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Optimal Investment Problem for an Insurer and a Reinsurer*

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Abstract This paper studies the optimal investment problem for an insurer and a reinsurer. The basic claim process is assumed to follow a Brownian motion with drift and the insurer can purchase proportional reinsurance from the reinsurer. The insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. Moreover, the authors consider the correlation between the claim process and the price process of the risky asset. The authors first study the optimization problem of maximizing the expected exponential utility of terminal wealth for the insurer. Then with the optimal reinsurance strategy chosen by the insurer, the authors consider two optimization problems for the reinsurer: The problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the ruin probability. By solving the corresponding Hamilton-Jacobi-Bellman equations, the authors derive the optimal reinsurance and investment strategies, explicitly. Finally, the authors illustrate the equality of the reinsurer's optimal investment strategies under the two cases.

Keywords Hamilton-Jacobi-Bellman equation, optimal reinsurance and investment strategies, proportional reinsurance, ruin probability, utility maximization.

1 Introduction

Since insurance companies are allowed to invest in the financial market, lots of researches study the investment problem for an insurer. Among these researches, the main investment objectives are ruin probability minimization and expected utility maximization. For example, Hipp and Plum[1] considered the optimal investment problem for an insurer whose risk process modelled as a compound Poisson process in the sense of minimizing the ruin probability.

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Schmidli[2] used a claim process that followed a Brownian motion with drift and obtained the optimal reinsurance strategy to minimize the ruin probability. Promislow and Young[3] discussed the problem of ruin probability minimization subject to both proportional reinsurance and investment strategies. Li and Wu[4] investigated the upper bound for finite-time ruin probability of an insurer in a Markov-modulated market. Liang and Guo[5] focused on the optimal proportional reinsurance and investment problem in the Sparre Andersen model and derived the explicit expression of the ruin probability or its lower bound when the claim sizes were exponentially. Yang and Zhang[6] studied an insurer's investment problem of exponential utility maximization with jump-diffusion risk process. Wang[7] obtained the optimal investment strategy for an insurer who was allowed to invest in multiple risky assets in the sense of maximizing the exponential utility of his/her reserve at a future time. Cao and Wan[8] considered the optimal reinsurance and investment problem of maximizing the expected utility of terminal wealth and gave the results for exponential and power utility functions. Bai and Guo[9] discussed the optimal excess-of-loss reinsurance and multidimensional portfolio selection problem. Liang, et al.[10] derived the optimal proportional reinsurance and investment strategies in a stock market with Ornstein-Uhlenbeck process. Liang and Bayraktar[11] investigated the optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution were not known but had to be inferred from the observations of claim arrivals. Besides the objectives of ruin probability minimization and expected utility maximization, Bai and Zhang[12], Bi, et al.[13], Zeng and Li[14], Li and Li[15] studied the optimal reinsurance and investment problem for an insurer under the mean-variance criterion. Moreover, some researchers introduced stochastic volatility models into the optimal reinsurance and investment problem, see, Gu, et al.[16], Lin and Li[17], Liang, et al.[18], Gu, et al.[19], Li, et al.[20], Zhao, et al.[21] and the references therein.

However, large numbers of articles only consider the optimization problem for an insurer. Actually, the management of the reinsurer is also meaningful. Thus, we focus on the optimal investment problem for an insurer and a reinsurer. In our model, the basic claim process is assumed to follow a Brownian motion with drift. The insurer can purchase proportional reinsurance from the reinsurer and both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. Moreover, we consider the correlation between the claim process and the price process of the risky asset. The effect of the net profit on proportional reinsurance is also taken into account. Firstly, we consider the optimization problem of maximizing the insurer's expected exponential utility of terminal wealth. By solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation, we derive the optimal reinsurance and investment strategies of the insurer, explicitly. Then according to the insurer's optimal reinsurance strategy, the reinsurer chooses the optimal investment strategy, i.e., the reinsurer increases his/her wealth via investment. Here, we consider two optimal investment problems for the reinsurer: The problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the ruin probability. Furthermore, we obtain the optimal investment strategies of the reinsurer. Finally, since Browne[22], Bai and Guo[23] both found that maximizing the expected exponential utility of terminal wealth and minimizing the ruin probability produced strategies with the same type for zero interest rate. We examine the equality of the reinsurer's optimal investment strategies under the two cases. The model we study in this paper has some practical meaning. In reality, the insurer usually plays a leading role. If the optimal reinsurance strategy of the insurer is smaller than that of the reinsurer, the reinsurer will accept the optimal reinsurance strategy chosen by the insurer, while in the opposite case, the reinsurer may not have enough economic strength to undertake the optimal reinsurance strategy chosen by the insurer, then the reinsurer will try to make more profits from investment, all of which can be described by our model.

The paper proceeds as follows. In Section 2, we introduce the formulation of the model. Section 3 provides the optimal proportional reinsurance and investment problem of maximizing expected exponential utility of terminal wealth for an insurer. In Section 4, with the optimal proportional reinsurance which has been obtained in Section 3, we derive the optimal investment strategies for a reinsurer in the sense of maximizing the expected exponential utility of terminal wealth and minimizing the ruin probability. Then we illustrate the equality between the two results. Section 5 concludes the paper.

2 Model Formulation

Consider a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ satisfying the usual condition, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration with $\mathcal{F} = \mathcal{F}_T$, and T is a fixed and finite time horizon. All stochastic processes introduced below are supposed to be adapted processes in this space.

In this paper, we model the claim process $C(t)$ according to a Brownian motion with drift as

$$dC(t) = adt - bdW_0(t), \tag{1}$$

where a and b are positive constants. $W_0(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$. Suppose that the premium is paid continuously at a constant rate $c = (1+\theta)a$ with the safety loading of the insurer $\theta > 0$. According to Equation (1), the surplus process of an insurer is given by

$$dR(t) = cdt - dC(t) = a\theta dt + bdW_0(t).$$

We assume that the insurer can purchase proportional reinsurance to reduce the underlying insurance risk and $p(t)$ represents the proportion reinsured at time t. Considering the proportional reinsurance, the surplus processes of the insurer and the reinsurer are

$$\begin{aligned} dR_1(t) &= (\theta - \eta p(t))adt + b(1-p(t))dW_0(t), \\ dR_2(t) &= \eta p(t)adt + bp(t)dW_0(t), \end{aligned}$$

where $\eta > \theta$ represents the safety loading of the reinsurer. The net profit of the insurer is $(1+\theta)a - (1+\eta)p(t)a - (1-p(t))a$ and that of the reinsurer is $(1+\eta)p(t)a - p(t)a$. Since the net profits of the insurer and the reinsurer are positive, $p(t)$ satisfies $0 \le p(t) \le \frac{\theta}{\eta} < 1$.

We consider a financial market consisting of a risk-free asset with the price process $S_0(t)$ given by

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1$$

and a risky asset with the price process $S(t)$ described by

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t)\right),$$

where r is the interest rate, μ, σ denote the appreciation rate and volatility of the risky asset's price process. $W(t)$ is another standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and $\operatorname{cov}[W_0(t), W(t)] = \rho_0 t$. As usual, we assume that $\mu > r$.

