J Syst Sci Complex (2015) 28: 830–847

On Observability and Detectability of Continuous-Time Stochastic Markov Jump Systems*

TAN Cheng · ZHANG Weihai

DOI: 10.1007/s11424-015-2253-y
Received: 19 November 2012 / Revised: 20 July 2013

Abstract This paper mainly studies observability and detectability for continuous-time stochastic Markov jump systems. Two concepts called W-observability and W-detectability for such systems are introduced, which are shown to coincide with various notions of observability and detectability reported recently in literature, such as exact observability, exact detectability and detectability. Besides, by introducing an accumulated energy function, some efficient criteria and interesting properties for both W-observability and W-detectability are obtained.

Keywords Continuous-time stochastic system, detectability, Markov jump, observability.

1 Introduction

Since a lot of practical systems can be described by stochastic models, stochastic control theory has attracted considerable interest and made great progress in a variety of the scientific and engineering fields over the past several decades. The study for stochastic systems has become an important research issue in the area of modern control theory and a great deal of control problems that have been studied well in deterministic systems are extended to stochastic systems, which include detectability and observability[1−10], stability and stabilization[2,3,10−14], H_2/H_∞ control problem[15] and LQ optimal control[16]. In particular, as one of the most basic dynamics models, stochastic Markov jump systems[4,5,8−12] have intimate connection with some practical systems which are vulnerable to component failures or repairs, abrupt changes in

TAN Cheng
School of Control Science and Engineering, Shandong University, Jinan 250061*, China.*
Email: tancheng1987love@163.com.
ZHANG Weihai
College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590*, China.* Email: w_hzhang@163.com.
*This research was supported by the Natural Science Foundation of China under Grant No. 61174078, the Research Fund for the Taishan Scholar Project of Shandong Province of China, the SDUST Research Fund under Grant No. 2011KYTD105, and the State Key Laboratory of Alternate Electrical Power System with Renewable Energy Sources under Grant No. LAPS13018.
◇ *This paper was recommended for publication by Editor ZHANG Jifeng.*

structure. Therefore, stochastic Markov jump systems can be used to represent some investment portfolio models and random failure processes in manufacturing industry and some networked control systems with packet dropout, which have been researched extensively.

On the other hand, observability and detectability are most essential and significant notions in modern control theory which play an important role in study of Kalman filtering problem, LQ optimal control, H_2/H_∞ optimal control and to some extent. In [2], Zhang, et al. generalized the classic notion of complete observability of deterministic systems to stochastic systems and gave the concept of exact observability, for which a necessary and sufficient condition called stochastic Popov-Belevitch-Hautus (PBH) criterion was developed by employing the spectrum technique. Moreover, it was shown that exact observability can be applied to study the stochastic stability based on the generalized Lyapunov equations (GLEs). Similarly, Damm[1], Zhang, et al.[3] extended the notion of detectability for deterministic systems to stochastic systems and gave the concepts of detectability and exact detectability, respectively. Some stochastic PBH criteria for detectability and exact detectability were also derived. Recently, a new mathematic approach, called the H-representation[17], was introduced to study the GLEs arising in stochastic control. Based on this method, several equivalent conditions can be obtained efficiently for stochastic stabilization, observability and detectability. Additionally, it is need to emphasize that there are still many different notions of detectability for stochastic systems reported in recently literature, such as MS-detectability, W-detectability and uniform detectability[4–7,18–20]. It is interesting that although many control problems for stochastic systems can be treated analogously to that of deterministic systems, there are still some concepts that are essentially different from each other. The notion of detectability is such case. For instance, although MS-detectability and exact detectability are equivalent to the notion of detectability in deterministic systems, exact detectability is weaker than MS-detectability in stochastic systems. The latter is dual to the notion of mean square stabilization. It should be pointed out that compared with other concepts of detectability, exact detectability seems to be served as the usual detectability concept for deterministic systems.

In [6, 7], Li, et al. used the concepts of W-detectability and W-observability to derive some unified treatments for various detectability and observability in both continuous- and discrete-time stochastic systems, which motivated us to generalize these two concepts to discrete-time[9] and continuous-time stochastic Markov jump systems and discuss the relationships between exact observability[2] and W-observability[4]; detectability[1], exact detectability[3] and W-detectability[4], respectively, to find out a similar unified treatment in the framework of stochastic Markov jump systems.

The objective of this paper is to study the notions of observability and detectability for continuous-time stochastic Markov jump systems and give an unified treatment for various notions of observability and detectability as done in [6, 7, 9]. In this paper, we extend the notions of W-observability and W-detectability to continuous-time stochastic Markov jump systems for the first time. These definitions adopt the following ideas:

(a) W-observability implies that both stable and unstable models could be reflected by the measurement output;

(b) W-detectability implies that any unstable model could be reflected by the measurement output.

The above ideas come from the standard concepts of observability and detectability for linear systems[18]. Moreover, it will be shown that W-observability is equivalent to exact observability. Likewise, W-detectability allows us to unify different concepts of detectability and exact detectability in the same framework. Besides, by defining an accumulated energy function, some efficient W-observability and W-detectability criteria and good properties for stochastic Markov jump systems can be derived. It is expected that the results of this paper will be useful in stochastic stability analysis, stochastic LQ optimal control problem and stochastic H_2/H_∞ control problem.

The outline of this paper is organized as follows. Section 2 introduces some notations and makes some important preliminaries which will be used throughout this paper. Additionally, we introduce the notions of asymptotical mean square stability and give some necessary and sufficient conditions for the stability of continuous-time stochastic Markov jump systems directly. In Section 3, the definition of W-observability of continuous-time stochastic Markov jump systems is presented, for which one efficient criterion is given. Then, an equivalent theorem among W-observability and exact observability is derived. In Section 4, W-detectability is defined, which is shown to coincide with detectability and exact detectability, and one criterion for W-detectability is also presented. Section 5 presents a simple numerical example to illustrate the main results of this paper and Section 6 concludes this paper with some remarks.

