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# Hierarchical Hypergames and Bayesian Games: A Generalization of the Theoretical Comparison of Hypergames and Bayesian Games Considering Hierarchy of Perceptions

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**Abstract** This paper discusses the relationship of two independently developed models of games with incomplete information, hierarchical hypergames and Bayesian games. It can be considered as a generalization of the previous study on the theoretical comparison of simple hypergames and Bayesian games (Sasaki and Kijima, 2012) by taking into account hierarchy of perceptions, i.e., an agent's perception about the other agents' perceptions, and so on. The authors first introduce the general way of transformation of any hierarchical hypergames into corresponding Bayesian games, which was called as the Bayesian representation of hierarchical hypergames. The authors then show that some equilibrium concepts for hierarchical hypergames can be associated with those for Bayesian games and discuss implications of the results.

**Keywords** Bayesian games, game theory, hierarchy of perceptions, hypergames, incomplete information.

## 1 Introduction

In game theory, a game is called complete information if all the components of the game are commonly known by all the agents (decision makers). Otherwise, if some or all of them lack full information about it, the game is called incomplete information.

The present study discusses two models of games with incomplete information: Hypergames and Bayesian games. Hypergame theory deals with agents who may misperceive some components of a game[1]. It is the basic idea of hypergames that each agent is assumed to have her

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own subjective view of the game, which is formulated as a normal form game called her subjective game, and make a decision based on it. The assumption allows agents to hold different perceptions about the game. Then a hypergame is defined as a collection of such subjective games. Since an agent may misperceive another agent's subjective game correctly, she may not know the whole structure of the hypergame in which she is involved. On the other hand, Bayesian games are usually considered as the standard model of games with incomplete information in game theory[2]. It is argued that, in Bayesian games, any kind of incompleteness of information can be captured by subjective probability distribution of each agent over the set of possible states. Each possibility is modeled as a type of an agent, and a game with incomplete information is reformulated as a game with complete (but imperfect) information called a Bayesian game by introducing the set of types as well as each type's belief, namely a probability distribution on the others' types.

These two models clearly deal with similar situations, nevertheless, as they have been established and developed independently, their relationship had been left ambiguous†. Recently Sasaki and Kijima[3] conducted a rigorous theoretical comparison of hypergames and Bayesian games. They discuss the simplest model of hypergames called simple hypergames and its relation to Bayesian games. A simple hypergame is given as the set of agents and the collection of each agent's subjective game[4, 5]. It has been shown that any simple hypergame can be reformulated as a Bayesian game according to Harsanyi's way to treat incompleteness of information, and some equilibrium concepts for simple hypergames can be associated with those for Bayesian games such as Bayesian Nash equilibrium.

In this article, we extend the comparative study by considering hierarchy of perceptions, that is, an agent's perception about another agent's perception, and so on. In a simple hypergame, explicitly or implicitly, an agent is assumed to believe her own perception is common knowledge. However, once an agent admits the possibility that the others may perceive the game in different ways, she would take into account it when formulating her decision problem. Furthermore, she might notice that the other agents may also notice such possibilities of misperceptions. Hierarchical hypergames explicitly model such hierarchies of perceptions of agents‡[6–9].

As the theoretical contribution of the present study, we clarify the relationship between hierarchical hypergames and Bayesian games. In a similar way to the previous study that deals with simple hypergames[3], we first propose the general way of transformation of a hierarchical hypergame into a Bayesian game. In order to express hierarchies of perceptions in terms of Bayesian games, we define types so that they capture an agent's belief about the other's beliefs, her belief about the other's beliefs about the other's beliefs, and so on[10]. Then we show some equilibrium concepts for hierarchical hypergames can be associated with those for Bayesian games.

Following this introduction, Section 2 introduces the models and concepts we discuss in the study. Section 3 presents the procedure of Bayesian representation of hierarchical hypergames,

†Bayesian games are standard in information economics nowadays, while hypergames have been developed and applied mainly in communities of operational research and systems engineering.

‡Hierarchical hypergames may be referred to as general hypergame or $n$-level hypergames in the literature.

the way of transformation of hierarchical hypergames into Bayesian games. Then Section 4 proves some propositions describing the relationships of equilibrium concepts for the both models. Based on the results, we discuss some implications in Section 5.

## 2 Models

### 2.1 Normal Form Games

Let us start with normal form games, the basis of hypergames and Bayesian games. A normal form game consists of three components: A set of agents, sets of actions available to them, and utility functions for each that associate real values (utilities) with outcomes. In what follows, we may simply say games as meaning normal form games. We do not deal with mixed extension of games in this paper.

**Definition 2.1** (Normal form games) $G = (I, A, u)$ is a normal form game, where

- $I$ is the finite set of agents.
- $A = \times_{i \in I} A_i$, where $A_i$ is the finite set of agent $i$'s actions. $a \in A$ is called an outcome.
- $u = (u_i)_{i \in I}$, where $u_i : A \to \Re$ is agent $i$'s utility function.

### 2.2 Hierarchical Hypergames

#### 2.2.1 The Framework

Hypergame theory deals with interactive situations where agents may misperceive some components of a game. Each agent is assumed to have her own subjective view of it, which is given as a normal form game and called the agent's subjective game, based on which she makes decisions.

Among several types of hypergame models, the hierarchical hypergame model the present study focuses on explicitly takes into account a hierarchy of perceptions, that is, an agent's perception about another agent's perception, and so on, by introducing the concept of viewpoints. A viewpoint indicates a specific perception in the hierarchy. For example, viewpoint $i$ means agent $i$'s view, and viewpoint $ji$ is agent $j$'s view perceived by agent $i$. In general, viewpoint $i_1 i_2 \cdots i_n$ is interpreted as agent $i_1$'s view perceived by agent $i_2 i_3 \cdots$ perceived by agent $i_n$. In a hierarchical hypergame, each perception of each viewpoint is given as a normal form game and called the viewpoint's subjective game. Then a hypergame is defined as the collection of all the subjective games.

