3+q^2+q)i_1 - i_2 - i_3$, we first make bound construction.



(a) Graph for the bound construction (b) Graph used to decrease i_1

Figure 5 Graph for the bound construction (Figure(a)) of class IV$_3$ in $PG(4,q)$ and graph used to decrease i_1 (Figure(b)), where $\langle l_i, H_j, R_k\rangle$ is body in $PG(4,q)$

Lemma 7.1 *Let*

$$i_1 = qi_0 - (q+1), \tag{14}$$

$$i_2 = qi_1 + q, \tag{15}$$

$$i_3 = q^2 i_1, \tag{16}$$

$$i_4 = (q^3+q^2+q)i_1 - i_2 - i_3, \tag{17}$$

where i_1, i_4 *are the upper bounds,* i_2, i_3 *are the lower bounds, then the bound sequence* $(i_0, i_1, i_2, i_3, i_4)$ *is the difference sequence.*

Proof Let p_5 be a given point on $\langle e_2, e_3\rangle\backslash\{e_2, e_3\}$, p_6 be a given point on $\langle e_2, e_4\rangle\backslash\{e_2, e_4\}$, p_7 be a given point on $\langle e_1, e_2\rangle\backslash\{e_1, e_2\}$ (see Figure 5(a)).

We construct the function $m(x)$ as follows:

$$m(x) = \begin{cases} i_0, & x = e_2, \\ i_0 - 2, & x \in (\langle p_7, e_3, e_4, e_5\rangle \cup \langle e_3, p_6\rangle \cup \{p_5\})\backslash\langle e_3, e_4\rangle, \\ i_0 - 1, & \text{others}, \end{cases}$$

where e_2 is p^*, $\langle e_1, e_3\rangle$ is l^*, $\langle e_1, e_3, e_4\rangle$ is P^*, and $\langle e_1, e_3, e_4, e_5\rangle$ is V^*. ∎

Lemma 7.2 *For all sequences* $(i_0, i_1, i_2, i_3, i_4)$ *satisfying* (15)–(17)*, if*

$$\frac{i_0}{q} + h_1(q) \le i_1 \le qi_0 - h_2(q), \tag{18}$$

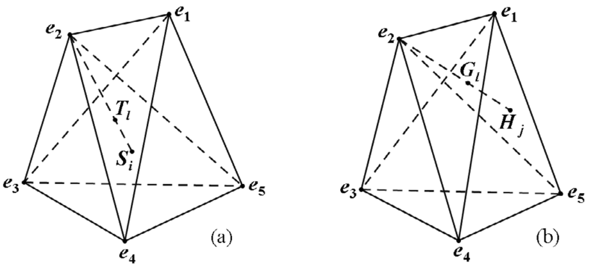
where $h_1(q) = q^9 - q^7 + q - 1$, $h_2(q) = (q^8 + 2q^7 + q)(q^2 - 1) + q + 1$, *then* $(i_0, i_1, i_2, i_3, i_4)$ *is the difference sequence.*

Proof Let l_i be line in $\langle e_1, e_3, e_4\rangle$ except $\langle e_1, e_3\rangle$, $0 \le i < q^2 + q$; $H_j \in \langle e_1, e_3, e_4, e_5\rangle \setminus \langle e_1, e_3, e_4\rangle$, $0 \le j < q^3$; $R_k \in V_4 \setminus (\langle e_1, e_3, e_4, e_5\rangle \cup \langle l_i, H_j, e_2\rangle)$, $0 \le k < q^4 - q^3$. $\langle l_i, H_j, R_k\rangle$ form a body set (see Figure 5(b)).

Based on $m(x)$, and modifying it slightly, we construct the function $m'(x)$ as follows:

$$m'(x) = \begin{cases} m(x) - 1, & x \in \langle l_i, H_j, R_k\rangle; \\ m(x), & \text{others}. \end{cases}$$

∎



(a) Graph used to increase i_2 (b) Graph used to increase i_3

Figure 6 Graph used to increase i_2 (Figure(a)) and graph used to increase i_3 (Figure(b)) in class IV$_3$

Lemma 7.3 *For all sequences* $(i_0, i_1, i_2, i_3, i_4)$ *satisfying* (16)–(18)*, if*

$$qi_1 + h_3(q) \le i_2 \le \frac{q^2}{q+1}(i_0 + i_1) - h_4(q), \tag{19}$$

where $h_3(q) = q^{10} - q^8 + 2q^3 - q^2 - q + 1,\ h_4(q) = q^8(q^2-1) + q^8(q-1) + q^2(q-1)$*, then* $(i_0, i_1, i_2, i_3, i_4)$ *is the difference sequence.*

Proof Let $S_i \in \langle e_1, e_3, e_4\rangle \setminus \langle e_1, e_3\rangle,\ 1 \le i \le q^2$; $T_l \in \langle e_2, S_i\rangle \setminus \{e_2, S_i\},\ 1 \le l \le q-1$ (see Figure 6(a)). We construct the function $m''(x)$ as follows:

$$m''(x) = \begin{cases} m'(x) + 1, & x = S_i, \\ m'(x) - 1, & x = T_l, \\ m'(x), & \text{others}, \end{cases}$$

where $m'(x)$ is the corresponding assignment function (after iterating repeatedly) after i_1 taking some value in (18). ∎

Theorem 7.4 *For q-ary linear codes of dimension* 5*, the sufficient conditions for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence of in class IV*$_3$ *are that*

(i) $h_0(q) \le i_0$;

(ii) $\frac{i_0}{q} + h_1(q) \le i_1 \le qi_0 - h_2(q)$;