2 Notations and Preliminaries

In this paper, let A^{T}, $\mathrm{Tr}(A)$ and $\mathrm{rank}(A)$ denote the transpose, the trace and the rank of the matrix A, respectively. I is the identity matrix and $\mathcal{I}_B$ is the indicator function of the set B. As usual, $A \geq 0\,(> 0)$ means that A is a positive semidefinite (positive definite) matrix and $A \geq B$ $(> B)$ means that $A - B \geq 0$ (> 0). Let $\otimes$ be the Kronecker product and $\sigma(L)$ represents the spectrum set of the linear operator or the matrix L. Let $\mathbb{N}$ denote the set of nonnegative integers, i.e., $\mathbb{N} = \{0, 1, \cdots\}$ and $\mathbb{C}^-$ the open left hand side complex plane. We set $\mathbb{R}^{+,0}$ to denote the interval $[0, \infty)$ in real plane and $\mathbb{R}^n$ refer to the linear space of all n-dimensional real vectors with usual 2-norm $\|\cdot\|$. Let $\mathbb{R}_{m\times n}$ be the space formed by all $m \times n$ real matrices and $\mathbb{S}_n$ the space of all $n \times n$ symmetric matrices. We use $\mathbb{S}_{n+}$ to denote the space of all positive semidefinite matrices. Let $\mathbb{R}^N_{m\times n}$ represent the linear space of all matrix groups $A = (A_1, A_2, \cdots, A_N)$ with $A_i \in \mathbb{R}_{m\times n}$. In the meanwhile, $\mathbb{S}^N_n$ and $\mathbb{S}^N_{n+}$ can also be defined similarly. We can derive that $\mathbb{S}^N_n = \{X = (X_1, X_2, \cdots, X_N), X_i \in \mathbb{S}_n, i = 1, 2, \cdots, N\}$ is a Hilbert space with the inner product

$$\langle U, V\rangle = \sum_{i=1}^{N} \mathrm{Tr}(U_i^{\mathrm{T}} V_i), \quad \forall U, V \in \mathbb{S}^N_n.$$

Without loss of generality, consider the following continuous-time stochastic Markov jump system

$$\begin{cases} dx(t) = A(\theta(t))x(t)dt + C(\theta(t))x(t)dw(t), \ x(0) = x_0, \\ y(t) = Q(\theta(t))x(t), \ t \in \mathbb{R}^{+,0} \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are the system state and the measurement output, respectively. $x(0) = x_0$ is the given initial condition. Furthermore, we present the following assumptions.

(a) $\mathcal{W} = \{w(t), t \in \mathbb{R}^{+,0}\}$ is an one-dimensional, standard Wiener process defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with $\mathcal{F}_t = \sigma\{\theta(s), w(s)|0 \leq s \leq t\}$.

(b) $\Theta = \{\theta(t), t \in \mathbb{R}^{+,0}\}$ is a homogeneous continuous-time Markov process with right continuous trajectories and its state space is a finite set defined by $\overline{N} = \{1, 2, \cdots, N\}$. In addition, $\theta(0) = \theta_0$ is the initial distribution. We also assume that

$$P(\theta(t + \Delta t) = j|\theta(t) = i) = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \lambda_{ii}\Delta t + o(\Delta t), & i = j, \end{cases} \tag{2}$$

where $\Lambda = [\lambda_{ij}], i, j \in \mathbb{N}$ is the stationary transition rate matrix with $\lambda_{ij} \geq 0$ $(i \neq j)$ and $0 \leq \lambda_i = -\lambda_{ii} = \sum_{i\neq j} \lambda_{ij} < \infty$, $o(\Delta t)$ refers to an infinitesimal of higher order w.r.t Δt.

(c) Wiener process $\mathcal{W} = \{w(t), t \in \mathbb{R}^{+,0}\}$ is independent with Markov process $\Theta = \{\theta(t), t \in \mathbb{R}^{+,0}\}$ and the initial distribution $\theta(0) = \theta_0$.

Define the following matrix groups A, $C \in \mathbb{R}^N_{n\times n}$ and $Q \in \mathbb{R}^N_{p\times n}$

$$A = (A_1, A_2, \cdots, A_N), \quad C = (C_1, C_2, \cdots, C_N), \quad Q = (Q_1, Q_2, \cdots, Q_N), \tag{3}$$

where $A_i = A(\theta(t) = i)$, $C_i = C(\theta(t) = i)$ and $Q_i = Q(\theta(t) = i)$ $(i \in \overline{N})$. For convenience, System (1) will be described as $[A, C; Q|\Lambda]$ hereinafter.

Besides, define matrices $X_i(t)$, $Y_i(t)$ $(t \in \mathbb{R}^{+,0}, i \in \overline{N})$ and their corresponding matrix groups $X(t)$, $Y(t)$ as follows

$$X_i(t) = E[x(t)x^{\mathrm{T}}(t)\mathcal{I}_{\{\theta(t)=i\}}], \quad Y_i(t) = E[y(t)y^{\mathrm{T}}(t)\mathcal{I}_{\{\theta(t)=i\}}], \tag{4}$$

$$X(t) = (X_1(t), X_2(t), \cdots, X_N(t)) \in \mathbb{S}^N_n, \quad Y(t) = (Y_1(t), Y_2(t), \cdots, Y_N(t)) \in \mathbb{S}^N_p. \tag{5}$$

At first, we introduce the following differential equations $X_i(t)$ and $Y_i(t)$ satisfying, which can be used to investigates the notions of detectability and observability below.

Lemma 2.1 (see [8]) *For system $[A, C; Q|\Lambda]$, $X_i(t)$ and $Y_i(t)$ $(i \in \overline{N})$ satisfy the following differential equations*

$$\begin{cases} \dot{X}_i(t) = A_iX_i(t) + X_i(t)A_i^{\mathrm{T}} + C_iX_i(t)C_i^{\mathrm{T}} + \sum_{j=1}^{N} \lambda_{ji}X_j(t), \quad X_i(0) = x_0x_0^{\mathrm{T}}\mathcal{I}_{\{\theta_0=i\}}, \\ Y_i(t) = Q_iX_i(t)Q_i^{\mathrm{T}}, \ t \in \mathbb{R}^{+,0}. \end{cases} \tag{6}$$

Definition 2.2 For system $[A, C; Q|\Lambda]$, Lyapunov operator $\mathcal{L}(\cdot) = (\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_N)$ from $\mathbb{S}^N_n$ to $\mathbb{S}^N_n$ is defined as follows

$$\mathcal{L}_i(X) = A_iX_i + X_iA_i^{\mathrm{T}} + C_iX_iC_i^{\mathrm{T}} + \sum_{j=1}^{N} \lambda_{ji}X_j, \ \forall X = (X_1, X_2, \cdots, X_N) \in \mathbb{S}^N_n, \ i \in \overline{N}. \tag{7}$$

The spectrum set of operator $\mathcal{L}(\cdot)$ is defined by $\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : \mathcal{L}(\widehat{X}) = \lambda\widehat{X}, \widehat{X} \in \mathbb{S}_n^N, \widehat{X} \neq 0\}$.

Remark 2.3 From Lemma 2.1 and Definition 2.2, for any $t \in \mathbb{R}^{+,0}$, we have $\dot{X}(t) = \mathcal{L}(X(t))$. Besides, it is easy to check that the following linear operator $\mathcal{L}^*(\cdot) = (\mathcal{L}_1^*, \mathcal{L}_2^*, \cdots, \mathcal{L}_N^*)$

$$\mathcal{L}_i^*(X) = A_i^{\mathrm{T}} X_i + X_i A_i + C_i^{\mathrm{T}} X_i C_i + \sum_{j=1}^N \lambda_{ij} X_j, \quad \forall X = (X_1, X_2, \cdots, X_N) \in \mathbb{S}_n^N, \quad i \in \overline{N} \quad (8)$$

is the adjoint operator of $\mathcal{L}(\cdot)$ with the inner product $\langle U, V\rangle = \sum_{j=1}^N Tr(U_i^{\mathrm{T}} V_i)$ for any $U, V \in \mathbb{S}_n^N$. As discussed in [1], it follows $\sigma(\mathcal{L}) = \sigma(\mathcal{L}^*)$.