**Definition 2.2** (Hierarchical hypergames) $H = (I, (G^\sigma)_{\sigma \in \Sigma})$ is a hierarchical hypergame, where $I$ is the set of agents involved in the situation and $\Sigma$ is the set of relevant viewpoints defined as below. For any $\sigma \in \Sigma$, $G^\sigma = (I^\sigma, A^\sigma, u^\sigma)$ is a normal form game called viewpoint $\sigma$'s subjective game, where

- $I^\sigma$ is the set of agents perceived by viewpoint $\sigma$.
- $A^\sigma = \times_{i \in I^\sigma} A_i^\sigma$, where $A_i^\sigma$ is the set of agent $i$'s actions perceived by viewpoint $\sigma$.

- $u^\sigma = (u_i^\sigma)_{i\in I^\sigma}$, where $u_i^\sigma : A^\sigma \to \Re$ is agent $i$'s utility function perceived by viewpoint $\sigma$.

Then $\Sigma$ is defined as $I \cup \bigcup_{n=2}^{\infty}\{i_1 i_2 \cdots i_n | i_n \in I, i_{k-1} \in I^{i_k i_{k+1}\cdots i_n} \setminus \{i_k\}$ for any $k = 2, 3, \cdots, n\}$.

We specify the set of viewpoints relevant in a hierarchical hypergame by $\Sigma$. A viewpoint is said to be relevant when it is actually taken into account in some agent's decision making. We assume that, in a hypergame, any viewpoint $\sigma$, when formulating the decision situation, considers views of all the agents who $\sigma$ thinks are participating in the game and does not consider views of anybody else. For example, when agent $i$ is in $I$, $i \in \Sigma$ by definition, and if another agent $j$ is in $I^i$, viewpoint $ji$ must be in $\Sigma$, and otherwise, it is not included in $\Sigma$. Furthermore, we suppose that a viewpoint does not contain any successive agents. For example, since considering agent $i$'s view perceived by agent $i$ is redundant, we do not consider viewpoint $ii$, that is, $ii \notin \Sigma$, and similarly, neither $jii$ nor $iij$ is included in $\Sigma$. In the subsequent discussion, when we refer to viewpoints, we only indicate viewpoints relevant in this sense.

We may deal with concatenations of viewpoints. For example, by $\sigma'\sigma$ with $\sigma = i_1 i_2 \cdots i_n$ and $\sigma' = j_1 j_2 \cdots j_m$ (with $j_m \neq i_1$) we mean viewpoint $j_1 j_2 \cdots j_m i_1 \cdots i_n$. When $\sigma = i_1 i_2 \cdots i_n$ with $n \geq 2$, any viewpoint $i_m i_{m+1} \cdots i_n$ with $n \geq m \geq 2$ is said to be higher than $\sigma$. On the other hand, any viewpoint $\tau\sigma$ with $\tau = j_1 j_2 \cdots j_l$ and $j_l \neq i_1$ is said to be lower than $\sigma$. For example, for viewpoint $ji$, viewpoint $i$ is higher than $ji$ while viewpoint $kji$ is lower than it. Furthermore, for $\sigma = i_1 i_2 \cdots i_n$, let us denote $i_1$ by $\sigma_1$. We say $\sigma_1$ is the lowest agent in viewpoint $\sigma$. For any $\sigma \in \Sigma$, let $\Sigma_\sigma = \sigma \cup \{\tau\sigma | \tau = j_1 j_2 \cdots j_l, j_l \neq \sigma_1$, and $\tau\sigma \in \Sigma\}$. $\Sigma_\sigma$ is the union of $\sigma$ itself and the set of viewpoints lower than $\sigma$.

Moreover, we assume that for any $\sigma \in \Sigma$, 1) $\sigma_1 \in I^\sigma$, and 2) $A_i^{i\sigma} \subseteq A_i^\sigma$ for all $i \in I^\sigma$. The first assumption means that, in a viewpoint, the lowest agent is always included in the agent set perceived by it, which is required for the second one to be well-defined. Then the second assumption means that if agent $i$ thinks another agent $j$ is aware that an action is available to $j$ herself, then $i$ never excludes the action from $j$'s action set in $i$'s subjective game[§].

### 2.2.2 Equilibrium Concepts

Among several equilibrium concepts for hierarchical hypergames, we select two of them because it will be shown later that these can be associated with equilibrium concepts for Bayesian games.

The first one is the notion of subjective rationalizability[¶][11].

**Definition 2.3** (Subjective rationalizability) Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hypergame. $a_\sigma^* \in A_{\sigma_1}^\sigma$ is called subjectively rationalizable for viewpoint $\sigma$ if and only if there exists $(a_\tau^*)_{\tau\in\Sigma_\sigma}$ in $\times_{\tau\in\Sigma_\sigma} A_{\tau_1}^\tau$ which satisfies

$$\forall \tau \in \Sigma_\sigma, \quad \forall a_\tau \in A_{\tau_1}^\tau, \quad u_{\tau_1}^\tau(a_\tau^*, a_{-\tau}^*) \geq u_{\tau_1}^\tau(a_\tau, a_{-\tau}^*),$$

where $a_{-\tau}^* = (a_{i\tau}^*)_{i\in I^\tau\setminus\{\tau_1\}}$. Then such $(a_\tau^*)_{\tau\in\Sigma_\sigma}$ is called a best response hierarchy of $\Sigma_\sigma$.

An action of the lowest agent in a viewpoint is called subjectively rationalizable for the viewpoint when it is a best response in the viewpoint's subjective game to some actions of the

[§]The assumption will be used in the proof of Lemma 3.1.

[¶]Note that, in Definition 2.3, $a_\tau^*$ is a particular action of $\tau_1$, the lowest agent of viewpoint $\tau$.

other agents, each of which is a best response in the subjective game of one step lower viewpoint to some actions of the other agents, and so on. The concept of subjective rationalizability can be understood based on the following idea: The lowest agent in a viewpoint would take a best response to actions which she thinks the other agents would choose. When expecting the choices of the others, she considers that each of the other agents takes a best response to actions which she thinks the agent thinks the other agents would choose, and such her inference goes on for further lower viewpoints. Subjective rationalizability can be regarded as an extension of rationalizability[12], a well-known concept in the standard game theory, to hierarchical hypergames. Under a condition called inside common knowledge[7], it has been shown that subjective rationalizability and (standard) rationalizability are equivalent[11].