(iii) $qi_1 + h_3(q) \le i_2 \le \frac{q^2}{q+1}(i_0 + i_1) - h_4(q)$;

(iv) $q^2 i_1 + h_5(q) \le i_3 \le \frac{q^2}{q+1}(i_1 + i_2) - h_6(q)$;

(v) $i_0 \le i_4 \le (q^3 + q^2 + q)i_1 - i_2 - i_3$,

where $h_0(q) = q^9 + 3q^8 + q^2 + 3,\ h_1(q) = q^9 - q^7 + q - 1,\ h_2(q) = (q^8 + 2q^7 + q)(q^2 - 1) + q + 1,\ h_3(q) = q^{10} - q^8 + 2q^3 - q^2 - q + 1,\ h_4(q) = q^8(q^2-1) + q^8(q-1) + q^2(q-1),\ h_5(q) = 0,\ h_6(q) = q^{11} - q^{10} + 2q^4 - 3q^3 + 2q^2 - 2q + 2.$

Proof Let $H_j \in \langle e_1, e_3, e_4, e_5\rangle \backslash \langle e_1, e_3, e_4\rangle,\ 0 \le j < q^3; G_l \in \langle e_2, H_j\rangle \backslash \{e_2, H_j\},\ 1 \le l \le q-1$ (see Fig.6(b)). We construct the function $m'''(x)$ as follows:

$$m'''(x) = \begin{cases} m''(x) + 1, & x = H_j, \\ m''(x) - 1, & x = G_l, \\ m''(x), & \text{others}, \end{cases}$$

where $m''(x)$ is the corresponding assignment function (after iterating repeatedly) after i_2 taking some value in (19). ▮

8 Results in Classes $\mathbf{IV}_4$, $\mathbf{IV}_5$ and $\mathbf{IV}_6$

Theorem 8.1 *For q-ary linear codes of dimension* 5*, the sufficient condition for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence of in class* IV_4 *are that*

(i) $f_6(q) \le i_0$;

(ii) $\frac{i_0}{q^2} + f_7(q) \le i_1 \le \frac{i_0}{q} - f_8(q)$;

(iii) $qi_1 + f_9(q) \le i_2 \le (q^2+q)i_1 - i_0 - f_{10}(q)$;

(iv) $i_0 + f_{11}(q) \le i_3 \le (q^2+q)i_1 - i_2 - f_{12}(q)$;

(v) $1 \le i_4 \le qi_3$,

where $f_6(q) = q^8 + 3q^6 + 5q^2 + 3q + 6,\ f_7(q) = q^7 - q^6 + 3q^5 - 3q^4 + 5q - 3,\ f_8(q) = 1,\ f_9(q) = q,\ f_{10}(q) = (q^8 + 3q^6 + 5q^2 + 2q)(q-1),\ f_{11}(q) = (q^8 + 3q^6 + 4q^2 + q)(q-1),\ f_{12}(q) = q^3 - q$.

Theorem 8.2 *For q-ary linear codes of dimension* 5*, the sufficient conditions for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence of in class* IV_5 *are that*

(i) $g_7(q) \le i_0$;

(ii) $\frac{i_0}{q^2} + g_8(q) \le i_1 \le \frac{i_0}{q} - g_9(q)$;

(iii) $qi_1 + g_{10}(q) \le i_2 \le (q^2+q)i_1 - i_0 - g_{11}(q)$;

(iv) $(q^2+q)i_1 - i_2 + g_{12}(q) \le i_3 \le q^2 i_1 - g_{13}(q)$;

(v) $i_0 \le i_4 \le (q^2+q)i_1 - i_2 + (q-1)i_3$,

where $g_7(q) = 2q^8 + 3q^5 + 7q^2 + 6q + 12,\ g_8(q) = 2q^7 - 2q^6 + 3q^4 - 3q^3 + 7q - 2,\ g_9(q) = 1,\ g_{10}(q) = (q^8 + q^5)(q-1) + 3q^3 + 2q^2,\ g_{11}(q) = (q^8 + 2q^5 + 4q^2 + q)(q-1) - q^2,\ g_{12}(q) = 1,\ g_{13}(q) = (q^8 + q^5)(q-1) + 3q^3 + 2q^2 - 1$.

Theorem 8.3 *For q-ary linear codes of dimension* 5*, the sufficient conditions for the sequence* $(i_0, i_1, i_2, i_3, i_4)$ *to be a difference sequence of in class* IV_6 *are that*

(i) $h_7(q) \le i_0$;

(ii) $\frac{i_0}{q^2} + h_8(q) \le i_1 \le \frac{i_0}{q} - h_9(q)$;

(iii) $qi_1 + h_{10}(q) \le i_2 \le (q^2+q)i_1 - i_0 - h_{11}(q)$;

(iv) $q^2 i_1 + h_{12}(q) \le i_3 \le \frac{q^2}{q+1}(i_1 + i_2) - h_{13}(q)$;

(v) $i_0 \le i_4 \le (q^3 + q^2 + q)i_1 - i_2 - i_3$,

where $h_7(q) = 2q^8 + 2q^2 + 2q + 4,\ h_8(q) = 2q^7 - 2q^6 + 2q - 1,\ h_9(q) = 1,\ h_{10}(q) = q^8 - q^7 + q^3 + 3q^2 + q + 1,\ h_{11}(q) = (q^8 + 3q^4 + q^3)(q-1) - q^2(q-2),\ h_{12}(q) = q^3,\ h_{13}(q) = q^9 - 2q^8 + 2q^7 - 2q^6 + 2q^5 - q^4 + 3q^3 - 3q^2 + 3q - 2$.

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