Definition 2.4 (see [1]) A linear operator $\mathcal{T}(\cdot) : H \to H$ is called to be resolvent positive if there exists a scalar $\alpha_0 \in \mathbb{R}$ such that for any $\alpha \geq \alpha_0$, the resolvent operator $(\alpha\mathscr{E} - \mathcal{T})^{-1}$ is positive, i.e.,

$$(\alpha\mathscr{E} - \mathcal{T})^{-1}(H_+) \subseteq H_+,$$

where $\mathscr{E}$ is the identity operator and H is a finite-dimensional space, ordered by the closed solid, pointed convex cone H_+.

Remark 2.5 According to [1], linear operator $\mathcal{L}(\cdot)$ is resolvent positive. Denote $\rho(\mathcal{L}) = \max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathcal{L})\}$. Then, there exists a nonzero $Z \in \mathbb{S}_{n+}^N$ such that $\mathcal{L}(Z) = \rho(\mathcal{L})Z$.

For the sake of discussion, we introduce another linear operator vec$(\cdot)$ as follows

$$\mathrm{vec}(A) = \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \\ \vdots \\ \mathrm{vec}(A_N) \end{bmatrix}, \quad \mathrm{vec}(A_j) = \begin{bmatrix} A_j^{(1)} \\ A_j^{(2)} \\ \vdots \\ A_j^{(n)} \end{bmatrix}, \quad j \in \overline{N},$$

where $A = (A_1, A_2, \cdots, A_N) \in \mathbb{R}_{m\times n}^N$, $A_j = (A_j^{(1)}, A_j^{(2)}, \cdots, A_j^{(n)})$ and $A_j^{(k)} \in \mathbb{R}^m$ is the k-th column vector of A_j. Using the operator vec$(\cdot)$, for any $U, V \in \mathbb{S}_n^N$, we have

$$\langle U, V\rangle = \sum_{i=1}^N \mathrm{Tr}(U_i^{\mathrm{T}} V_i) = \mathrm{vec}^{\mathrm{T}}(U)\mathrm{vec}(V). \quad (9)$$

Moreover, let $\zeta = n^2 N$ and define vector $\chi(t) = \mathrm{vec}(X(t))$. Taking operator vec$(\cdot)$ on both sides of $\dot{X}(t) = \mathcal{L}(X(t))$ with $\mathrm{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\mathrm{vec}(X)$, we have

$$\dot{\chi}(t) = \mathscr{A}\chi(t), \quad (10)$$

where $\mathscr{A} = \mathrm{diag}(A_1 \otimes I_n + I_n \otimes A_1 + C_1 \otimes C_1, A_2 \otimes I_n + I_n \otimes A_2 + C_2 \otimes C_2 \cdots, A_N \otimes I_n + I_n \otimes A_N + C_N \otimes C_N) + (\Lambda^{\mathrm{T}} \otimes I_{n^2}) \in \mathbb{R}_{\zeta\times\zeta}$. Therefore, System (1) is reduced to a simple linear system (10), which will be used to investigate the concepts of stability, observability, detectability for system $[A, C; Q|\Lambda]$.

Definition 2.6 System $[A, C; Q|\Lambda]$ is said to be asymptotically mean square stable, if for any $x_0 \in \mathbb{R}^n$ and the initial distribution θ_0, $\lim_{t\to\infty} E\|x(t)\|^2 = 0$.

According to [2], we can get that system $[A, C; Q|\Lambda]$ is asymptotically mean square stable, iff one of the following statements holds.

(a) $\sigma(\mathcal{L}) \subseteq \mathbb{C}^-$; (b) $\sigma(\mathscr{A}) \subseteq \mathbb{C}^-$; (c) $\rho(\mathcal{L}) < 0$.

Generally speaking, for system $[A, C; Q|\Lambda]$, we can compute the matrix $\mathscr{A}$ and its spectrum set $\sigma(\mathscr{A})$ firstly. Then judge its asymptotical mean square stability based on the item (c) as above.

In order to derive the criteria for W-observability and W-detectability, we define the norm $\|X(t)\|_E = E\|x(t)\|^2$ and introduce the following accumulated energy function

$$W^h(X(0)) = \int_0^h \langle X(\tau), Q^{\mathrm{T}}Q\rangle d\tau, \quad \forall h \geq 0, \tag{11}$$

where $X(0) = (X_1(0), X_2(0), \cdots, X_N(0)) \in \mathbb{S}^N_{n+}$ and $Q^{\mathrm{T}}Q = (Q_1^{\mathrm{T}}Q_1, Q_2^{\mathrm{T}}Q_2, \cdots, Q_N^{\mathrm{T}}Q_N)$.

Remark 2.7 Notice that the function (11) has the physical interpretation of the accumulated energy of the output process $y(t)$. Indeed, for any $h \geq 0$, we have

$$\begin{aligned} W^h(X(0)) = \int_0^h \langle X(\tau), Q^{\mathrm{T}}Q\rangle d\tau &= \int_0^h \sum_{i=1}^N \mathrm{Tr}(X_i^{\mathrm{T}}(\tau)Q_i^{\mathrm{T}}Q_i)d\tau \\ &= \int_0^h \sum_{i=1}^N \mathrm{Tr}(Y_i(\tau))d\tau \\ &= \int_0^h E\left[\sum_{i=1}^N \mathrm{Tr}(y(\tau)y^{\mathrm{T}}(\tau))\mathcal{I}_{\theta(\tau)=i}\right]d\tau \\ &= \int_0^h E\|y(\tau)\|^2 d\tau \\ &= \int_0^h \|Y(\tau)\|_E d\tau, \end{aligned}$$

where $X(\tau) = (X_1(\tau), X_2(\tau), \cdots, X_N(\tau)) \in \mathbb{S}^N_{n+}$, $X_i(\tau) = E[x(\tau)x^{\mathrm{T}}(\tau)\mathcal{I}_{\{\theta(\tau)=i\}}]$ and $Y_i(\tau) = E[y(\tau)y^{\mathrm{T}}(\tau)\mathcal{I}_{\{\theta(\tau)=i\}}]$ $(i \in \overline{N}, 0 \leq \tau \leq T)$.