When agent $i$ makes decision in this way, her choice can be predicted as a subjectively rationalizable action of viewpoint $i$. Thus, we may also say that it is agent $i$'s subjectively rationalizable action. Let us denote the set of subjectively rationalizable actions of agent $i$ in a hierarchical hypergame $H$ by $R_i(H)$. As mentioned above, if agent $i$ has inside common knowledge, that is, if she believes $G^i$ is common knowledge, then $R_i(H)$ is equal to the set of her rationalizable actions in $G^i$.

Next, the other equilibrium concept we discuss is best response equilibrium, which is an extension of it defined for simple hypergames[3] to hierarchical hypergames.

**Definition 2.4** (Best response equilibrium) Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hierarchical hypergame. $a^* = (a_i^*, a_{-i}^*) \in \times_{i\in I} A_i^i$ is a best response equilibrium of $H$ iff for all $i \in I$,

$$\forall a_i \in A_i^i, \quad u_i^i(a^*) \geq u_i^i(a_i, a_{-i}^*).$$

Best response equilibrium can only exist in a specific class of hypergame satisfying the first condition in the definition above, which is required for the utility functions in the second condition to be appropriately defined. Then, in such a hypergame, when each agent chooses a best response to the choices of the others, such an outcome is called a best response equilibrium. Although the definition apparently looks like Nash equilibrium for normal form games, the implication is largely different. Best response equilibrium does not assure that, from a particular agent's point of view, the other's choices are also their best responses. Let us denote the set of best response equilibria of $H$ by $BE(H)$. When such an utility shown in the definition cannot be defined, necessarily the hypergame cannot have any best response equilibrium. For example, if $I^i \subset I$ or $a_j^* \notin A_j^i$ for some $j(\neq i)$, then $u_i^i(a^*)$ is not defined, hence $BE(H) = \phi$.

### 2.3 Bayesian Games

#### 2.3.1 The Framework

Bayesiang games are defined as follows.

**Definition 2.5** (Bayesian games) $G^b = (I, A, T, p, u)$ is a Bayesian game, where

- $I$ is the set of agents.
- $A = \times_{i\in I} A_i$, where $A_i$ is the set of agent $i$'s actions.

- $T = \times_{i \in I} T_i$, where $T_i$ is the set of agent $i$'s types.
- $p = (p_i)_{i \in I}$, where $p_i$ is agent $i$'s subjective prior, which is a joint probability distribution on $T_{-i}$ for each $t_i \in T_i$, where $T_{-i} = \times_{j \in I \setminus \{i\}} T_j$.
- $u = (u_i)_{i \in I}$, where $u_i : A \times T \to \Re$ is agent $i$'s utility function.

Bayesian games capture any uncertainty about any components of the game by introducing types which are characterized by subjective priors and utility functions. That is, each type is assumed to have a probability distribution on the types of the other agents, and utility is determined not only by choices of the agents but also by their types.

### 2.3.2 Equilibrium Concepts

To define equilibrium concepts for Bayesian games, we need to introduce "action plans" for each type of each agent, which we call her strategy so that we avoid the confusion with the concept of action. A strategy of agent $i$, $s_i$, is a mapping from her types to her actions, namely, $s_i : T_i \to A_i$. Let us denote the set of agent $i$'s strategies by $S_i$ and let $S = \times_{i \in I} S_i$. We may write $s_{-i}(t_{-i})$ as meaning $(s_j(t_j))_{j \in I \setminus \{i\}}$ with $s_j \in S_j$ and $t_j \in T_j$. Then Bayesian Nash equilibrium is defined as follows

**Definition 2.6** (Bayesian Nash equilibrium) Let $G^b = (I, A, T, p, u)$ be a Bayesian game. $s^* = (s_i^*, s_{-i}^*) \in S$ is a Bayesian Nash equilibrium of $G^b$ iff $\forall i \in I$, $\forall t_i \in T_i$, $\forall s_i \in S_i$,

$$\sum_{t_{-i} \in T_{-i}} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i})) p_i(t_{-i}|t_i) \geq \sum_{t_{-i} \in T_{-i}} u_i((s_i(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i})) p_i(t_{-i}|t_i).$$

In a Bayesian Nash equilibrium, each agent maximizes her expected utility given her belief (subjective prior). Let us denote the set of Bayesian Nash equilibria of a Bayesian game $G^b$ by $BN(G^b)$.

We also consider a joint probability distribution $p^o$ on the type set $T$, which describes probabilities with which a particular combination of types for each agent is chosen actually. We call it the objective prior of the game. In particular, we say subjective priors are consistent in a Bayesian game when each agent's subjective probability distribution is given as the conditional probability distribution computed from the objective prior by Bayes formula[‖].

Whether or not a prior is consistent, we can also formulate a Bayesian game by using objective priors as $G^b = (I, A, T, p^o, u)$, and define Nash equilibrium of it as follows.

**Definition 2.7** (Nash equilibrium of Bayesian games) Let $G^b = (I, A, T, p^o, u)$ be a Bayesian game (with objective prior). $s^* = (s_i^*, s_{-i}^*) \in S$ is a Nash equilibrium of $G^b$ iff $\forall i \in I, \forall s_i \in S_i$,

$$\sum_{t \in T} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), t) p^o(t) \geq \sum_{t \in T} u_i((s_i(t_i), s_{-i}^*(t_{-i})), t) p^o(t).$$

In a Nash equilibrium of a Bayesian game, each agent maximizes her expected utility given an objective prior. Let us denote the set of Nash equilibria of $G^b$ by $N(G^b)$.

[‖] That is, $(p_i)_{i \in I}$ are consistent iff there exists a probability distribution $p^o$ on $T$ such that $\forall i \in I, \forall t \in T, p_i(t_{-i}|t_i) = p^o(t) / \Sigma_{t_{-i} \in T_{-i}} p^o(t_i, t_{-i})$.

We introduced two formulations of Bayesian games, one with subjective priors and the other with objective priors**. Both of them are written as $G^b$, and when we write $BN(G^b)$ and $N(G^b)$, we suppose $(p_i)_{i \in I}$ and $p^o$, respectively. Furthermore we may say that $p^o$ is the objective prior of $G^b = (I, A, T, p, u)$.