Suppose that the insurer and the reinsurer are allowed to invest their surplus in the financial market. Let $\pi_1(t)$, $\pi_2(t)$ represent the amounts invested in the risky asset by the insurer and the reinsurer at time t, respectively. Corresponding to the reinsurance and investment, the surplus processes of the insurer and the reinsurer $X(t)$ and $Y(t)$ are

$$dX(t) = [rX(t) + \pi_1(t)(\mu - r) + (\theta - \eta p(t))a]dt + \pi_1(t)\sigma dW(t) + b(1 - p(t))dW_0(t), \quad (2)$$

$$dY(t) = [rY(t) + \pi_2(t)(\mu - r) + \eta p(t)a]dt + \pi_2(t)\sigma dW(t) + bp(t)dW_0(t). \quad (3)$$

$(p(t), \pi_1(t))$ and $\pi_2(t)$ are said to be admissible if they are $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable and satisfy $(p(t), \pi_1(t)) \in \Pi_1$, $\pi_2(t) \in \Pi_2$, where $\Pi_1 = \{(p(t), \pi_1(t)) : 0 \le p(t) \le \frac{\theta}{\eta}, \mathrm{E}[\int_0^T (\pi_1(t))^2 dt] < \infty\}$ and $\Pi_2 = \{\pi_1(t) : \mathrm{E}[\int_0^T (\pi_2(t))^2 dt] < \infty\}$.

3 Maximizing the Utility Function for the Insurer

Suppose that the insurer has a utility function $U_1(x)$ which is strictly concave and continuously differentiable on $(-\infty, \infty)$. In this section, we try to find a strategy $(p(t), \pi_1(t))$ that maximizes the expected utility of the insurer's terminal wealth, i.e.,

$$\max_{(p,\pi_1)\in\Pi_1} \mathrm{E}[U_1(X(T))].$$

The corresponding HJB equation associated with the optimization problem is

$$V_t + \sup_{(p,\pi_1)\in\Pi_1} \left\{ [rx + \pi_1(\mu - r) + a(\theta - \eta p)]V_x + \frac{1}{2}\left[\sigma^2\pi_1^2 + b^2(1-p)^2 + 2\sigma\pi_1 b(1-p)\rho_0\right]V_{xx} \right\} = 0 \quad (4)$$

with $V(T, x) = U_1(x)$ and V_t, V_x, V_{xx} denote partial derivatives of the first and second orders with respect to (w.r.t.) t and x.

The first order maximizing condition for the optimal investment strategy is

$$\pi_1^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{V_x}{V_{xx}} - \frac{b\rho_0(1-p)}{\sigma}. \quad (5)$$

Putting Equation (5) into HJB equation (4), after simplification, we have

$$V_t + rxV_x + a\theta V_x - ap\eta V_x - \frac{b\rho_0(1-p)(\mu - r)}{\sigma}V_x + \frac{b^2(1-p)^2(1-\rho_0^2)}{2}V_{xx} - \frac{(\mu-r)^2}{2\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0 \quad (6)$$

with $V(T,x)=U_1(x)$.

Differentiating w.r.t. $p(t)$ in Equation (6) gives

$$p^0 = 1+\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2\sigma(1-\rho_0^2)}\cdot\frac{V_x}{V_{xx}}. \tag{7}$$

If $0\le p^0(t)\le\frac{\theta}{\eta}$, the optimal reinsurance proportion $p^*(t)$ coincides with $p^0(t)$. If $p^0(t)\le 0$, we simply let $p^*(t)$ be 0. And if $p^0(t)\ge\frac{\theta}{\eta}$, we set $p^*(t)=\frac{\theta}{\eta}$.

Assume that the insurer has an exponential utility function $U_1(x)$

$$U_1(x)=-\frac{1}{m_1}\mathrm{e}^{-m_1x},\qquad m_1>0, \tag{8}$$

which has a constant absolute risk aversion parameter m_1 and plays an important role in insurance mathematics and actuarial practice.

By applying stochastic control theory, we can derive the optimal reinsurance and investment strategies and the corresponding value function when maximizing the insurer's expected exponential utility of terminal wealth, which are given in the following theorem. Details are shown in Appendix.

Theorem 3.1 *When maximizing the expected exponential utility of terminal wealth, the optimal reinsurance and investment strategies and the corresponding value function of the insurer are as follows:*

1) *If* $\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}>1$ *and* $\mathrm{e}^{rT}>\frac{[a\sigma\eta-b\rho_0(\mu-r)]\eta}{b^2m_1\sigma(1-\rho_0^2)(\eta-\theta)}$, *the optimal reinsurance and investment strategies are*

$$p^*(t)=\begin{cases}\dfrac{\theta}{\eta}, & 0\le t<t_0,\\[2ex] 1-\dfrac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}, & t_0\le t<t_1,\\[2ex] 0, & t_1\le t\le T,\end{cases}$$

$$\pi_1^*(t)=\begin{cases}\dfrac{\mu-r}{\sigma^2m_1}\mathrm{e}^{-r(T-t)}-\dfrac{b\rho_0(\eta-\theta)}{\sigma\eta}, & 0\le t<t_0,\\[2ex] \dfrac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\dfrac{\mu-r}{m_1}-\dfrac{a\rho_0\sigma\eta-b\rho_0^2(\mu-r)}{m_1b(1-\rho_0^2)}\right], & t_0\le t<t_1,\\[2ex] \dfrac{\mu-r}{\sigma^2m_1}\mathrm{e}^{-r(T-t)}-\dfrac{b\rho_0}{\sigma}, & t_1\le t\le T,\end{cases}$$

and the value function is given by Equation (34).

2) *If* $\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}>1$ *and* $\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\le\mathrm{e}^{rT}\le\frac{[a\sigma\eta-b\rho_0(\mu-r)]\eta}{b^2m_1\sigma(1-\rho_0^2)(\eta-\theta)}$, *the optimal reinsurance and investment strategies are*

$$p^*(t)=\begin{cases}1-\dfrac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}, & 0\le t<t_1,\\[2ex] 0, & t_1\le t\le T,\end{cases}$$

$$\pi_1^*(t) = \begin{cases} \dfrac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\dfrac{\mu-r}{m_1} - \dfrac{a\rho_0\sigma\eta - b\rho_0^2(\mu-r)}{m_1 b(1-\rho_0^2)}\right], & 0 \le t < t_1, \\ \dfrac{\mu-r}{\sigma^2 m_1}\mathrm{e}^{-r(T-t)} - \dfrac{b\rho_0}{\sigma}, & t_1 \le t \le T, \end{cases}$$

and the value function is given in Equation (38)*.*

3) *If* $\frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} > 1$ *and* $\mathrm{e}^{rT} < \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)}$*, the optimal reinsurance and investment strategies are*

$$p^*(t) = 0, \quad 0 \le t \le T,$$
$$\pi_1^*(t) = \frac{\mu-r}{\sigma^2 m_1}\mathrm{e}^{-r(T-t)} - \frac{b\rho_0}{\sigma}, \quad 0 \le t \le T,$$

and the value function is given by Equation (39)*.*

4) *If* $\frac{\eta-\theta}{\eta} < \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} \le 1$ *and* $\mathrm{e}^{rT} > \frac{[a\sigma\eta - b\rho_0(\mu-r)]\eta}{b^2 m_1\sigma(1-\rho_0^2)(\eta-\theta)}$*, the optimal reinsurance and investment strategies are*