Define a series of matrix group $S(t) = (S_1(t), S_2(t), \cdots, S_N(t)) \in \mathbb{S}^N_{n+}$ $(t \in \mathbb{R}^{+,0})$ satisfying the following coupled-Riccati equations

$$\dot{S}_i(t) = Q_i^{\mathrm{T}}Q_i + \mathcal{L}_i^*(S(t)) = Q_i^{\mathrm{T}}Q_i + A_i^{\mathrm{T}}S_i(t) + S_i(t)A_i + C_i^{\mathrm{T}}S_i(t)A_i + \sum_{j=1}^N \lambda_{ji}S_j(t), \quad i \in \overline{N} \tag{12}$$

with the initial condition $S(0) = 0$. Together with the linear operator $\mathcal{L}^*(\cdot)$, we have $\dot{S}(t) = Q^{\mathrm{T}}Q + \mathcal{L}^*(S(t))$. In what follows, we give some good properties for matrix group $S(t)$ which will be used in the proof of the main results.

Lemma 2.8 *For any $h \geq 0$, $S(t)$ and $X(0)$ satisfy the following equation*

$$W^h(X(0)) = \langle X(0), S(h)\rangle. \tag{13}$$

Proof Let $\alpha(t) = \mathrm{vec}(S(t))$ and $\beta = \mathrm{vec}(Q^{\mathrm{T}}Q)$. Using $\dot{\chi}(t) = \mathscr{A}\chi(t)$ and $\chi(0) = \chi_0 = \mathrm{vec}(X(0))$, we have $\chi(t) = \mathrm{e}^{\mathscr{A}t}\chi_0$. It is easily straightforward to check that for any $h \geq 0$,

$$\begin{aligned} W^h(X(0)) = \int_0^h \langle X(\tau), Q^{\mathrm{T}}Q\rangle d\tau &= \int_0^h \chi^{\mathrm{T}}(\tau)\beta d\tau \\ &= \int_0^h (\mathrm{e}^{\mathscr{A}\tau}\chi_0)^{\mathrm{T}}\beta d\tau \\ &= \chi_0^{\mathrm{T}} \int_0^h \mathrm{e}^{\mathscr{A}^{\mathrm{T}}\tau}\beta d\tau. \end{aligned} \tag{14}$$

On the other hand, by taking operator $\mathrm{vec}(\cdot)$ on both sides of (12), we can get

$$\dot{\alpha}(t) = \mathrm{vec}(\dot{S}(t)) = \mathrm{vec}(Q^{\mathrm{T}}Q + \mathcal{L}^*(S(t))) = \beta + \mathscr{A}^{\mathrm{T}}\alpha(t), \tag{15}$$

where $\mathscr{A}^{\mathrm{T}} = \mathrm{diag}(A_1^{\mathrm{T}} \otimes I_n + I_n \otimes A_1^{\mathrm{T}} + C_1^{\mathrm{T}} \otimes C_1^{\mathrm{T}}, A_2^{\mathrm{T}} \otimes I_n + I_n \otimes A_2^{\mathrm{T}} + C_2^{\mathrm{T}} \otimes C_2^{\mathrm{T}}, \cdots, A_N^{\mathrm{T}} \otimes I_n + I_n \otimes A_N^{\mathrm{T}} + C_N^{\mathrm{T}} \otimes C_N^{\mathrm{T}}) + (\varLambda \otimes I_{n^2}) \in \mathbb{R}_{\zeta\times\zeta}$. Then, since $\alpha(0) = \mathrm{vec}(S(0)) = 0$, for any $h \geq 0$, it follows that

$$\alpha(h) = \int_0^h \mathrm{e}^{\mathscr{A}^{\mathrm{T}}(h-\tau)}\beta d\tau = \int_h^0 \mathrm{e}^{\mathscr{A}^{\mathrm{T}}l}\beta d(h-l) = \int_0^h \mathrm{e}^{\mathscr{A}^{\mathrm{T}}l}\beta dl. \tag{16}$$

Combining (10) with (16), we can derive

$$\langle X(0), S(h)\rangle = \chi_0^{\mathrm{T}}\alpha(h) = \chi_0^{\mathrm{T}} \int_0^h \mathrm{e}^{\mathscr{A}^{\mathrm{T}}l}\beta dl. \tag{17}$$

Compared (14) with (17), we can derive $W^h(X(0)) = \langle X(0), S(h)\rangle = \chi_0^{\mathrm{T}}\alpha(h)$ conclusively. ∎

Lemma 2.9 *Let $S(t) = (S_1(t), S_2(t), \cdots, S_N(t)) \in \mathbb{S}_{n+}^N$ be the solution of the matrix differential equations* (12)*. Then, for any $h_1 \geq h_2 \geq 0$, $S(h_1) \geq S(h_2)$ holds.*

Proof From Lemma 2.8, for any $X(0) \in \mathbb{S}_{n+}^N$, it follows that for any $h_1 \geq h_2 \geq 0$,

$$W^{h_1}(X(0)) = \langle X(0), S(h_1)\rangle \geq \langle X(0), S(h_2)\rangle = W^{h_2}(X(0)) \geq 0, \tag{18}$$

from which we have $\langle X(0), S(h_1) - S(h_2)\rangle \geq 0$. Thus, $S(h_1) \geq S(h_2)$ due to the arbitrariness of $X(0)$. ∎

3 Observability

In this section, the notion of W-observability is defined, which is shown to be equivalent with exact observability of stochastic systems. Besides, an efficient W-observability criterion is also derived. To this end, we give the following definition firstly.

Definition 3.1 System $[A, C; Q|\Lambda]$ is said to be W-observable, if for each $x_0 \in \mathbb{R}^n$ and the initial distribution $\theta(0) = \theta_0$, there exist scalars $N_d \geq 0$ and $\gamma > 0$ such that $W^{N_d}(X(0)) \geq \gamma \|X(0)\|_E$.

Lemma 3.2 *For system $[A, C; Q|\Lambda]$, the following statements are equivalent:*
(a) $W^h(X(0)) = \langle X(0), S(h)\rangle = \chi_0^{\mathrm{T}}\alpha(h) = 0$ *for some* $h > 0$*;*
(b) $\chi_0^{\mathrm{T}}\big[\frac{d^k}{dt^k}\alpha(t)|_{t=0}\big] = 0$ *for* $k = 1, 2, \cdots, \zeta$*;*
(c) $\chi_0^{\mathrm{T}}(\mathscr{A}^{\mathrm{T}})^{k-1}\beta = 0$ *for* $k = 1, 2, \cdots, \zeta$*;*
(d) $\chi_0^{\mathrm{T}}\alpha(t) = 0$ *for any* $t \in \mathbb{R}^{+,0}$.