## 3 Bayesian Representation of Hierarchical Hypergames

This section presents a general way to transform hierarchical hypergames into Bayesian games, which we call Bayesian representation of hierarchical hypergames.

### 3.1 Extension of Subjective Games

First, we extend each viewpoint's subjective game in a similar way to extensions of subjective games in simple hypergames[3]. It is originally based on Harsanyi's[2] claim that any kinds of uncertainties about a game as well as perceptual differences among agents can be modeled in an unified way, which goes on as follows††

- (Agents) Whether an agent is participating in the game can be converted into what the agent's action set is, by allowing her only one action, "non-participation" (NP), when she is supposed to be out of the game.
- (Actions) Whether a particular action is feasible for an agent can in turn be converted into what the agent's utility function is, by saying that she will get some very low utility whenever she takes the action that is supposed to be infeasible.
- (Utility functions) This way, any uncertainty or perceptual differences about agents as well as actions can be reduced to those about utility functions, if any. Then by regarding each possible utility function of each agent as a type of the agent, the game can be modeled as a Bayesian game.

Let us call the above argument Harsanyi's claim. When we apply it to situations represented as hierarchical hypergames, it can be interpreted as follows. For example, suppose first that viewpoint $\sigma$ thinks that an agent $i$, not $\sigma_1$, does not participate in the game, though agent $i$ actually is in the game. Then the claim argues that $\sigma$'s exclusion of agent $i$ is game-theoretically equivalent to saying that $\sigma$ includes $i$ in the set of the agents and allows $i$ to use only one action,

**Harsanyi[2] originally named the latter (with objective prior) "Bayesian games", while the former (Definition 2.5) "the standard form of games with incomplete information", though nowadays the former is also referred to as Bayesian games in textbooks of game theory. The two kinds of equilibrium concepts are also often mixed up. However, since most studies assume consistency of priors, those distinctions make practically no difference as it has been shown that, in any Bayesian games with consistent priors, the set of Bayesian Nash equilibria coincides with that of Nash equilibria. On the other hand, our study deals with Bayesian games that may not hold consistency. Consistency of priors still controversial[13, 14], while we do not go into the detail of this topic. To understand Harsanyi's original definition of Bayesian games as well as discussion on consistency of priors, see also Myerson[15].

††For a brief guide of the claim, see e.g., Myerson[16].

"non-participation." This way allows every viewpoint to see the common set of agents, which coincides with the set of all the agents actually involved in the hypergame.

Next, suppose viewpoint $\sigma$ thinks that a particular action of agent $i$, not of $\sigma_1$, is not feasible for the agent, while it is actually included in agent $i$'s action set. Then the claim argues that this is equivalent to saying that $\sigma$ considers that the action is surely in agent $i$'s action set but gives the agent very low utility whenever it is used. Consequently, every viewpoint sees the same action set of a particular agent, which is the union of the agent's action set originally conceived by each viewpoint. As a result, perceptual differences in agents as well as actions among viewpoints are resolved, and those only in utility functions remain.

Based on the discussion above, we define extension of subjective games in hierarchical hypergames as follows.

**Definition 3.1** (Extended subjective games) Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hierarchical hypergame. For any $\sigma \in \Sigma$, a normal form game $\overline{G}^\sigma = (\overline{I}^\sigma, \overline{A}^\sigma, \overline{u}^\sigma)$ is called viewpoint $\sigma$'s extended subjective game iff it satisfies all of the following conditions:

- $\overline{I}^\sigma = I$.
- $\overline{A}^\sigma = \times_{i\in I}\overline{A}_i^\sigma$, where $\forall i \in I$, $\overline{A}_i^\sigma = \bigcup_{\sigma'\in\Sigma} A_i^{\sigma'}$ if $i \in I^\sigma$ for any $\sigma \in \Sigma$, $\overline{A}_i^\sigma = \bigcup_{\sigma'\in\Sigma} A_i^{\sigma'} \cup \{NP\}$ otherwise.
- $\overline{u}^\sigma = (\overline{u}_i^\sigma)_{i\in I}$, where $\overline{u}_i^\sigma : \overline{A}^\sigma \to \Re$. For any $i \in I$ and $a = (a_i, a_{-i}) \in \overline{A}^\sigma$, $\overline{u}_i^\sigma(a)$ is defined as follows, where $c$ is a real constant bigger than $-\infty$

$$\overline{u}_i^\sigma(a) = \begin{cases} u_i^\sigma((a_j)_{j\in I^\sigma}), & \text{if } i \in I^\sigma \wedge a_k = NP \text{ for any } k \in I \setminus I^\sigma \wedge (a_j)_{j\in I^\sigma} \in A^\sigma, \\ -\infty, & \text{if } (i \in I^\sigma \wedge a_i \notin A_i^\sigma) \vee (i \notin I^\sigma \wedge a_i \neq NP), \\ c, & \text{otherwise.} \end{cases}$$

Then $\overline{H} = (I, (\overline{G}^\sigma)_{\sigma\in\Sigma})$, the collection of extended subjective games of all the viewpoints, is called the extended hierarchical hypergame (induced from $H$). Conversely, we may say that $H$ is the original hierarchical hypergame of $\overline{H}$ and $G^\sigma$ is the original subjective game of $\overline{G}^\sigma$.

The way of extension of subjective games follows Harsanyi's claim. Roughly speaking, utility functions are determined according to the next three principles. First, any outcomes modeled in the original subjective game assign the same utilities to each agent in its extension as well. Second, when someone takes an action that is not modeled in the original subjective game, the agent always gets extremely low utility, $-\infty$. Third, in the other cases, an agent is just supposed to get some utility $c$ because we cannot determine any specific utility value for such cases based on the original hypergame and this assumption is sufficient for us. Note that the extension is unique up to $c \in \Re$. Then let us denote $\overline{I}^\sigma$ and $\overline{A}^\sigma$ by $\overline{I}$ and $\overline{A}$, respectively, because they are by definition identical for all $\sigma \in \Sigma$.

The next lemma assures that subjectively rationalizable actions of the agents in a hierarchical hypergame are "preserved" in its extension, and vice versa.