$$p^*(t) = \begin{cases} \dfrac{\theta}{\eta}, & 0 \le t < t_1, \\ 1 - \dfrac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}, & t_1 \le t \le T, \end{cases}$$
$$\pi_1^*(t) = \begin{cases} \dfrac{\mu-r}{\sigma^2 m_1}\mathrm{e}^{-r(T-t)} - \dfrac{b\rho_0(\eta-\theta)}{\sigma\eta}, & 0 \le t < t_1, \\ \dfrac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\dfrac{\mu-r}{m_1} - \dfrac{a\rho_0\sigma\eta - b\rho_0^2(\mu-r)}{m_1 b(1-\rho_0^2)}\right], & t_1 \le t \le T, \end{cases}$$

and the value function is given in Equation (40)*.*

5) *If* $\frac{\eta-\theta}{\eta} < \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} \le 1$ *and* $\mathrm{e}^{rT} \le \frac{[a\sigma\eta - b\rho_0(\mu-r)]\eta}{b^2 m_1\sigma(1-\rho_0^2)(\eta-\theta)}$*, the optimal reinsurance and investment strategies are*

$$p^*(t) = 1 - \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}, \quad 0 \le t \le T,$$
$$\pi_1^*(t) = \frac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\frac{\mu-r}{m_1} - \frac{a\rho_0\sigma\eta - b\rho_0^2(\mu-r)}{m_1 b(1-\rho_0^2)}\right], \quad 0 \le t \le T,$$

and the value function is given by Equation (43)*.*

6) *If* $\frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} \le \frac{\eta-\theta}{\eta}$*, the optimal reinsurance and investment strategies are*

$$p^*(t) = \frac{\theta}{\eta}, \quad 0 \le t \le T,$$
$$\pi_1^*(t) = \frac{\mu-r}{\sigma^2 m_1}\mathrm{e}^{-r(T-t)} - \frac{b\rho_0(\eta-\theta)}{\sigma\eta}, \quad 0 \le t \le T,$$

and the value function is given in Equation (44)*.*

The proof can be seen in the appendix.

4 Optimal Investment Problems for the Reinsurer Under Two Cases

In reality, the insurer usually plays a leading role. Through the similar method, we can also derive the optimal reinsurance strategy of the reinsurer. If the optimal reinsurance strategy of the insurer is smaller than that of the reinsurer, the reinsurer will accept the optimal reinsurance strategy chosen by the insurer, while in the opposite case, the reinsurer may not have enough economic strength to undertake the optimal reinsurance strategy chosen by the insurer, then the reinsurer will try to make more profits from investment. In this section, with the optimal reinsurance strategy of the insurer, we consider two optimization problems for the reinsurer: The problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the ruin probability.

4.1 Maximizing the Expected Exponential Utility of Terminal Wealth

In this subsection, we will find the optimal investment strategy in the case of maximizing the reinsurer's expected exponential utility of terminal wealth. Suppose that the reinsurer has a utility function $U_2(y)$ which is strictly concave and continuously differentiable on $(-\infty,\infty)$. The objective of this problem is to find the optimal value function

$$\max_{\pi_2\in\varPi_2} \mathrm{E}[U_2(Y(T))].$$

The corresponding HJB equation is

$$H_t+\sup_{\pi_2\in\varPi_2}\left\{[ry+\pi_2(\mu-r)+ap^*\eta]H_y+\frac{1}{2}[\sigma^2\pi_2^2+b^2(p^*)^2+2\sigma\pi_2 bp^*\rho_0]H_{yy}\right\}=0 \tag{9}$$

with $H(T,y)=U_2(y)$, where H_t,H_y,H_{yy} denote partial derivatives of the first and second orders w.r.t. t and y.

The first order maximizing condition for the optimal investment strategy of the reinsurer is

$$\pi_2^*=-\frac{\mu-r}{\sigma^2}\cdot\frac{H_y}{H_{yy}}-\frac{bp^*\rho_0}{\sigma}. \tag{10}$$

Plugging Equation (10) into HJB equation (9) yields

$$H_t+ryH_y+a\eta p^*H_y-\frac{b\rho_0p^*(\mu-r)}{\sigma}H_y+\frac{b^2(p^*)^2(1-\rho_0^2)}{2}H_{yy}-\frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}}=0 \tag{11}$$

with $H(T,y)=U_2(y)$.

Suppose that the reinsurer also has an exponential utility function $U_2(y)$

$$U_2(y)=-\frac{1}{m_2}\mathrm{e}^{-m_2y},\qquad m_2>0. \tag{12}$$

Following the exponential utility function described by Equation (12), we assume that the solution to Equation (11) has the following form

$$H(t,y)=-\frac{1}{m_2}\mathrm{e}^{-m_2y\mathrm{e}^{r(T-t)}+g(t)} \tag{13}$$

with the boundary condition given by $g(T)=0$.

Then

$$H_t=(m_2rye^{r(T-t)}+g_t)H,\quad H_y=-m_2e^{r(T-t)}H,\quad H_{yy}=m_2^2e^{2r(T-t)}H,$$

where g_t denotes the derivative of $g(t)$ w.r.t. t.

Now we try to find the solution to Equation (11) and recover $\pi_2^*(t)$ from derivatives of $H(t,y)$ for different reinsurance proportion.

If $p^*(t)=p^0(t)$, Equation (11) is simplified into

$$\begin{aligned}&H_t+ryH_y+\left[1-\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}e^{-r(T-t)}\right]a\eta H_y\\&-\left[1-\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}e^{-r(T-t)}\right]\cdot\frac{b\rho_0(\mu-r)H_y}{\sigma}\\&+\left[1-\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}e^{-r(T-t)}\right]^2\frac{b^2(1-\rho_0^2)H_{yy}}{2}-\frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}}=0,\end{aligned}\tag{14}$$

if $p^*(t)=0$, Equation (11) becomes

$$H_t+ryH_y-\frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}}=0,\tag{15}$$

and if $p^*(t)=\frac{\theta}{\eta}$, Equation (11) is transformed into

$$H_t+ryH_y+a\theta H_y-\frac{b\rho_0\theta(\mu-r)}{\sigma\eta}H_y+\frac{b^2\theta^2(1-\rho_0^2)}{2\eta^2}H_{yy}-\frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}}=0.\tag{16}$$

The procedures of solving Equations (14)–(16) are similar to those for an insurer in Section 3, so we omit them here.

From the above analysis, we propose the optimal investment strategies of the reinsurer who aims to maximize the expected exponential utility of terminal wealth with the optimal reinsurance strategy of the insurer in the following theorem.