Proof (a) $\Rightarrow$(b). From Lemma 2.8, we have that $W^t(X(0)) = \langle X(0), S(t)\rangle = \chi_0^{\mathrm{T}}\alpha(t) \geq 0$ $(\forall t \in \mathbb{R}^{+,0})$. If there exists a $h > 0$ such that $W^h(X(0) = 0$, it follows from Lemma 2.9 that

$$0 \leq \chi_0^{\mathrm{T}}\alpha(t) = \langle X(0), S(t)\rangle \leq \langle X(0), S(h)\rangle = \chi_0^{\mathrm{T}}\alpha(h) = 0, \qquad t \in [0, h]. \tag{19}$$

Therefore, we have that $W^t(X(0)) = \langle X(0), S(t)\rangle = \chi_0^{\mathrm{T}}\alpha(t) = 0$ for all $t \in [0, h]$, which leads to

$$\chi_0^{\mathrm{T}}\left[\frac{d^k}{dt^k}\alpha(t)\bigg|_{t=0}\right] = 0, \quad k \in \mathbb{N}. \tag{20}$$

(b)$\Rightarrow$ (a). This result is trivial.

(b)$\Leftrightarrow$ (c). Using the equation (15) and $\alpha(0) = 0$ recursively, we can get

$$\frac{d^k}{dt^k}\alpha(t)\bigg|_{t=0} = \frac{d^{k-1}}{dt^{k-1}}\left[\frac{d}{dt}\alpha(t)\right]\bigg|_{t=0} = \frac{d^{k-1}}{dt^{k-1}}\left[\beta + \mathscr{A}^{\mathrm{T}}\alpha(t)\right]\bigg|_{t=0} = \cdots = (\mathscr{A}^{\mathrm{T}})^{k-1}\beta. \tag{21}$$

Then, for each integer $1 \leq k \leq \zeta$,

$$\chi_0^{\mathrm{T}}\left[\frac{d^k}{dt^k}\alpha(t)\bigg|_{t=0}\right] = 0 \Leftrightarrow \chi_0^{\mathrm{T}}(\mathscr{A}^{\mathrm{T}})^{k-1}\beta = 0.$$

(c)$\Rightarrow$(d). Assume that $\chi_0^{\mathrm{T}}(\mathscr{A}^{\mathrm{T}})^{k-1}\beta = 0$ holds for $k = 1, 2, \cdots, \zeta$. Using the Caylay-Hamilton theorem and the equation (16), for any $t \in \mathbb{R}^{+,0}$,

$$\begin{aligned}\alpha(t) = \int_0^t \mathrm{e}^{\mathscr{A}^{\mathrm{T}}\tau}\beta d\tau &= \int_0^t \sum_{k=1}^{\zeta}\delta_k(\tau)(\mathscr{A}^{\mathrm{T}})^{k-1}\beta d\tau \\ &= \sum_{k=1}^{\zeta}(\mathscr{A}^{\mathrm{T}})^{k-1}\beta\int_0^t \delta_k(\tau)d\tau \\ &= \sum_{k=1}^{\zeta}\widehat{\delta}_k(t)(\mathscr{A}^{\mathrm{T}})^{k-1}\beta,\end{aligned} \tag{22}$$

where $\delta_k(t)$ and $\widehat{\delta}_k(t) = \int_0^t \delta_k(\tau)d\tau$ are some bounded scalar functions. Then, from item (c), for any $t \in \mathbb{R}^{+,0}$,

$$\chi_0^{\mathrm{T}}\alpha(t) = \sum_{k=1}^{\zeta}\widehat{\delta}_k(t)[\chi_0^{\mathrm{T}}(\mathscr{A}^{\mathrm{T}})^{k-1}\beta] = 0. \tag{23}$$

(d)$\Rightarrow$(a). This result is trivial. ∎

Based on Lemma 3.2, we can get the following criterion for W-observability.

Theorem 3.3 *System $[A, C; Q|\Lambda]$ is W-observable iff there exists a scalar $h > 0$ such that $S(h) > 0$, i.e. $S_i(h) > 0$, $(i \in \overline{N})$.*

Proof Sufficiency. If there exists a scalar $h > 0$ such that $S(h) = (S_1(h), S_2(h), \cdots, S_N(h)) > 0$, by the accumulated energy equation (11) with (12), we have

$$\begin{aligned} W^h(X(0)) = \langle X(0), S(h)\rangle &= \sum_{i=1}^{N} \mathrm{Tr}(X_i^{\mathrm{T}}(0)S_i(h)) \\ &\geq \lambda_{\min}(S_i(h)) \sum_{i=1}^{N} \mathrm{Tr}(X_i(0)) \\ &= \lambda_{\min}(S_i(h)) \sum_{i=1}^{N} \mathrm{Tr}(x_0 x_0^{\mathrm{T}})\mathcal{I}_{\{\theta_0 = i\}} \\ &= \lambda_{\min}(S_i(h))\|X(0)\|_E, \end{aligned} \tag{24}$$

which indicates that system $[A, C; Q|\Lambda]$ is W-observable with $N_d = h$ and $\gamma = \lambda_{\min}(S_i(h))$.

Necessity. By Definition 3.1, if system $[A, C; Q|\Lambda]$ is W-observable, there exist scalars $N_d \geq 0$ and $\gamma > 0$ such that $W^{N_d}(X(0)) \geq \gamma\|X(0)\|_E$. In view of Lemma 2.8, we can write

$$\begin{aligned} \gamma\|X(0)\|_E \leq W^{N_d}(X(0)) = \langle X(0), S(N_d)\rangle &= \sum_{i=1}^{N} \mathrm{Tr}(X_i^{\mathrm{T}}(0)S_i(N_d)) \\ &= \sum_{i=1}^{N} (x_0^{\mathrm{T}} S_i(N_d) x_0)\mathcal{I}_{\{\theta_0 = i\}}. \end{aligned} \tag{25}$$

Owing to the arbitrariness of $x_0 \in \mathbb{R}^n$ and the initial distribution θ_0, we have $S(N_d) > 0$. ∎

Due to the difficulty of determining the appropriate scalar $h > 0$, it is not convenient to use Theorem 3.2 in practical applications. To overcome this difficulty, we construct the following observable matrix $\mathscr{O}_i$ and give a rank criterion for W-observability. Define the matrix group $O(k) = (O_1(k), O_2(k), \cdots, O_N(k)) \in \mathbb{S}_{n+}^N$ $(k \in \mathbb{N})$ and the observable matrix $\mathscr{O}_i \in \mathbb{R}_{n\zeta \times n}$

$$\mathscr{O}_i = [O_i^{\mathrm{T}}(0) \vdots O_i^{\mathrm{T}}(1) \vdots \cdots \vdots O_i^{\mathrm{T}}(\zeta - 1)]^{\mathrm{T}}, \quad i \in \overline{N}, \tag{26}$$

where $O(k)$ satisfying the difference equation $O(k+1) = \mathcal{L}^*(O(k))$ with $O(0) = Q^{\mathrm{T}}Q$. Therefore, $O_i(k)$ satisfies the following equation

$$O_i(k+1) = \mathcal{L}_i^*(O(k)) = A_i^{\mathrm{T}} O_i(k) + O_i(k) A_i + C_i^{\mathrm{T}} O_i(k) C_i + \sum_{j=1}^{N} \lambda_{ij} O_j(k). \tag{27}$$

