A Descent Method for Mixed Variational Inequalities*

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Abstract A new descent method for solving mixed variational inequalities is developed based on the auxiliary principle problem. Convergence of the proposed method is also demonstrated.

Keywords Auxiliary principle problem, descent direction, mixed variational inequalities.

1 Introduction

For solving the classical optimization problems, Cohen[1, 2] and Cohen and Zhu[3] introduced the auxiliary problem principle and used it as a general framework to develop various computational algorithms, including the gradient algorithm, the subgradient algorithm, and the decomposition/coordination algorithms. Cohen[4] extended this framework to develop the methods of searching solutions to the variational inequalities. Recently, Zhu and Marcotte[5] studied the convergence of the iterative schemes developed on the basis of the auxiliary problem principle. Verma[6] further generalized the auxiliary problem principle to develop the methods of solving a class of nonlinear variational inequality.

In recent years, to facilitate the development of variational inequality theory, the variational inequality problem has been generalized along with various directions. A particularly important generalization is the mixed variational inequalities. It is well known that the projection method and its various variations, including the Wiener-Hopf equations, are not efficient for developing

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the numerical methods of solving nonlinear mixed variational inequalities. To address this issue, in this paper we explore the nonlinear mixed variational inequality problem by utilizing the auxiliary problem principle. With this study, we propose a new descent method for solving the monotone mixed variational inequalities. Furthermore, we also demonstrate that this method is convergent with some reasonable conditions.

2 Preliminaries

Let H be a real Hilbert space, and the associated inner product and norm be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let X be a nonempty closed and convex subset of H and F be a continuous mapping from H to itself. In addition, let $\phi(x) : X \to R \cup \{+\infty\}$ be a proper, convex and continuous function. Then, the mixed variational inequality (MVI) problem is to find a vector $x^* \in X$, such that

$$\langle F(x^*), x - x^* \rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in X. \tag{2.1}$$

We denote by X^* the set of all the solutions to MVI(X, F).

It is easy to see that for $\phi \equiv 0$, MVI reduces to finding $x^* \in X$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in X,$$

which is actually the classical variational inequality introduced by Stampacchia[7] and Facchinei and Pang[8]. Noor[9–11] used the resolvent operator technique to develop some effective iterative methods for solving the monotone mixed variational inequalities. For solving the MVI, we introduce the following auxiliary problem (AP):

$$\langle \varGamma(\omega(x)) - \varGamma(x) + \lambda F(x), y - \omega(x) \rangle + \lambda[\phi(y) - \phi(\omega(x))] \geq 0, \quad \forall y \in X, \tag{2.2}$$

where λ is a positive constant and $\varGamma(x)$ is a strongly monotone function with parameter β. Since $\varGamma$ is strongly monotone on X, the solution of (2.2), denoted as $\omega(x)$, is unique. In addition, it is easy to know that the mapping $\omega(x)$ is continuous, and when the auxiliary mapping $\varGamma$ possesses some special structures, the auxiliary variational inequality (2.2) is easy to be solved.

For $x \in X$, we define the residue function as follows:

$$R(x) = x - \omega(x). \tag{2.3}$$

It is easy to see that $x^* \in X^*$ if and only if $R(x^*) = 0$. For ease of exposition, we present a well-known result as follows.

Lemma 2.1[12] (i) *If $x^* \in X$ is a solution of the following problem*

$$\langle F(x), x - x^* \rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in X, \tag{2.4}$$

then it is also a solution of (2.1).

(ii) *The solution set of Problem* (2.4) *is convex and closed.*

(iii) *If F is monotone, the problem* (2.1) *is equivalent to* (2.4).

In addition, we assume throughout the paper that:

(H_1) the solution set X^* is nonempty.

(H_2) $\|F(x) - F(\omega(x))\| \le \frac{\alpha}{\lambda}\|x - \omega(x)\|, \quad \forall\, x \in X, \exists\, \alpha \in (0, \beta)$.

3 Main Results

In this section, we will develop a descent method for the mixed variational inequalities. Furthermore, we will show that this descent method is convergent with some reasonable conditions.