**Lemma 3.1** (Subjective rationalizability in extended hierarchical hypergames) *Let $\overline{H}$ be the extended hierarchical hypergame of a hierarchical hypergame $H = (I, (G^\sigma)_{\sigma \in \Sigma})$. Then we have, for any $i \in I$, $R_i(\overline{H}) = R_i(H)$.*

*Proof* (Proof of $R_i(\overline{H}) \supseteq R_i(H)$) Suppose $a_i^* \in R_i(H)$, which means, there exists $(a_\sigma^*)_{\sigma \in \Sigma_i}$ such that $a_\sigma^* \in A_{\sigma_1}^\sigma$ for all $\sigma \in \Sigma_i$ which satisfies

$$\forall \sigma \in \Sigma_i, \quad \forall a_\sigma \in A_{\sigma_1}^\sigma, \quad u_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*) \geq u_{\sigma_1}^\sigma(a_\sigma, a_{-\sigma}^*),$$

where $a_{-\sigma}^* = (a_{j\sigma}^*)_{j \in I^\sigma \setminus \{\sigma_1\}}$. In $\overline{H}$, $\forall \sigma \in \Sigma_i, \overline{A}_{\sigma_1} \supseteq A_{\sigma_1}^\sigma$. For any $\sigma \in \Sigma_i$, if $a_{\sigma_1} \in \overline{A}_{\sigma_1} \setminus A_{\sigma_1}^\sigma$,

$$\overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*) > \overline{u}_{\sigma_1}^\sigma(a_{\sigma_1}, a_{-\sigma}^*) = -\infty,$$

otherwise, i.e., if $a_{\sigma_1} \in A_{\sigma_1}^\sigma$,

$$\overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*) \geq \overline{u}_{\sigma_1}^\sigma(a_{\sigma_1}, a_{-\sigma}^*).$$

Therefore, we have

$$\forall \sigma \in \Sigma_i, \quad \forall a_\sigma \in \overline{A}_{\sigma_1}, \quad \overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*) \geq \overline{u}_{\sigma_1}^\sigma(a_\sigma, a_{-\sigma}^*).$$

This means $a_i^* \in R_i(\overline{H})$. Hence $R_i(\overline{H}) \supseteq R_i(H)$.

(Proof of $R_i(\overline{H}) \subseteq R_i(H)$) Next, suppose $a_i^* \in R_i(\overline{H})$, which means, there exists $(a_\sigma^*)_{\sigma \in \Sigma_i}$ such that $a_\sigma^* \in \overline{A}_{\sigma_1}$ for all $\sigma \in \Sigma_i$ which satisfies

$$\forall \sigma \in \Sigma_i, \quad \forall a_\sigma \in \overline{A}_{\sigma_1}, \quad \overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*) \geq \overline{u}_{\sigma_1}^\sigma(a_\sigma, a_{-\sigma}^*),$$

where $a_{-\sigma}^* = (a_{j\sigma}^*)_{j \in \overline{I} \setminus \{\sigma_1\}}$. Now suppose that there exists $\sigma \in \Sigma_i$ such that $a_\sigma^* \notin A_{\sigma_1}^\sigma$. But if so, for such $\sigma$, $\forall a_{-\sigma_1} \in \times_{j \in \overline{I} \setminus \{\sigma_1\}} \overline{A}_j, \overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma_1}) = -\infty$, which is smaller than $\overline{u}_{\sigma_1}^\sigma(a_{\sigma_1}, a_{-\sigma_1})$ for any $a_{\sigma_1} \in A_{\sigma_1}^\sigma (\subseteq \overline{A}_{\sigma_1}^\sigma)$, and this contradicts the requirement of subjective rationalizability, i.e., $\forall a_\sigma \in \overline{A}_{\sigma_1}, \overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*) \geq \overline{u}_{\sigma_1}^\sigma(a_\sigma, a_{-\sigma}^*)$. Thus, we have $\forall \sigma \in \Sigma_i, a_\sigma^* \in A_{\sigma_1}^\sigma$. Since we have assumed that for any $\sigma \in \Sigma$, $\forall j \in I^\sigma, A_j^{j\sigma} \subseteq A_j^\sigma$, for any $\sigma \in \Sigma$, $\forall k \in I^\sigma \setminus \{i\}, a_{k\sigma}^* \in A_k^\sigma$. Therefore, for any $\sigma \in \Sigma_i$,

$$\forall a_\sigma \in A_{\sigma_1}^\sigma, \quad u_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*)(= \overline{u}_{\sigma_1}^\sigma(a_\sigma^*, a_{-\sigma}^*)) \geq u_{\sigma_1}^\sigma(a_\sigma, a_{-\sigma}^*)(= \overline{u}_{\sigma_1}^\sigma(a_\sigma, a_{-\sigma}^*)).$$

This means $a_i^* \in R_i(H)$. Hence $R_i(\overline{H}) \subseteq R_i(H)$. ∎

Lemma 3.1 says that an action of an agent is subjectively rationalizable in a hierarchical hypergame if and only if it is subjectively rationalizable in its extension as well.

On the other hand, the next lemma refers to the relationship between best response equilibria in an extended hierarchical hypergame and those in its original hypergame.

**Lemma 3.2** (Best response equilibria in extended hierarchical hypergames) *Let $H = (I, (G^\sigma)_{\sigma \in \Sigma})$ be a hierarchical hypergame and $\overline{H}$ be its extension. Then we have $BE(H) \subseteq BE(\overline{H})$. Particularly equality holds if* (i) *$\forall i \in I, I^i = I$, and* (ii) *$\forall i, j \in I, A_j^j \subseteq A_j^i$.*

*Proof* Let $\overline{G}^\sigma = (\overline{I}, \overline{A}, \overline{u}^\sigma)$ be the extended subjective game of $\sigma \in \Sigma$ induced from $H$. Consider a particular outcome of the hypergame $H$, $a^* = (a_i^*, a_{-i}^*) \in \times_{i \in I} A_i^i$. If the hypergame

$H$ and this $a^*$ do not satisfy the first condition in Definition 2.4, then $BE(H) = \phi$, and thus $BE(H) \subseteq BE(\overline{H})$.