Lemma 3.4 *Define the matrix group $O(k)$ $(k \in \mathbb{N})$ and $S(t)$ $(t \in \mathbb{R}^{+,0})$ as above. Then*

$$O(k) = \frac{d^{k+1}}{dt^{k+1}} S(t)\bigg|_{t=0}. \tag{28}$$

Proof We derive this result by induction. When $k = 0$, for each $i \in \overline{N}$, from (12), we have

$$\frac{d}{dt}S_i(t)\bigg|_{t=0} = [Q_i^{\mathrm{T}}Q_i + \mathcal{L}_i^*(S(t))]|_{t=0} = Q_i^{\mathrm{T}}Q_i + \mathcal{L}_i^*(S(0)) = Q_i^{\mathrm{T}}Q_i = O_i(0).$$

Assume the equation (28) is true for $k = n$, i.e.,

$$O_i(n) = \frac{d^{n+1}}{dt^{n+1}}S(t)\bigg|_{t=0}, \quad \forall i \in \overline{N},$$

we show that the equation (28) is also true for $k = n + 1$. It follows from (27) that

$$\begin{aligned} O_i(n+1) &= \mathcal{L}_i^*(O(n)) \\ &= A_i^{\mathrm{T}}O_i(n) + O_i(n)A_i + C_i^{\mathrm{T}}O_i(n)C_i + \sum_{j=1}^{N}\lambda_{ij}O_j(n) \\ &= A_i^{\mathrm{T}}\left(\frac{d^{n+1}}{dt^{n+1}}S_i(t)\bigg|_{t=0}\right) + \left(\frac{d^{n+1}}{dt^{n+1}}S_i(t)\bigg|_{t=0}\right)A_i + C_i^{\mathrm{T}}\left(\frac{d^{n+1}}{dt^{n+1}}S_i(t)\bigg|_{t=0}\right)C_i \\ &\quad + \sum_{j=1}^{N}\lambda_{ij}\left(\frac{d^{n+1}}{dt^{n+1}}S_j(t)\bigg|_{t=0}\right) \\ &= \frac{d^{n+1}}{dt^{n+1}}\left[A_i^{\mathrm{T}}S_i(t) + S_i(t)A_i + C_i^{\mathrm{T}}S_i(t)C_i + \sum_{j=1}^{N}\lambda_{ij}S_j(t)\right]\bigg|_{t=0} \\ &= \frac{d^{n+1}}{dt^{n+1}}\left[\frac{d}{dt}S_i(t)\right]\bigg|_{t=0} \\ &= \frac{d^{n+2}}{dt^{n+2}}S_i(t)\bigg|_{t=0}. \end{aligned} \tag{29}$$

This proof is completed by inductive hypothesis. ∎

Below, based on the observable matrix $\mathscr{O}_i$ we give the rank criterion for W-detectability of system $[A, C; Q|\Lambda]$ as follows, which is easy to verify.

Proposition 3.5 *For system $[A, C; Q|\Lambda]$, assume the matrix group $S(t)$ and the observability matrix $\mathscr{O}_i$ $(i \in \overline{N})$ are defined as above. The following statements are equivalent:*

(a) $S(h) > 0$ *for some* $h > 0$*;*

(b) $S(t) > 0$ *for any* $t > 0$*;*

(c) *The observable matrix $\mathscr{O}_i$ has full column rank for each $i \in \overline{N}$.*

Proof (b)⇒(a). This result is trivial.

(a)⇒(b). We show this result by contradiction. Assume there exists a scalar $t^* > 0$ such that the matrix group $S(t^*) = (S_1(t^*), S_2(t^*), \cdots, S_N(t^*)) \in \mathbb{S}_{n+}^N$ is not strictly positive definite. Without loss of generality, we assume that $S_j(t^*)$ $(j \in \overline{N})$ is not strictly positive definite. Thus, there exists a nonzero $x_0 \in \mathbb{R}^n$ such that $x_0^{\mathrm{T}}S_j(t^*)x_0 = 0$. Define the matrix group $X^* = (X_1^*, X_2^*, \cdots, X_N^*) \in \mathbb{S}_{n+}^N$ with

$$X_i^* = \begin{cases} x_0x_0^{\mathrm{T}}\mathcal{I}_{\{\theta_0^*=i\}} = 0, & i \neq j, \\ x_0x_0^{\mathrm{T}}\mathcal{I}_{\{\theta_0^*=j\}} = x_0x_0^{\mathrm{T}}, & i = j. \end{cases} \tag{30}$$

By Lemmas 2.8 and 3.2, we can derive

$$W^{t^*}(X^*) = \langle X^*, S(t^*)\rangle = \sum_{i=1}^{N} \mathrm{Tr}((X_i^*)^{\mathrm{T}} S_i(t^*)) = \mathrm{Tr}((X_j^*)^{\mathrm{T}} S_j(t^*)) = x_0^{\mathrm{T}} S_j(t^*) x_0 = 0,$$

which is equivalent to $W^t(X^*) = 0$ for any $t \geq 0$. Therefore, there does not exist a scalar $h > 0$ such that $S(h) > 0$, which is contradictory to the item (a).

(b)⇒(c). We show this by contradiction. Without loss of generality, we assume the matrix $\mathscr{O}_j$ $(j \in \overline{N})$ is not of full column rank. Then there exists a nonzero $x_0 \in \mathbb{R}^n$ such that

$$x_0^{\mathrm{T}} \mathscr{O}_j x_0 = x_0^{\mathrm{T}} \begin{bmatrix} O_j(0) \\ O_j(1) \\ \vdots \\ O_j(\zeta - 1) \end{bmatrix} x_0 = 0.$$

Define the matrix group $X^* = (X_1^*, X_2^*, \cdots, X_N^*) \in \mathbb{S}_{n+}^N$ as (30) and let $\chi_0^* = \mathrm{vec}(X^*)$. Employing the inner product (9) and Lemma 3.2, for $k = 1, 2, \cdots, \zeta$, we have

$$\begin{aligned} (\chi_0^*)^{\mathrm{T}} \left[\frac{d^k}{dt^k} \alpha(t) \bigg|_{t=0} \right] &= \mathrm{vec}^{\mathrm{T}}(X^*) \mathrm{vec} \left[\frac{d^k}{dt^k} S(t) \bigg|_{t=0} \right] \\ &= \mathrm{vec}^{\mathrm{T}}(X^*) \mathrm{vec}[O(k-1)] \\ &= \langle X^*, O(k-1) \rangle \\ &= \sum_{i=1}^{N} \mathrm{Tr}((X_i^*)^{\mathrm{T}} O_i(k-1)) \\ &= \sum_{i=1}^{N} [x_0^{\mathrm{T}} O_i(k-1) x_0] \mathcal{I}_{\{\theta_0^* = i\}} = x_0^{\mathrm{T}} O_j(k-1) x_0 = 0. \end{aligned} \tag{31}$$

Thus, for any $t \geq 0$,

$$W^t(X^*) = (\chi_0^*)^{\mathrm{T}} \alpha(t) = \langle X^*, S(t) \rangle = \sum_{i=1}^{N} \mathrm{Tr}(X_i^* S_i(t)) = x_0^{\mathrm{T}} S_j(t) x_0 = 0. \tag{32}$$

This equation indicates that $S(t)$ is not strictly positive definite which is a contradiction.