Proposition 3.1 *Suppose that F is monotone and (H_1) and (H_2) hold, then $-\{\Gamma(x) - \Gamma(\omega(x)) + \lambda[F(\omega(x)) + \gamma(x)]\}$ is a descent direction of the merit function $\frac{1}{2}\|x - x^*\|^2$ at x, where $\gamma(x) \in \partial\phi(x)$ and $\partial(\phi(x))$ is the subdifferential of $\phi(x)$ at x.*

Proof By putting $y = x^*$ in (2.2), we have

$$\langle \Gamma(\omega(x)) - \Gamma(x) + \lambda F(x), x^* - \omega(x)\rangle + \lambda[\phi(x^*) - \phi(\omega(x))] \ge 0. \tag{3.1}$$

Then we get

$$\langle \Gamma(x) - \Gamma(\omega(x)), x - x^* - (x - \omega(x))\rangle - \langle \lambda F(x), \omega(x) - x^*\rangle + \lambda[\phi(x^*) - \phi(\omega(x))] \ge 0.$$

So

$$\begin{aligned}
&\langle \Gamma(x) - \Gamma(\omega(x)), x - x^*\rangle \\
\ge&\langle \Gamma(x) - \Gamma(\omega(x)), x - \omega(x)\rangle + \lambda\langle F(x), \omega(x) - x^*\rangle + \lambda[\phi(\omega(x)) - \phi(x^*)] \\
\ge&\beta\|x - \omega(x)\|^2 + \lambda\langle F(x), \omega(x) - x^*\rangle + \lambda[\phi(\omega(x)) - \phi(x^*)].
\end{aligned} \tag{3.2}$$

Since F is monotone, we have

$$\langle \lambda F(\omega(x)), x - x^*\rangle \ge \langle \lambda F(\omega(x)), x - \omega(x)\rangle + \lambda\{\phi(x^*) - \phi(\omega(x))\}. \tag{3.3}$$

By (3.2) and (3.3), it follows that

$$\begin{aligned}
&\langle \Gamma(x) - \Gamma(\omega(x)) + \lambda F(\omega(x)), x - x^*\rangle \\
\ge&\beta\|x - \omega(x)\|^2 + \lambda\langle F(x), \omega(x) - x^*\rangle + \lambda\langle F(\omega(x)), x - \omega(x)\rangle \\
=&\beta\|x - \omega(x)\|^2 - \lambda\langle F(x) - F(\omega(x)), x - \omega(x)\rangle + \lambda\langle F(x), x - x^*\rangle \\
\ge&\beta\|x - \omega(x)\|^2 - \lambda\|F(x) - F(\omega(x))\|\|x - \omega(x)\| + \lambda\langle F(x), x - x^*\rangle \\
\ge&(\beta - \alpha)\|x - \omega(x)\|^2 + \lambda[\phi(x) - \phi(x^*)].
\end{aligned}$$

Therefore,

$$\langle \Gamma(x) - \Gamma(\omega(x)) + \lambda[F(\omega(x)) + \gamma(x)], x - x^*\rangle \ge (\beta - \alpha)\|\omega(x) - x\|^2, \gamma(x) \in \partial\phi(x). \tag{3.4}$$

Since (H_2) holds, it follows that $\beta > \alpha$ and $-\{\Gamma(x) - \Gamma(\omega(x)) + \lambda[F(\omega(x)) + \gamma(x)]\}$ is a descent direction of the merit function $\frac{1}{2}\|x - x^*\|^2$. Thus, the desired result follows and the proof is completed. ∎

Based on Proposition 3.1, an algorithm is stated as follows:

Algorithm 3.1

Step 0 (Initialization) Given an arbitrary x^0, $\beta > \alpha$, $\delta \in (0,2)$, $k := 0$, an error tolerance $\varepsilon > 0$.