Theorem 4.1 *According to the optimal reinsurance strategy chosen by the insurer, the optimal investment strategy of the reinsurer in the sense of maximizing the expected exponential utility of terminal wealth is given as follows:*

1) *If the optimal reinsurance proportion of the insurer is* $p^*(t)=p^0(t)$, *then*

$$\pi_2^*(t)=\frac{e^{-r(T-t)}}{\sigma^2}\left[\frac{\mu-r}{m_2}+\frac{a\rho_0\sigma\eta-b\rho_0^2(\mu-r)}{m_1b(1-\rho_0^2)}\right]-\frac{b\rho_0}{\sigma}.$$

2) *If* $p^*(t)=0$, *we have*

$$\pi_2^*(t)=\frac{\mu-r}{\sigma^2m_2}e^{-r(T-t)}.$$

3) *If* $p^*(t)=\frac{\theta}{\eta}$, *the reinsurer's optimal investment strategy is*

$$\pi_2^*(t)=\frac{\mu-r}{\sigma^2m_2}e^{-r(T-t)}-\frac{b\rho_0\theta}{\sigma\eta}.$$

Proof 1) If $p^*(t) = p^0(t)$, from Equations (10) and (13), we obtain

$$\begin{aligned}\pi_2^* &= -\frac{\mu - r}{\sigma^2} \cdot \frac{H_x}{H_{xx}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu - r}{\sigma^2} \cdot \frac{\mathrm{e}^{-r(T-t)}}{-m_2} - \frac{b\rho_0}{\sigma} \cdot \left[1 - \frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2 m_1 \sigma(1-\rho_0^2)} \mathrm{e}^{-r(T-t)}\right] \\ &= \frac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\frac{\mu - r}{m_2} + \frac{a\rho_0\sigma\eta - b\rho_0^2(\mu - r)}{m_1 b(1-\rho_0^2)}\right] - \frac{b\rho_0}{\sigma}.\end{aligned}$$

2) If $p^*(t) = 0$, according to Equations (10) and (13), we derive

$$\pi_2^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{H_x}{H_{xx}} - \frac{bp^*\rho_0}{\sigma} = -\frac{\mu - r}{\sigma^2} \cdot \frac{\mathrm{e}^{-r(T-t)}}{-m_2} = \frac{\mu - r}{\sigma^2 m_2}\mathrm{e}^{-r(T-t)}.$$

3) If $p^*(t) = \frac{\theta}{\eta}$, from Equations (10) and (13), we have

$$\pi_2^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{H_x}{H_{xx}} - \frac{bp^*\rho_0}{\sigma} = -\frac{\mu - r}{\sigma^2} \cdot \frac{\mathrm{e}^{-r(T-t)}}{-m_2} - \frac{b\rho_0\theta}{\sigma\eta} = \frac{\mu - r}{\sigma^2 m_2}\mathrm{e}^{-r(T-t)} - \frac{b\rho_0\theta}{\sigma\eta}.$$

The proof is completed. ∎

4.2 Minimizing the Ruin Probability

In this subsection, we want to find the reinsurer's optimal investment strategy in the sense of minimizing the ruin probability with the optimal reinsurance strategy of the insurer. Let $\tau^{\pi_2} = \inf\{t : Y(t) < 0\}$ be the first time when the surplus process of the reinsurer becomes negative. Since the underlying risk model is a Brownian motion with drift, it is known that $\tau^{\pi_2} = \inf\{t : Y(t) = 0\}$ with probability 1. Denote the ruin probability, given the initial reserve y, by

$$\phi^{\pi_2}(y) = P(\tau^{\pi_2} < \infty | Y_0 = y),$$

and the minimum ruin probability by

$$\phi(y) = \inf_{\pi_2} \phi^{\pi_2}(y).$$

$\phi(y)$ is convex and decreasing on $y \in (0, \infty)$. Our goal is to find the minimum ruin probability $\phi(y)$ and the optimal investment strategy $\pi_2^*(t)$ such that $\phi(y) = \phi^{\pi_2^*}(y)$.

To solve the above problem, we consider the following HJB equation

$$\min_{\pi_2 \in \Pi_2}\left\{[ry + \pi_2(\mu - r) + ap^*\eta]\phi_y + \frac{1}{2}[\sigma^2\pi_2^2 + b^2(p^*)^2 + 2\sigma\pi_2 bp^*\rho_0]\phi_{yy}\right\} = 0 \tag{17}$$

with the boundary conditions $\phi(0) = 1$ and $\phi(\infty) = 0$.

Assume that there exists a solution ϕ satisfying $\phi_y < 0$, $\phi_{yy} > 0$ on the interval $y \in (0, \infty)$ and $\phi(0) = 1, \phi(\infty) = 0$. Differentiating w.r.t. $\pi_2(t)$ in Equation (17), we obtain

$$\pi_2^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{\phi_y}{\phi_{yy}} - \frac{b\rho_0 p^*}{\sigma}. \tag{18}$$

Introducing Equation (18) into HJB equation (17), after simplification, we have

$$ry\phi_y + ap^*\eta\phi_y - \frac{b\rho_0 p^*(\mu - r)}{\sigma}\phi_y + \frac{b^2(p^*)^2(1-\rho_0^2)}{2}\phi_{yy} - \frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{\phi_y^2}{\phi_{yy}} = 0 \tag{19}$$

with $\phi(0) = 1, \phi(\infty) = 0$.

From the surplus process of the reinsurer (3), we find that when $p^*(t) = 0$, $\pi_2(t) = 0$ and $y \geq 0$, the ruin of the reinsurer will not occur, that is $\phi(y) = 0$. So in the following part, we derive the explicit expressions for $\phi(y)$ in the cases that $p^*(t) = p^0(t)$ and $p^*(t) = \frac{\theta}{\eta}$.

If $p^*(t) = p^0(t)$, Equation (19) is simplified into

$$\begin{aligned} &ry\phi_y + a\eta\phi_y - \frac{b\rho_0(\mu-r)}{\sigma}\phi_y - \frac{(a\sigma\eta - b\rho_0(\mu-r))^2}{m_1 b^2\sigma^2(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}\phi_y \\ &+ \left[1 - \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}\right]^2 \frac{b^2(1-\rho_0^2)\phi_{yy}}{2} - \frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{\phi_y^2}{\phi_{yy}} = 0 \end{aligned} \tag{20}$$

with the boundary conditions $\phi(0) = 1$ and $\phi(\infty) = 0$. Set $h_1(y) = \frac{\phi_y}{\phi_{yy}}$, from Equation (20) we derive

$$\begin{aligned} &\frac{(\mu-r)^2}{2\sigma^2}h_1(y)^2 - \left[ry + a\eta - \frac{b\rho_0(\mu-r)}{\sigma} - \frac{(a\sigma\eta - b\rho_0(\mu-r))^2}{m_1 b^2\sigma^2(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}\right] h_1(y) \\ &- \left[1 - \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}\right]^2 \frac{b^2(1-\rho_0^2)}{2} = 0. \end{aligned}$$

Let

$$N_1 = ry + a\eta - \frac{b\rho_0(\mu-r)}{\sigma} - \frac{(a\sigma\eta - b\rho_0(\mu-r))^2}{m_1 b^2\sigma^2(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}$$

and due to $\phi_y < 0, \phi_{yy} > 0$, we obtain

$$h_1(y) = \frac{\sigma^2 N_1 - \sigma^2\sqrt{N_1^2 + \frac{b^2(\mu-r)^2(1-\rho_0^2)(p^0)^2}{\sigma^2}}}{(\mu-r)^2} \tag{21}$$

and then we derive

$$\phi(y) = c_1 + c_2\int_0^y \exp\left(\int_0^z \frac{1}{h_1(s)}ds\right)dz.$$

In terms of the boundary conditions $\phi(0) = 1, \phi(\infty) = 0$, we have $c_1 = 1$ and

$$c_2 = -\frac{1}{\int_0^\infty \exp\left(\int_0^z \frac{1}{h_1(s)}ds\right)dz}.$$