On the other hand, suppose they satisfy the condition. Then let us suppose $a^* \in BE(H)$, which means, as the second condition in Definition 2.4 says,

$$\forall i \in I, \quad \forall a_i \in A_i^i, \quad u_i^i(a^*) \geq u_i^i(a_i, a_{-i}^*).$$

Then we have

$$\forall i \in I, \quad \forall a_i \in \overline{A}_i, \quad \overline{u}_i^i(a^*) \geq \overline{u}_i^i(a_i, a_{-i}^*)$$

because $\overline{u}_i^i(a^*) = u_i^i(a^*)$, and $\overline{u}_i^i(a_i, a_{-i}^*) = u_i^i(a_i, a_{-i}^*)$ if $a_i \in A_i^i$, while $\overline{u}_i^i(a_i, a_{-i}^*) = -\infty$ otherwise. Since all the agents see the common set of agents and their actions in the extended hypergame $\overline{H}$, $\overline{H}$ and $a^*$ necessarily satisfy the first condition in Definition 2.4. Therefore the equation above is equivalent to $a^* \in BE(\overline{H})$. Hence we have $BE(H) \subseteq BE(\overline{H})$.

Next, let us assume $H$ satisfies (i) $\forall i \in I, I^i = I$, and (ii) $\forall i, j \in I, A_j^j \subseteq A_j^i$. Suppose $a^* = (a_i^*, a_{-i}^*) \in BE(\overline{H})$, which means,

$$\forall i \in I, \quad \forall a_i \in \overline{A}_i, \quad \overline{u}_i^i(a^*) \geq \overline{u}_i^i(a_i, a_{-i}^*).$$

Since $\forall i \in I, \forall a_i \in \overline{A}_i \setminus A_i^i, \forall a_{-i} \in \times_{j \in \overline{I} \setminus \{i\}} \overline{A}_j, \overline{u}_i^i(a_i, a_{-i}) = -\infty$, we have $\forall i \in I, a_i^* \in A_i^i$. By (ii), then $\forall i, j \in I, a_j^* \in A_j^i$. Thus we have

$$\forall i \in I, \quad \forall a_i \in A_i^i, \quad u_i^i(a^*)(= \overline{u}_i^i(a^*)) \geq u_i^i(a_i, a_{-i}^*)(= \overline{u}_i^i(a_i, a_{-i}^*)).$$

This means $a^* \in BE(H)$. Under the conditions (i) and (ii), we have both $BE(H) \subseteq BE(\overline{H})$ and $BE(\overline{H}) \subseteq BE(H)$, hence $BE(H) = BE(\overline{H})$. ∎

Lemma 3.2 means that if an outcome is a best response equilibrium in a hierarchical hypergame, then it is so in its extension as well. Although the converse may not always hold, the lemma also specifies a sufficient condition for it to be true.

### 3.2 Bayesian Representation

We define Bayesian representations of hierarchical hypergames, transformed Bayesian games from hypergames, as below. Every viewpoint's subjective game has common sets of agents and actions in an extended hierarchical hypergame, so we capture perceptual differences in utility functions as types as Harsanyi's claim suggests. The types also reflect the hierarchy of perceptions in the hypergame[‡‡].

**Definition 3.2** (Bayesian representation of hierarchical hypergames) Let $H = (I, (G^\sigma)_{\sigma \in \Sigma})$ be a hierarchical hypergame and $\overline{G}^\sigma = (\overline{I}, \overline{A}, \overline{u}^\sigma)$ be the extended subjective game of $\sigma \in \Sigma$. $G^b(H) = (I, A, T, p, u)$ is called the Bayesian representation of $H$ iff it satisfies all of the following conditions:

- $I$ in $H$ and $I$ in $G^b(H)$ are identical (and equal to $\overline{I}$).

‡‡In this regard, Mertens and Zamir[10] presented the general way to deals with belief about the other's belief, belief about the other's belief about the other's belief, and so on, in Bayesian games. See also Myerson[16].

- $A = \overline{A}$.
- $T = \times_{i \in I} T_i$. For all $i \in I$ and $\sigma \in \Sigma$, $T_i = \{t_i^\sigma | \sigma \in \Sigma \wedge \sigma_1 = i\}$, where $t_i^\sigma \in T_i$ is a type of agent $i$ to whose view is associated with $\overline{G}^\sigma$.
- $p = (p_i)_{i \in I}$, where $p_i(\cdot|t_i)$ is agent $i$'s subjective prior, which is a joint probability distribution on $T_{-i}$ for each $t_i \in T_i$ such that for any $t_i^\sigma \in T_i$, $p_i(t_{-i}|t_i^\sigma) = 1$ if $t_{-i} = (t_j^{j\sigma})_{j \in I \setminus \{i\}}$, while $p_i(t_{-i}|t_i^\sigma) = 0$ otherwise.
- $u = (u_i)_{i \in I}$, where $u_i : A \times T \to \Re$ such that for any $a \in A, t_i^\sigma \in T_i$ and $t_{-i} \in T_{-i}$, $u_i(a, (t_i^\sigma, t_{-i})) = \overline{u}_i^\sigma(\overline{a})$, where $a = \overline{a}$.

It is the basic idea of the definition of types that $t_i^\sigma \in T_i$ is supposed to be a type of agent $i$ who believes that the game is $\overline{G}^\sigma$, viewpoint $\sigma$'s extended subjective game, and any agent $j(\neq i)$ believes the game is not $\overline{G}^\sigma$ but $\overline{G}^{j\sigma}$. Hence $t_i^\sigma$ assigns probability 1 to the combination of types of the other agents each of which is $t_j^{j\sigma}$ for any $j \neq i$ while assigning probability 0 to any other combinations, and has the same utility function as that in $\overline{G}^\sigma$, i.e., $\overline{u}_i^\sigma$. Types of an agent are defined for each viewpoint $\sigma$ whose lowest agent, $\sigma_1$, is the agent herself. Thus, for instance, $t_i^{ij} \in T_i$ but $t_i^{ji} \notin T_i$. This is because, in the hierarchical hypergame, nobody thinks that agent $i$ sees $\overline{G}^{ji}$. We do not include such types that every agent thinks "impossible" in the type set of the Bayesian representation. Since we deal with an infinite perception hierarchy in a hierarchical hypergame, the type space constructed in this way is also infinite. Note that the transformation into a Bayesian game is unique from a given hierarchical hypergame.