(c)⇒(b). We prove this by contradiction. Similar to the proof above, we assume there exists a scalar $t^* > 0$ such that the matrix group $S(t^*) = (S_1(t^*), S_2(t^*), \cdots, S_N(t^*))$ is not strictly positive definite. Assume $S_j(t^*)$ $(j \in \overline{N})$ is not positive definite, and there exists a nonzero $x_0 \in \mathbb{R}^n$ such that $x_0^{\mathrm{T}} S_j(t^*) x_0 = 0$. Then, by Lemma 3.2 and the equations (30)–(31), we can get that for each $k = 1, 2, \cdots, \zeta$

$$W^{t^*}(X^*) = (\chi_0^*)^{\mathrm{T}} \alpha(t^*) = 0 \Leftrightarrow (\chi_0^*)^{\mathrm{T}} \left[\frac{d^k}{dt^k} \alpha(t) \bigg|_{t=0} \right] = x_0^{\mathrm{T}} O_j(k-1) x_0 = 0, \tag{33}$$

where $X^* = (X_1^*, X_2^*, \cdots, X_N^*) \in \mathbb{S}_{n+}^N$ and $\chi_0^* = \mathrm{vec}(X^*)$ are defined as above. Thus, the matrix $\mathscr{O}_j$ is not of full column rank, which is a contradiction. ∎

In [2], Zhang, et al. introduced the notion of exact observability and extended PBH criterion of deterministic systems to stochastic systems. Below, we generalize this notion to continuous-time stochastic Markov jump systems and derive the main result of this section.

Definition 3.6 System $[A, C; Q|\Lambda]$ is said to be exactly observable, if

$$y(t) = 0 \ \text{ a.s.}, \quad 0 \leq t \leq T, \quad \forall T > 0 \Rightarrow x_0 = 0. \tag{34}$$

Theorem 3.7 *For system* $[A, C; Q|\Lambda]$*, the following statements are equivalent:*

(a) *System* $[A, C; Q|\Lambda]$ *is W-observable in the sense of Definition* 3.1*.*

(b) *System* $[A, C; Q|\Lambda]$ *is exactly observable in the sense of Definition* 3.6*.*

(c) *There exists some* $h > 0$ *such that* $S(h)$ *is positive definite.*

(d) $S(t) > 0$ *for any* $t > 0$*.*

(e) *The matrix* $\mathscr{O}_i$ *has full column rank for each* $i \in \overline{N}$*.*

(f) (*Stochastic PBH Criterion*) *There does not exist nonzero* $Z = (Z_1, Z_2, \cdots, Z_N) \in \mathbb{S}_{n+}^N$ *such that*

$$\mathcal{L}(Z) = \lambda Z, \ (Q_1 Z_1, Q_2 Z_2, \cdots, Q_N Z_N) = 0, \quad \lambda \in \mathbb{C}. \tag{35}$$

Proof Clearly, (a) $\Leftrightarrow$ (c) follows from Theorem 3.3 and (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follows from Proposition 3.5. (b) $\Leftrightarrow$ (f) is Theorem 3.1 in [8]. Below, we will show (a) $\Leftrightarrow$ (b).

(b) $\Rightarrow$ (a). Since $y(t) = 0$ a.s.$, 0 \leq t \leq T, \ \forall T > 0 \Rightarrow x_0 = 0$ is equivalent to that for any $x_0 \neq 0$ there exists a $t^* \in [0, T]$ such that $y(t^*) \neq 0$. It follows from the continuity of $y(t)$ that there exists a sufficient small scalar $\varepsilon > 0$ such that $y(t) \neq 0$, $\forall t \in (t^* - \varepsilon, t^* + \varepsilon) \subseteq [0, T]$. By Remark 2.7, we have

$$W^{\mathrm{T}}(X(0)) = \int_0^T \|Y(\tau)\|_E d\tau \geq \int_{t^*-\varepsilon}^{t^*+\varepsilon} \|Y(\tau)\|_E d\tau > 0. \tag{36}$$

If we set

$$\gamma_0 = \frac{\int_{t^*-\varepsilon}^{t^*+\varepsilon} \|Y(\tau)\|_E d\tau}{\|X(0)\|_E},$$

then the equation (36) indicates that there exist scalars $N_d = T > 0$ and $\gamma_0 > 0$ such that $W^{N_d}(X(0)) \geq \gamma_0 \|X(0)\|_E$, i.e., system $[A, C; Q|\Lambda]$ is W-observable.

(a) $\Rightarrow$ (b). Assume that $y(t) = 0$ *a.s.*$, 0 \leq t \leq T, \ \forall T > 0$. Then, by Remark 2.7, we have

$$W^{\mathrm{T}}(X(0)) = \int_0^T \|Y(\tau)\|_E d\tau = 0. \tag{37}$$

We conclude from Definition 3.1 that $X(0) = 0$, which is equivalent to $x_0 = 0$. This proof is completed. ∎

Remark 3.8 From Theorem 3.7, the notions of W-observability and exact observability are equivalent in the framework of continuous-time stochastic Markov jump systems. For simplicity, system $[A, C; Q|\Lambda]$ is said to be observable if it satisfies one of these definitions above. On the other hand, the item (e) of Theorem 3.7 provides an efficient rank criterion to check the observability of $[A, C; Q|\Lambda]$ by the column rank of the matrix $\mathscr{O}_i$, which is analogous to the rank criterion for observability in linear system theory.

4 Detectability

In this section, we will study the properties of W-detectability, exact detectability and detectability and unify those different notions in the framework of continuous-time stochastic Markov jump systems. Moreover, we derive an efficient W-detectability criterion for such systems. First of all, we introduce the following definition.

Definition 4.1 System $[A, C; Q|\Lambda]$ is said to be W-detectable, if there exist scalars N_d, $t_d \geq 0$, $0 \leq \delta < 1$, $\gamma > 0$ such that $W^{N_d}(X(0)) \geq \gamma \|X(0)\|_E$ whenever $\|X(t_d)\|_E \geq \delta \|X(0)\|_E$.