If $p^*(t) = \frac{\theta}{\eta}$, Equation (19) is transformed into

$$ry\phi_y + a\theta\phi_y - \frac{b\rho_0\theta(\mu-r)}{\sigma\eta}\phi_y + \frac{b^2\theta^2(1-\rho_0^2)}{2\eta^2}\phi_{yy} - \frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{\phi_y^2}{\phi_{yy}} = 0 \tag{22}$$

with the boundary conditions $\phi(0) = 1$ and $\phi(\infty) = 0$. Similarly, set $h_2(y) = \frac{\phi_y}{\phi_{yy}}$ and we obtain

$$\frac{(\mu-r)^2}{2\sigma^2}h_2(y)^2 - \left[ry + a\theta - \frac{b\rho_0\theta(\mu-r)}{\sigma\eta}\right]h_2(y) - \frac{b^2\theta^2(1-\rho_0^2)}{2\eta^2} = 0.$$

Let

$$N_2 = ry + a\theta - \frac{b\rho_0\theta(\mu - r)}{\sigma\eta}$$

and due to $\phi_y < 0, \phi_{yy} > 0$, we derive

$$h_2(y) = \frac{\sigma^2 N_2 - \sigma^2\sqrt{N_2^2 + \frac{b^2\theta^2(\mu-r)^2(1-\rho_0^2)}{\sigma^2\eta^2}}}{(\mu - r)^2} \tag{23}$$

and

$$\phi(y) = c_3 + c_4 \int_0^y \exp\left(\int_0^z \frac{1}{h_2(s)} ds\right) dz.$$

Considering the boundary conditions $\phi(0) = 1, \phi(\infty) = 0$, we have $c_3 = 1$ and

$$c_4 = -\frac{1}{\int_0^\infty \exp\left(\int_0^z \frac{1}{h_2(s)} ds\right) dz}.$$

Finally, the following theorem summarizes the above analysis.

Theorem 4.2 *According to the optimal reinsurance strategy chosen by the insurer, the optimal investment strategy of the reinsurer in the sense of minimizing the ruin probability is as follows:*

1) *If the optimal reinsurance proportion of the insurer is $p^*(t) = p^0(t)$, then*

$$\pi_2^*(y) = -\frac{\mu - r}{\sigma^2} h_1(y) - b\rho_0\sigma\left[1 - \frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2 m_1 \sigma(1 - \rho_0^2)} e^{-r(T-t)}\right],$$

where $h_1(y)$ is given in Equation (21)*.*

2) *If $p^*(t) = 0$, we have $\pi_2^*(y) = 0$.*

3) *If $p^*(t) = \frac{\theta}{\eta}$, the reinsurer's optimal investment strategy is*

$$\pi_2^*(y) = -\frac{\mu - r}{\sigma^2} h_2(y) - \frac{b\rho_0\theta}{\sigma\eta},$$

where $h_2(y)$ is given by Equation (23)*.*

Proof 1) If $p^*(t) = p^0(t)$, according to Equations (18) and (21), we obtain

$$\begin{aligned}\pi_2^* &= -\frac{\mu - r}{\sigma^2} \cdot \frac{\phi_y}{\phi_{yy}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu - r}{\sigma^2} h_1(y) - b\rho_0\sigma\left[1 - \frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2 m_1 \sigma(1 - \rho_0^2)} e^{-r(T-t)}\right].\end{aligned}$$

2) From Equation (3), we find that if $p^*(t) = 0$, $\pi_2(y) = 0$ and $y \geq 0$, the ruin of the reinsurer will not occur, so $\pi_2^*(y) = 0$.

3) If $p^*(t) = \frac{\theta}{\eta}$, according to Equations (18) and (23), we derive

$$\begin{aligned}\pi_2^* &= -\frac{\mu - r}{\sigma^2} \cdot \frac{\phi_y}{\phi_{yy}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu - r}{\sigma^2} h_2(y) - \frac{b\rho_0\theta}{\sigma\eta}.\end{aligned}$$

The proof is completed. ∎

4.3 The Equality of the Reinsurer's Optimal Investment Strategies Under the Two Cases

Browne[22], Bai and Guo[23] both found that with zero interest rate, the insurer's optimal investment strategy that maximized the expected exponential utility of terminal wealth also minimized the ruin probability. We now examine the equality of the reinsurer's optimal investment strategies under these two cases when there is no risk-free asset.

For the reason that when reinsurance proportion $p^*(t) = 0$, $\pi_2^*(y) = 0$ for $y \geq 0$ and the minimum ruin probability is $\phi(y) = 0$. Thus, this subsection only shows the equality of the reinsurer's optimal investment strategies when $p^*(t) = p^0(t)$ and $p^*(t) = \frac{\theta}{\eta}$.

If $r = 0$, for $p^*(t) = p^0(t)$, the optimal investment strategy of the reinsurer in the sense of maximizing the expected exponential utility of terminal wealth is

$$\pi_2^*(t) = \frac{\mu}{\sigma^2 m_2} - \frac{b\rho_0 p^0(t)}{\sigma} \tag{24}$$

and that of minimizing the ruin probability is

$$\pi_2^*(t) = -\frac{\mu}{\sigma^2} h_1 - \frac{b\rho_0 p^0(t)}{\sigma}, \tag{25}$$

where

$$p^0(t) = 1 - \frac{a\sigma\eta - b\rho_0\mu}{b^2 m_1 \sigma(1-\rho_0^2)}, \quad N_1 = a\eta - \frac{b\rho_0\mu}{\sigma} - \frac{(a\sigma\eta - b\rho_0\mu)^2}{m_1 b^2 \sigma^2 (1-\rho_0^2)},$$
$$h_1 = \frac{\sigma^2 N_1 - \sigma^2\sqrt{N_1^2 + \frac{b^2\mu^2(1-\rho_0^2)(p^0)^2}{\sigma^2}}}{\mu^2}.$$

Let $m_2 = -\frac{1}{h_1}$, we see that Equations (24) and (25) are equivalent.

For $p^*(t) = \frac{\theta}{\eta}$, the optimal investment strategy of the reinsurer in the sense of maximizing the expected exponential utility of terminal wealth and minimizing the ruin probability are

$$\pi_2^*(t) = \frac{\mu}{\sigma^2 m_2} - \frac{b\rho_0\theta}{\sigma\eta}, \tag{26}$$

$$\pi_2^*(t) = -\frac{\mu}{\sigma^2} h_2 - \frac{b\rho_0\theta}{\sigma\eta}, \tag{27}$$

where

$$N_2 = a\theta - \frac{b\rho_0\theta\mu}{\sigma\eta}, \quad h_2 = \frac{\sigma^2 N_2 - \sigma^2\sqrt{N_2^2 + \frac{b^2\theta^2(\mu-r)^2(1-\rho_0^2)}{\sigma^2\eta^2}}}{(\mu-r)^2}.$$

Let $m_2 = -\frac{1}{h_2}$, we find that Equations (26) and (27) are equivalent.

For $r \neq 0$, there is no equality between the reinsurer's optimal investment strategies under the two cases. Because if $r \neq 0$, the optimal investment strategy which considers maximizing the expected exponential utility of terminal wealth is related to the time t, on the other hand, the optimal investment strategy that minimizes the ruin probability is related to the wealth of the reinsurer.