We call, for each $i \in I$, $t_i^i$ is the actual type of agent $i$ because, from an objective point of view, agent $i$ makes decision based on $G^i$ and believes any other agent $j$ plays according to $G^{ji}$. Therefore, the objective prior of Bayesian representations of a hierarchical hypergame is defined as follows.

**Definition 3.3** (Objective priors) Let $G^b(H) = (I, A, T, p, u)$ be the Bayesian representation of a hierarchical hypergame $H$. Then $p^o$ is called the objective prior of $G^b(H)$ iff for any $t \in T$,

$$p^o(t) = \begin{cases} 1, & \text{if } \forall i \in I, t_i = t_i^i, \\ 0, & \text{otherwise.} \end{cases}$$

The objective prior assigns probability 1 to the combination of actual types for each agent, while assigns probability 0 to any other combinations of types. Hence Bayesian representation of a hierarchical hypergame obviously does not assure that agents have consistent subjective priors.

## 4 Comparisons of Equilibrium Concepts

In this section, we show some propositions that describe relations between the equilibrium concepts for hierarchical hypergames and Bayesian games.

### 4.1 Subjective Rationalizability and Bayesian Nash Equilibria

Our first result refers to the relation between equilibrium concepts of hierarchical hypergames and their Bayesian representations (with subjective priors).

**Proposition 4.1** (Subjective rationalizability and Bayesian nash equilibria) *Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hierarchical hypergame and $G^b(H) = (I, A, T, p, u)$ be its Bayesian representation. Then $\forall i \in I$, $a_i^* \in R_i(H)$ iff there exists $s^* = (s_i^*, s_{-i}^*) \in BN(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$.*

*Proof* Let $\overline{G}^\sigma = (\overline{I}, \overline{A}, \overline{u}^\sigma)$ be the extended subjective game of $\sigma \in \Sigma$. Suppose $(s_i^*, s_{-i}^*) \in BN(G^b(H))$, which means $\forall i \in I, \forall t_i \in T_i, \forall s_i \in S_i$,

$$\sum_{t_{-i}\in T_{-i}} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i}))p_i(t_{-i}|t_i)) \geq \sum_{t_{-i}\in T_{-i}} u_i((s_i(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i}))p_i(t_{-i}|t_i).$$

In $G^b(H)$, this is equivalent to: $\forall i \in I, \forall t^\sigma \in T_i, \forall s_i \in S_i$,

$$u_i((s_i^*(t_i^\sigma), (s_j^*(t_j^{j\sigma}))_{j\in I\setminus\{i\}}), (t_i^\sigma, (t_j^{j\sigma})_{j\in I\setminus\{i\}})) \geq u_i((s_i(t_i^\sigma), (s_j^*(t_j^{j\sigma}))_{j\in I\setminus\{i\}}), (t_i^\sigma, (t_j^{j\sigma})_{j\in I\setminus\{i\}})),$$

which is equivalent to: $\forall i \in I, \forall t^\sigma \in T_i, \forall s_i \in S_i$,

$$\overline{u}_i^\sigma(s_i^*(t_i^\sigma), (s_j^*(t_j^{j\sigma}))_{j\in I\setminus\{i\}}) \geq \overline{u}_i^\sigma((s_i(t_i^\sigma), (s_j^*(t_j^{j\sigma}))_{j\in I\setminus\{i\}})),$$

which is equivalent to: $\forall \sigma \in \Sigma, \forall s_{\sigma_1} \in S_{\sigma_1}$,

$$\overline{u}_{\sigma_1}^\sigma(s_{\sigma_1}^*(t_{\sigma_1}^\sigma), (s_j^*(t_j^{j\sigma}))_{j\in I\setminus\{\sigma_1\}}) \geq \overline{u}_{\sigma_1}^\sigma((s_{\sigma_1}(t_{\sigma_1}^\sigma), (s_j^*(t_j^{j\sigma}))_{j\in I\setminus\{\sigma_1\}})),$$

which is equivalent to the fact that, for any $i \in I$, $s_{\sigma_1}^*(t_{\sigma_1}^\sigma)$ for each $\sigma \in \Sigma_i$ constitutes a best response hierarchy in $\overline{H}$, the extension of $H$, and hence $\forall i \in I, s_i^*(t_i^i) \in R_i(\overline{H})$.

After all, $\forall i \in I, a_i^* = R_i(\overline{H})$ iff there exists $(s_i^*, s_{-i}^*) \in BN(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$. Since $R_i(\overline{H}) = R_i(H)$ for any $i \in I$ (due to Lemma 3.1), we have $\forall i \in I, a_i^* = R_i(H)$. Hence we have the proposition. ∎

Proposition 4.1 claims that a hierarchical hypergame has such an outcome in which every agent chooses each subjectively rationalizable action if and only if its Bayesian representation has such a Bayesian Nash equilibrium in which the actual type of each agent chooses the same action as the agent's subjectively rationalizable action. It implies that, from an analyzer's point of view, the both formulations, hypergames and Bayesian games, predict same outcomes by using these equilibrium concepts.

### 4.2 Best Response Equilibria and Nash Equilibria of Bayesian Games

Next, if we consider Bayesian representations with objective priors, we have next proposition.

**Proposition 4.2** (Best response equilibria (in extended hypergames) and nash equilibria of Bayesian games) *Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hierarchical hypergame, $\overline{H}$ be its extension, and $G^b(H) = (I, A, T, p^o, u)$ be the Bayesian representation of $H$. Then $a^* = (a_i^*, a_{-i}^*) \in BE(\overline{H})$ iff there exists $s^* = (s_i^*, s_{-i}^*) \in N(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$.*

*Proof* Let $\overline{G}^{\sigma} = (\overline{I}, \overline{A}, \overline{u}^{\sigma})$ be the extended subjective game of $\sigma \in \Sigma$ induced from $H$. Suppose $(s_i^*, s_{-i}^*) \in N(G^b(H))$, which means $\forall i \in I, \forall s_i \in S_i$,

$$\sum_{t \in T} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), t)p^o(t) \geq \sum_{t \in T} u_i((s_i(t_i), s_{-i}^*(t_{-i})), t)p^o(t).$$