Step 1 For x^k, solving AP(Γ, X, x^k) and finding $\omega(x^k)$ such that

$$\langle \Gamma(\omega(x^k)) - \Gamma(x^k) + \lambda F(x^k), y - \omega(x^k)\rangle + \lambda[\phi(y) - \phi(\omega(x^k))] \geq 0, \quad \forall y \in X.$$

Step 2 (Stopping criterion) If $\|\omega(x^k) - x^k\|^2 \leq \varepsilon$, stop;
else, set $d(x^k) = \Gamma(x^k) - \Gamma(\omega(x^k)) + \lambda[F(\omega(x^k)) + \gamma(x^k)], \gamma(x^k) \in \partial\phi(x^k)$.

Step 3 Set $x^{k+1} = P_K[x^k - \delta t_k d(x^k)]$ and $t_k = \frac{(\beta-\alpha)\|x^k-\omega(x^k)\|^2}{\|d(x^k)\|^2}$, $k := k+1$, go to Step 1.

Theorem 3.1 *Let x^* be a solution of Problem* (2.1) *and $\{x^k\}$ be a sequence generated by Algorithm* 3.1*, then $\{x^k\}$ is bounded.*

Proof Let x^* be a solution of Problem (2.1), and it follows from (3.4), then we can obtain that:

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^* - \delta t_k d(x^k)\|^2 \\
&= \|x^k - x^*\|^2 - 2\delta t_k \langle d(x^k), x^k - x^*\rangle + (\delta t_k)^2 \|d(x^k)\|^2 \\
&\leq \|x^k - x^*\|^2 - 2\delta t_k (\beta - \alpha)\|x^k - \omega(x^k)\|^2 + (\delta t_k)^2 \|d(x^k)\|^2 \\
&= \|x^k - x^*\|^2 - 2\delta \frac{(\beta-\alpha)\|x^k - \omega(x^k)\|^2}{\|d(x^k)\|^2}(\beta - \alpha)\|x^k - \omega(x^k)\|^2 \\
&\quad + \left(\delta \frac{(\beta-\alpha)\|x^k - \omega(x^k)\|^2}{\|d(x^k)\|^2}\right)^2 \|d(x^k)\|^2 \\
&= \|x^k - x^*\|^2 - (2-\delta)\delta \frac{(\beta-\alpha)^2\|x^k - \omega(x^k)\|^4}{\|d(x^k)\|^4}.
\end{aligned} \tag{3.5}$$

Since $\delta \in (0,2)$ and $\beta > \alpha$, we have $\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 \leq \cdots \leq \|x^0 - x^*\|^2$, by which it is easy to see that the sequence $\{x^k\}$ is bounded. Thus, the proof is completed. ∎

Theorem 3.2 *The sequence $\{x^k\}$ generated by Algorithm* 3.1 *converges to a solution of Problem* (2.1).

Proof It follows from (3.5) that

$$\sum_{k=0}^{\infty} \|x^k - \omega(x^k)\| < \infty,$$

which implies that

$$\lim_{k\to\infty} \|x^k - \omega(x^k)\| = 0. \tag{3.6}$$

Let $\overline{x}$ be a cluster point of $\{x^k\}$ and $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$, which converges to $\overline{x}$. Since $R(x) = x - \omega(x)$ is continuous, we have $R(\overline{x}) = \lim_{j\to\infty} R(x^{k_j}) = 0$, which leads to that $\overline{x}$ is a solution of Problem (2.1). Again, from (3.5) it follows that $\|x^{k+1} - \overline{x}\|^2 \leq \|x^k - \overline{x}\|^2$. Therefore, the sequence $\{x^k\}$ has only one cluster point $\overline{x}$ and $\lim_{k\to\infty} x^k = \overline{x}$. Thus the desired result follows and the proof is completed. ∎

Remark 3.1 If $\Gamma(x) = x$ and $\phi(x)$ is the indictor function on X, then we can obtain the results of Xiu, et al.[13], which is developed for solving the classical variational inequalities.

4 Conclusions

In this paper, we have developed a new descent method for solving the monotone mixed variational inequalities. Furthermore, we have also demonstrated that this method is convergent under some reasonable conditions. In the future, we will further explore the applications of this method in various areas, such as in some complex systems, including the transportation system, the electricity system, etc.

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