5 Conclusion

In this paper, we focus on the optimal reinsurance and investment problem for an insurer and a reinsurer. In our model, the basic claim process is assumed to follow a Brownian motion with drift. The insurer can purchase proportional reinsurance from the reinsurer and both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. Moreover, we consider the correlation between the claim process and the price process of the risky asset. The effect of the net profit on proportional reinsurance is also taken into account. Firstly, we consider the optimization problem of maximizing the insurer's expected exponential utility of terminal wealth. By solving the corresponding HJB equation, we derive the optimal reinsurance and investment strategies of the insurer, explicitly. Then according to the insurer's optimal reinsurance strategy, the reinsurer chooses the optimal investment strategy, i.e., the reinsurer increases his/her wealth via investment. We consider two optimal investment problems for the reinsurer: the problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the ruin probability. Furthermore, we obtain the optimal investment strategies of the reinsurer. Finally, we find that, with zero interest rate, the optimal strategies of the reinsurer under two cases are equivalent.

References

[1] Hipp C and Plum M, Optimal investment for insurers, *Insurance: Mathematics and Economics*, 2000, **27**(2): 215–228.

[2] Schmidli H, Optimal proportional reinsurance policies in a dynamic setting, *Scandinavian Actuarial Journal*, 2001, **2001**(1): 55–68.

[3] Promislow S D and Young V R, Minimizing the probability of ruin when claims follow Brownian motion with drift, *North American Actuarial Journal*, 2005, **9**(3): 110–128.

[4] Li J and Wu R, Upper bound for finite-time ruin probability in a Markov-modulated market, *Journal of Systems Science and Complexity*, 2011, **24**(2): 308–316.

[5] Liang Z and Guo J, Optimal investment and proportional reinsurance in the Sparre Andersen model, *Journal of Systems Science and Complexity*, 2012, **25**(5): 926–941.

[6] Yang H and Zhang L, Optimal investment for insurer with jump-diffusion risk process, *Insurance: Mathematics and Economics*, 2005, **37**(3): 615–634.

[7] Wang N, Optimal investment for an insurer with exponential utility preference, *Insurance: Mathematics and Economics*, 2007, **40**(1): 77–84.

[8] Cao Y and Wan N, Optimal proportional reinsurance and investment based on Hamilton-Jacobi-Bellman equation, *Insurance: Mathematics Economics*, 2009, **45**(2): 157–162.

[9] Bai L and Guo J, Optimal dynamic excess-of-loss reinsurance and multidimensional portfolio selection, *Science China Mathematics*, 2010, **53**(7): 1787–1804.

[10] Liang Z, Yuen K C, and Guo J, Optimal proportional reinsurance and investment in a stock market with Ornstein-Uhlenbeck process, *Insurance: Mathematics and Economics*, 2011, **49**(2): 207–215.

[11] Liang Z and Bayraktar E, Optimal reinsurance and investment with unobservable claim size and intensity, *Insurance: Mathematics and Economics*, 2014, **55**: 156–166.

[12] Bai L and Zhang H, Dynamic mean-variance problem with constrained risk control for the insurers, *Mathematical Methods of Operations Research*, 2008, **68**(1): 181–205.

[13] Bi J, Guo J, and Bai L, Optimal multi-asset investment with no-shorting constraint under mean-variance criterion for an insurer, *Journal of Systems Science and Complexity*, 2011, **24**(2): 291–307.

[14] Zeng Y and Li Z, Optimal time-consistent investment and reinsurance policies for mean-variance insurers, *Insurance: Mathematics and Economics*, 2011, **49**(1): 145–154.

[15] Li Y and Li Z, Optimal time-consistent investment and reinsurance strategies for mean-variance insurers with state dependent risk aversion, *Insurance: Mathematics and Economics*, 2013, **53**(1): 86–97.

[16] Gu M, Yang Y, Li S, and Zhang J, Constant elasticity of variance model for proportional reinsurance and investment strategies, *Insurance: Mathematics and Economics*, 2010, **46**(3): 580–587.

[17] Lin X and Li Y, Optimal reinsurance and investment for a jump diffusion risk process under the CEV model, *North American Actuarial Journal*, 2011, **15**(3): 417–431.

[18] Liang Z, Yuen K C, and Cheung K C, Optimal reinsurance-investment problem in a constant elasticity of variance stock market for jump-diffusion risk model, *Applied Stochastic Models in Business and Industry*, 2012, **28**(6): 585–597.

[19] Gu A, Guo X, Li Z, and Zeng Y, Optimal control of excess-of-loss reinsurance and investment for insurers under a CEV model, *Insurance: Mathematics and Economics*, 2012, **51**(3): 674–684.

[20] Li Z, Zeng Y, and Lai Y, Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model, *Insurance: Mathematics and Economics*, 2012, **51**(1): 191–203.

[21] Zhao H, Rong X, and Zhao Y, Optimal excess-of-loss reinsurance and investment problem for an insurer with jump-diffusion risk process under the Heston model, *Insurance: Mathematics and Economics*, 2013, **53**(3): 504–514.

[22] Browne S, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research*, 1995, **20**(4): 937–958.

[23] Bai L and Guo J, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, *Insurance: Mathematics and Economics*, 2008, **42**(3): 968–975.

Appendix

Proof of Theorem 3.1 According to the exponential utility function described by Equation (8), we try to obtain the solution to Equation (6) in the following way

$$V(t,x) = -\frac{1}{m_1}\mathrm{e}^{-m_1 x \mathrm{e}^{r(T-t)} + f(t)} \tag{28}$$

with the boundary condition given by $f(T) = 0$.

Then

$$V_t = (m_1 r x \mathrm{e}^{r(T-t)} + f_t)V, \quad V_x = -m_1 \mathrm{e}^{r(T-t)} V, \quad V_{xx} = m_1^2 \mathrm{e}^{2r(T-t)} V,$$

where f_t denotes the derivative of $f(t)$ w.r.t. t.

From Equations (7) and (28), the corresponding $p^0(t)$ is given by

$$p^0(t) = 1 - \frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2 m_1 \sigma(1-\rho_0^2)} \mathrm{e}^{-r(T-t)}. \tag{29}$$

If $\frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} > 0$, Equation (29) shows that $p^0(t) \in [0, \frac{\theta}{\eta}]$ is equivalent to

$$t \le t_1 = T - \frac{1}{r} \ln \frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2 m_1\sigma(1-\rho_0^2)},$$
$$t \ge t_0 = T - \frac{1}{r} \ln \frac{[a\sigma\eta - b\rho_0(\mu - r)]\eta}{b^2 m_1\sigma(1-\rho_0^2)(\eta-\theta)}.$$

Now we find the solution to Equation (6) and recover $p^*(t), \pi_1^*(t)$ from derivatives of $V(t,x)$.

Case 1 $\frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} > 1$.