In $G^b(H)$, this is equivalent to: $\forall i \in I, \forall s_i \in S_i$,

$$u_i((s_i^*(t_i^i), (s_j^*(t_j^j))_{j \in I \setminus \{i\}}), (t_i^i, (t_j^j)_{j \in I \setminus \{i\}})) \geq u_i((s_i(t_i^i), (s_j^*(t_j^j))_{j \in I \setminus \{i\}}), (t_i^i, (t_j^j)_{j \in I \setminus \{i\}})),$$

which is equivalent to: $\forall i \in I$, $\forall s_i \in S_i$,

$$\overline{u}_i^i(s_i^*(t_i^i), (s_j^*(t_j^j))_{j \in I \setminus \{i\}}) \geq \overline{u}_i^i(s_i(t_i^i), (s_j^*(t_j^j))_{j \in I \setminus \{i\}}).$$

Thus, $\forall i \in I$, $\forall a_i \in \overline{A}_i, \overline{u}_i^i(a_i^*, a_{-i}^*) \geq \overline{u}_i^i(a_i, a_{-i}^*)$ iff there exists $s^* = (s_i^*, s_{-i}^*) \in N(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$. The former of the statement is by definition equivalent to $(a_i^*, a_{-i}^*) \in BE(\overline{H})$. Hence we have the proposition. ∎

Proposition 4.2 says that an extended hierarchical hypergame has a best response equilibrium if and only if its Bayesian representation (with objective prior) has such a Nash equilibrium in which the actual type of each agent chooses the same action as the one in the best response equilibrium. Combined with Lemma 3.2, it implies the next proposition.

**Proposition 4.3** (Best response equilibria (in original hypergames) and nash equilibria of Bayesian games)) *Let $H$ be a hierarchical hypergame and $G^b(H) = (I, A, T, p^o, u)$ be its Bayesian representation. If $a^* = (a_i^*, a_{-i}^*) \in BE(H)$, then there exists $s^* = (s_i^*, s_{-i}^*) \in N(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$. Particularly, if $H$ satisfies the sufficient condition of Lemma* 3.2*, the converse also holds.*

*Proof* The proposition is straightforward from Lemma 3.2 and Proposition 4.2. ∎

Proposition 4.3 states that if an outcome is a best response equilibrium in a hierarchical hypergame, then its Bayesian representation has such a Nash equilibrium in which the actual type of each agent chooses the same action as the one in the best response equilibrium. Particularly, if the hierarchical hypergame satisfies the sufficient condition of Lemma 3.2, the converse is also true.

## 5 Conclusions and Discussions

We have studied the relationship of two independently developed models of games with incomplete information, hierarchical hypergames and Bayesian games. We first proposed the general way of transformation of hierarchical hypergames into Bayesian games, which we call Bayesian representation of hierarchical hypergames, and showed that any hierarchical hypergames can be reformulated in terms of Bayesian games which may have inconsistent subjective priors. We then proved some propositions that associate equilibrium concepts for hierarchical hypergames with those for Bayesian games.

We interpret the results presented in the previous section as follows. Any hierarchical hypergames can be analyzed in terms of Bayesian games as long as our interest is in the two equilibrium concepts of hypergames we discussed, namely subjective rationalizability and best response equilibrium. (The converse obviously does not hold: Bayesian games may not be transformed into and analyzed in terms of hypergames in general.) As Proposition 4.1 suggests, predicted outcomes are the same whether we analyze the situation in question as a hypergame with subjective rationalizability or as a Bayesian game with Bayesian Nash equilibrium. On the other hand, Proposition 4.3 implies that best response equilibria in a hypergame is always included in Nash equilibria of its Bayesian representation. It also shows a sufficient condition under which the equality holds.

Hypergame theorists may say it would not be realistic to accept Harsanyi's claim in some situations. This might be convincing from an agent's point of view. However, from an analyzer's point of view, the propositions above imply that it is harmless to analyze Bayesian representations of hirerarchical hypergames as if we accept Harsanyi's claim. Hence Bayesian games are general enough in theory, as the standard arguments suggest, in that any hypergame situations can be formulated as Bayesian games.

We, however, point out that there are at least three advantages of choosing the hypergame formulation. First, the hypergame structure is more intuitive and simpler and thus requires us less tasks in modeling. Furthermore, with regard to Proposition 4.1, the computational complexity in calculating equilibria is also much less in analyzing hypergames than Bayesian games when studying hypergames with inside common knowledge. The difference in the computational complexity comes from the fact that, when analyzing subjective rationalizability in hypergames with inside common knowledge, we need only investigate each agent's behavior in not all but just some subjective games (see Subsection 2.3.2), while we have to consider infinitely many types' best responses in order to calculate Bayesian Nash equilibrium. To understand this, for instance, consider a hierarchical hypergame in which agent $i$ has inside common knowledge, that is, she believes $G^i$ is common knowledge. Then $R_i(H)$ is equal to the set of her (standard) rationalizable actions in $G^i$, which can be easily calculated by using the standard game theory's technique. On the other hand, if we analyze this situation as a Bayesian game, we have to identify best responses of all the types associated to viewpoints lower than $i$ in order to Bayesian Nash equilibrium.

Second, as Proposition 4.1 suggests the connection of subjective rationalizability and Bayesian Nash equilibrium, some stronger solution concept than subjective rationalizability can provide us unique insights that cannot be captured by Bayesian games with their existing equilibrium concepts. For simple hypergames, since it has been shown that hyper Nash equilibrium and Bayesian Nash equilibrium are connected in a similar manner, stable hyper Nash equilibrium[17], a more strict concept than hyper Nash equilibrium, can lead to unique insights as discussed in the previous study[3]. Although any concept that can deal with the kind of stability that stable hyper Nash equilibrium can capture has not yet been proposed for hierarchical hypergames, such a concept would enhance uniqueness of hypergame analysis.

Third, since the Bayesian game modeling requires all the agents to be aware of all the possibilities relevant to the situation, i.e., types, if we want to describe an agent's unawareness[18], the hypergame modeling would be more convincing from the epistemic point of view. In this case, the hypergame can technically be transformed into and analyzed in terms of a Bayesian game as mentioned above, but Harsanyi's claim that requires full awareness would become incompatible with the epistemic supposition.

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