1) If $\mathrm{e}^{rT} > \frac{[a\sigma\eta - b\rho_0(\mu-r)]\eta}{b^2 m_1\sigma(1-\rho_0^2)(\eta-\theta)}$ and $t_1 \le t \le T$, we have $p^0(t) \le 0$. So $p^*(t) = 0$ and Equation (6) is transformed into

$$V_t + rxV_x + a\theta V_x - \frac{b\rho_0(\mu-r)}{\sigma} V_x + \frac{b^2(1-\rho_0^2)}{2} V_{xx} - \frac{(\mu-r)^2}{2\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0. \tag{30}$$

When $t_0 \le t < t_1$, $p^0(t) \in [0, \frac{\theta}{\eta}]$, then $p^*(t) = p^0(t)$. Equation (6) becomes

$$V_t + rxV_x + a(\theta - \eta)V_x - \frac{(\mu-r)^2}{2\sigma^2} \cdot \frac{V_x^2}{V_{xx}} - \frac{[a\sigma\eta - b\rho_0(\mu-r)]^2}{2\sigma^2 b^2(1-\rho_0^2)} \cdot \frac{V_x^2}{V_{xx}} = 0. \tag{31}$$

And when $0 \le t < t_0$, we obtain $p^0(t) \ge \frac{\theta}{\eta}$. Let $p^*(t) = \frac{\theta}{\eta}$, Equation (6) is simplified into

$$V_t + rxV_x - \frac{b\rho_0(\eta-\theta)(\mu-r)}{\sigma\eta} V_x + \frac{b^2(\eta-\theta)^2(1-\rho_0^2)}{2\eta^2} V_{xx} - \frac{(\mu-r)^2}{2\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0. \tag{32}$$

Equations (30)–(32) can be solved by the same procedure and we demonstrate the procedure with Equation (30) only. Introducing V_t, V_x, V_{xx} into Equation (30), we obtain

$$f_t - \left[a m_1\theta - \frac{b\rho_0 m_1(\mu-r)}{\sigma}\right] \mathrm{e}^{r(T-t)} + \frac{b^2 m_1^2(1-\rho_0^2)}{2} \mathrm{e}^{2t(T-t)} - \frac{(\mu-r)^2}{2\sigma^2} = 0. \tag{33}$$

Considering the boundary condition $f(T) = 1$, the solution to Equation (33) is

$$f(t) = \left[\frac{a m_1\theta}{r} - \frac{b\rho_0 m_1(\mu-r)}{r\sigma}\right](1 - \mathrm{e}^{r(T-t)}) - \frac{b^2 m_1^2(1-\rho_0^2)}{4r}(1 - \mathrm{e}^{2r(T-t)}) - \frac{(\mu-r)^2}{2\sigma^2}(T-t).$$

Similarly, we can derive the solutions to Equations (33) and (32), explicitly. Therefore, the optimal reinsurance and investment strategies of the insurer are

$$p^*(t) = \begin{cases} \frac{\theta}{\eta}, & 0 \le t < t_0, \\ 1 - \frac{a\sigma\eta - b\rho_0(\mu-r)}{b^2 m_1\sigma(1-\rho_0^2)} \mathrm{e}^{-r(T-t)}, & t_0 \le t < t_1, \\ 0, & t_1 \le t \le T, \end{cases}$$

$$\pi_1^*(t) = \begin{cases} \dfrac{\mu - r}{\sigma^2 m_1}\mathrm{e}^{-r(T-t)} - \dfrac{b\rho_0(\eta - \theta)}{\sigma\eta}, & 0 \le t < t_0, \\ \dfrac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\dfrac{\mu - r}{m_1} - \dfrac{a\rho_0\sigma\eta - b\rho_0^2(\mu - r)}{m_1 b(1-\rho_0^2)}\right], & t_0 \le t < t_1, \\ \dfrac{\mu - r}{\sigma^2 m_1}\mathrm{e}^{-r(T-t)} - \dfrac{b\rho_0}{\sigma}, & t_1 \le t \le T. \end{cases}$$

Noting that $V(t,x)$ is continuous at $t = t_0$ and $t = t_1$, we obtain

$$V(t,x) = \begin{cases} -\dfrac{1}{m_1}\mathrm{e}^{-m_1 x \mathrm{e}^{r(T-t)} + f_3(t)}, & 0 \le t < t_0, \\ -\dfrac{1}{m_1}\mathrm{e}^{-m_1 x \mathrm{e}^{r(T-t)} + f_2(t)}, & t_0 \le t < t_1, \\ -\dfrac{1}{m_1}\mathrm{e}^{-m_1 x \mathrm{e}^{r(T-t)} + f_1(t)}, & t_1 \le t \le T, \end{cases} \tag{34}$$

where

$$\begin{aligned} f_1(t) = {} & \left[\frac{am_1\theta}{r} - \frac{b\rho_0 m_1(\mu - r)}{r\sigma}\right](1 - \mathrm{e}^{r(T-t)}) \\ & - \frac{b^2 m_1^2(1-\rho_0^2)}{4r}(1 - \mathrm{e}^{2r(T-t)}) - \frac{(\mu - r)^2}{2\sigma^2}(T - t), \end{aligned} \tag{35}$$

$$\begin{aligned} f_2(t) = {} & \frac{am_1(\theta - \eta)}{r}(\mathrm{e}^{r(T-t_1)} - \mathrm{e}^{r(T-t)}) - \left[\frac{(a\sigma\eta - b\rho_0(\mu - r))^2}{2\sigma^2 b^2(1-\rho_0^2)} + \frac{(\mu - r)^2}{2\sigma^2}\right](t_1 - t) \\ & + \left[\frac{am_1\theta}{r} - \frac{b\rho_0 m_1(\mu - r)}{r\sigma}\right](1 - \mathrm{e}^{r(T-t_1)}) \\ & - \frac{b^2 m_1^2(1-\rho_0^2)}{4r}(1 - \mathrm{e}^{2r(T-t_1)}) - \frac{(\mu - r)^2}{2\sigma^2}(T - t_1), \end{aligned} \tag{36}$$

$$\begin{aligned} f_3(t) = {} & \frac{bm_1\rho_0(\eta - \theta)(\mu - r)}{r\sigma\eta}(\mathrm{e}^{r(T-t)} - \mathrm{e}^{r(T-t_0)}) \\ & + \frac{b^2 m_1^2(\theta - \eta)^2(1-\rho_0^2)}{4r\eta^2}(\mathrm{e}^{2r(T-t)} - \mathrm{e}^{2r(T-t_0)}) \\ & - \frac{(\mu - r)^2}{2\sigma^2}(t_0 - t) + \frac{am_1(\theta - \eta)}{r}(\mathrm{e}^{r(T-t_1)} - \mathrm{e}^{r(T-t_0)}) \\ & - \left[\frac{(a\sigma\eta - b\rho_0(\mu - r))^2}{2\sigma^2 b^2(1-\rho_0^2)} + \frac{(\mu - r)^2}{2\sigma^2}\right](t_1 - t_0) \\ & + \left[\frac{am_1\theta}{r} - \frac{b\rho_0 m_1(\mu - r)}{r\sigma}\right](1 - \mathrm{e}^{r(T-t_1)}) \\ & - \frac{b^2 m_1^2(1-\rho_0^2)}{4r}(1 - \mathrm{e}^{2r(T-t_1)}) - \frac{(\mu - r)^2}{2\sigma^2}(T - t_1). \end{aligned} \tag{37}$$

2) If $\frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2 m_1\sigma(1-\rho_0^2)} \le \mathrm{e}^{rT} \le \frac{[a\sigma\eta - b\rho_0(\mu - r)]\eta}{b^2 m_1\sigma(1-\rho_0^2)(\eta - \theta)}$ and $t_1 \le t \le T$, we find $p^0(t) \le 0$. Therefore, we let $p^*(t) = 0$. When $0 \le t < t_1$, we have $p^0(t) \in [0, \frac{\theta}{\eta}]$, so $p^*(t) = p^0(t)$, i.e., the optimal

reinsurance and investment strategies of the insurer are

$$p^*(t)=\begin{cases}1-\dfrac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}, & 0\le t<t_1,\\ 0, & t_1\le t\le T,\end{cases}$$

$$\pi_1^*(t)=\begin{cases}\dfrac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\dfrac{\mu-r}{m_1}-\dfrac{a\rho_0\sigma\eta-b\rho_0^2(\mu-r)}{m_1b(1-\rho_0^2)}\right], & 0\le t<t_1,\\ \dfrac{\mu-r}{\sigma^2m_1}\mathrm{e}^{-r(T-t)}-\dfrac{b\rho_0}{\sigma}, & t_1\le t\le T.\end{cases}$$

Similar to the above-mentioned derivations, we obtain the value function as follows

$$V(t,x)=\begin{cases}-\dfrac{1}{m_1}\mathrm{e}^{-m_1x\mathrm{e}^{r(T-t)}+f_2(t)}, & 0\le t<t_1,\\ -\dfrac{1}{m_1}\mathrm{e}^{-m_1x\mathrm{e}^{r(T-t)}+f_1(t)}, & t_1\le t\le T,\end{cases}\tag{38}$$

where $f_1(t)$ and $f_2(t)$ are given by Equations (35) and (36).

3) If $\mathrm{e}^{rT}<\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}$, we have $p^0(t)\le 0$ for $0\le t\le T$. Thus the optimal reinsurance and investment strategies of the insurer are

$$p^*(t)=0,\quad 0\le t\le T,$$
$$\pi_1^*(t)=\frac{\mu-r}{\sigma^2m_1}\mathrm{e}^{-r(T-t)}-\frac{b\rho_0}{\sigma},\quad 0\le t\le T,$$

and the value function is

$$V(t,x)=-\frac{1}{m_1}\mathrm{e}^{-m_1x\mathrm{e}^{r(T-t)}+f_1(t)},\quad 0\le t\le T,\tag{39}$$

where $f_1(t)$ is given in Equation (35).

Case 2 $\frac{\eta-\theta}{\eta}<\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\le 1$.

1) If $\mathrm{e}^{rT}>\frac{[a\sigma\eta-b\rho_0(\mu-r)]\eta}{b^2m_1\sigma(1-\rho_0^2)(\eta-\theta)}$ and $t_0\le t\le T$, we derive $p^0(t)\in[0,\frac{\theta}{\eta}]$. So $p^*(t)=p^0(t)$. When $0\le t<t_0$, $p^0(t)\ge\frac{\theta}{\eta}$. We have $p^*(t)=\frac{\theta}{\eta}$, i.e., the optimal reinsurance and investment strategies of the insurer are

$$p^*(t)=\begin{cases}\dfrac{\theta}{\eta}, & 0\le t<t_1,\\ 1-\dfrac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)}, & t_1\le t\le T,\end{cases}$$

$$\pi_1^*(t)=\begin{cases}\dfrac{\mu-r}{\sigma^2m_1}\mathrm{e}^{-r(T-t)}-\dfrac{b\rho_0(\eta-\theta)}{\sigma\eta}, & 0\le t<t_1,\\ \dfrac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\dfrac{\mu-r}{m_1}-\dfrac{a\rho_0\sigma\eta-b\rho_0^2(\mu-r)}{m_1b(1-\rho_0^2)}\right], & t_1\le t\le T.\end{cases}$$

By similar derivations, we have

$$V(t,x)=\begin{cases}-\dfrac{1}{m_1}\mathrm{e}^{-m_1 x\mathrm{e}^{r(T-t)}+f_5(t)}, & 0\le t<t_1,\\ -\dfrac{1}{m_1}\mathrm{e}^{-m_1 x\mathrm{e}^{r(T-t)}+f_4(t)}, & t_1\le t\le T,\end{cases}\tag{40}$$

where

$$f_4(t)=\frac{am_1(\theta-\eta)}{r}(1-\mathrm{e}^{r(T-t)})-\left[\frac{(a\sigma\eta-b\rho_0(\mu-r))^2}{2\sigma^2b^2(1-\rho_0^2)}+\frac{(\mu-r)^2}{2\sigma^2}\right](T-t),\tag{41}$$

$$\begin{aligned}f_5(t)=&\frac{bm_1\rho_0(\eta-\theta)(\mu-r)}{r\sigma\eta}(\mathrm{e}^{r(T-t)}-\mathrm{e}^{r(T-t_1)})\\&+\frac{b^2m_1^2(\eta-\theta)^2(1-\rho_0^2)}{4r\eta^2}(\mathrm{e}^{2r(T-t)}-\mathrm{e}^{2r(T-t_1)})-\frac{(\mu-r)^2}{2\sigma^2}(t_1-t)\\&+\frac{am_1(\theta-\eta)}{r}(1-\mathrm{e}^{r(T-t_1)})-\left[\frac{(a\sigma\eta-b\rho_0(\mu-r))^2}{2\sigma^2b^2(1-\rho_0^2)}+\frac{(\mu-r)^2}{2\sigma^2}\right](T-t_1).\end{aligned}\tag{42}$$

2) If $\mathrm{e}^{rT}\le\frac{[a\sigma\eta-b\rho_0(\mu-r)]\eta}{b^2m_1\sigma(1-\rho_0^2)(\eta-\theta)}$, we derive $p^0(t)\in[0,\frac{\theta}{\eta}]$ for $0\le t\le T$, then the optimal reinsurance and investment strategies of the insurer are

$$p^*(t)=1-\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\mathrm{e}^{-r(T-t)},\quad 0\le t\le T,$$

$$\pi_1^*(t)=\frac{\mathrm{e}^{-r(T-t)}}{\sigma^2}\left[\frac{\mu-r}{m_1}-\frac{a\rho_0\sigma\eta-b\rho_0^2(\mu-r)}{m_1b(1-\rho_0^2)}\right],\quad 0\le t\le T,$$

and the value function is

$$V(t,x)=-\frac{1}{m_1}\mathrm{e}^{-m_1x\mathrm{e}^{r(T-t)}+f_4(t)},\quad 0\le t\le T,\tag{43}$$

where $f_4(t)$ is given by Equation (41).

Case 3 $\frac{a\sigma\eta-b\rho_0(\mu-r)}{b^2m_1\sigma(1-\rho_0^2)}\le\frac{\eta-\theta}{\eta}$.

For $0\le t\le T$, we obtain $p^0(t)\ge\frac{\theta}{\eta}$. Therefore, the optimal reinsurance and investment strategies of the insurer are

$$p^*(t)=\frac{\theta}{\eta},\quad 0\le t\le T,$$

$$\pi_1^*(t)=\frac{\mu-r}{\sigma^2m_1}\mathrm{e}^{-r(T-t)}-\frac{b\rho_0(\eta-\theta)}{\sigma\eta},\quad 0\le t\le T.$$

Meanwhile, we can derive the optimal value function in the following form

$$V(t,x)=-\frac{1}{m_1}\mathrm{e}^{-m_1x\mathrm{e}^{r(T-t)}+f_6(t)},\quad 0\le t\le T,\tag{44}$$

where

$$\begin{aligned}f_6(t)=&-\frac{bm_1\rho_0(\eta-\theta)(\mu-r)}{r\sigma\eta}(1-\mathrm{e}^{r(T-t)})\\&-\frac{b^2m_1^2(\eta-\theta)^2(1-\rho_0^2)}{4r\eta^2}(1-\mathrm{e}^{2r(T-t)})-\frac{(\mu-r)^2}{2\sigma^2}(T-t).\end{aligned}\tag{45}$$

The proof of Theorem 3.1 is completed. ∎