Remark 4.2 This definition of W-detectability is based on the standard concept of detectability in linear time-varying systems that any unstable model could be reflected by the output process[18]. Whereas, by Definition 3.1, W-observability requires that both stable and unstable models could be reflected by the output process. Thus, W-observability is a special case of W-detectability with $\delta = 0$ in the sense of Definition 4.1.

Lemma 4.3 *For the matrix group $S(k)$ and $X(0)$ defined as above, the following statements are equivalent:*

(a) $W^h(X(0)) = 0$ *for some* $h > 0$*;*

(b) $W^t(X(0)) = 0$ *for any* $t \geq 0$*;*

(c) $W^t(X(l)) = 0$ *for any* $t, l \geq 0$.

Proof (a) $\Rightarrow$ (c) Noticing that

$$W^t(X(0)) = \langle X(0), S(t) \rangle = \chi_0^{\mathrm{T}} \alpha(t) = 0, \quad \forall t \geq 0, \tag{38}$$

which is the item (d) in Lemma 3.2, thus we have that (a) $\Leftrightarrow$ (c).

(c) $\Rightarrow$ (a). This result is obvious.

(b) $\Rightarrow$ (c). Since $W^t(X(0)) = 0$ for any $t \geq 0$, it is easy to get that for any $t, l \geq 0$, $W^{t+l}(X(0)) = 0$ holds. Notice that the accumulated energy function $W^t(X(0))$ is a strictly increasing function. Then, we can derive that

$$0 \leq W^t(X(l)) = \int_0^t \langle X(\tau + l), Q^{\mathrm{T}} Q \rangle d\tau = \int_l^{l+t} \langle X(\tau), Q^{\mathrm{T}} Q \rangle d\tau \leq W^{l+t}(X(0)) = 0. \tag{39}$$

Therefore, $W^t(X(l)) = 0$ for any $t, l \geq 0$ which ends this proof. ∎

Below, with the help of the accumulated energy function $W^h(X)$ and the above Lemma 4.3, we can get the following W-detectability criterion.

Theorem 4.4 *System $[A, C; Q|\Lambda]$ is W-detectable iff for each $x_0 \in \mathbb{R}^n$ and the initial distribution $\theta(0) = \theta_0$, there exists some $h > 0$ such that*

$$W^h(X(0)) = 0 \ \Rightarrow \lim_{t \to \infty} \|X(t)\|_E = 0. \tag{40}$$

Proof Necessity. If system $[A, C; Q|\Lambda]$ is W-detectable and there exists some $h > 0$ such that $W^h(X(0)) = 0$ for each $x_0 \in \mathbb{R}^n$ and $\theta(0) = \theta_0$, by Lemma 4.3, we can get that for all $t, l \geq 0$, $W^t(X(l)) = 0$ holds. Then, from Definition 4.1, it is easy to get that there exists a scalar $0 \leq \delta < 1$ such that for some scalar $t_d \geq 0$ and each $t \geq 0$, the inequality

$$\|X(t + t_d)\|_E < \delta \|X(t)\|_E \tag{41}$$

holds. Otherwise, without loss of generality, if there exist scalars $t_d \geq 0$ and $0 \leq \delta < 1$ such that $\|X(t_d)\|_E \geq \delta \|X(0)\|_E$, it follows from the definition of W-detectability that there exist scalars $N_d \geq 0$ and $\gamma > 0$ such that $W^{N_d}(X(0)) \geq \gamma \|X(0)\|_E > 0$ which contradicts $W^{N_d}(X(0)) = 0$. Therefore, the inequality (41) is true. For arbitrary $t = t^* + mt_d \in \mathbb{R}^{+,0}$ $(0 \leq t^* < t_d, m \in \mathbb{N})$, we can derive the following inequality recursively

$$\|X(t)\|_E = \|X(t^* + mt_d)\|_E < \delta \|X(t^* + (m-1)t_d)\|_E < \cdots < \delta^m \|X(t^*)\|_E. \tag{42}$$

Taking limit on both sides of the above inequality (42), we have

$$\lim_{t\to\infty} \|X(t)\|_E \leq \lim_{m\to\infty} \max_{0\leq t^* < t_d} \|X(t^* + mt_d)\|_E \leq \lim_{m\to\infty} \delta^m \big(\max_{0\leq t^* < t_d} \|X(t^*)\|_E \big) = 0, \tag{43}$$

which yields

$$\lim_{t\to\infty} E\|x(t)\|^2 = \lim_{t\to\infty} \|X(t)\|_E = 0. \tag{44}$$

Sufficiency. Denote $\mathbb{Z} = \{X(0) : X(0) \in \mathbb{S}_{n+}^N,\ \|X(0)\|_E = 1,\ W^h(X(0)) = 0\}$, where $X(0)$ is defined as above. Due to the statement

$$W^h(X(0)) = 0\ \Rightarrow \lim_{t\to\infty} \|X(t)\|_E = 0, \tag{45}$$

it follows that $\lim_{t\to\infty} \|X(t)\|_E = 0$ for all $X(0) \in \mathbb{Z}$. Thus, for each $X(0) \in \mathbb{Z}$, there exist some scalars $0 < \delta < 1$ and $t_d > 0$ such that $\|X(t_d)\|_E < \delta \|X(0)\|_E = \delta$. Denote another set $\mathbb{M} = \{X(0) : X(0) \in \mathbb{S}_{n+}^N,\ \|X(0)\|_E = 1,\ \|X(t_d)\| < \delta\}$ and the corresponding complement set is $\overline{\mathbb{M}} = \{X(0) :\ X(0) \in \mathbb{S}_{n+}^N,\ \|X(0)\|_E = 1,\ \|X(t_d)\| \geq \delta\}$. Since $X(0) \in \mathbb{Z} \Rightarrow \|X(t_d)\| < \delta$, we have $\mathbb{Z} \subseteq \mathbb{M}$.

Below, we will show that system $[A, C; Q|\Lambda]$ is W-detectable by contradiction. Assume system $[A, C; Q|\Lambda]$ is not W-detectable, i.e., for each $N_d > 0$ and $\gamma > 0$,

$$X(0) \in \overline{\mathbb{M}} \Rightarrow W^{N_d}(X(0)) < \gamma. \tag{46}$$

Let $N_d = h$ and choose a sequence $X^{(i)}(0) \in \overline{\mathbb{M}}$ $(i = 1, 2, \cdots)$ such that $W^h(X^{(i)}(0)) < \gamma_i$ with $\lim_{i\to\infty} \gamma_i = 0$. From the compactness of $\overline{\mathbb{M}}$, there exists some $X^*(0) \in \overline{\mathbb{M}}$ and a subsequence of $X^{(i)}(0)$ such that $\lim_{k\to\infty} X^{(i_k)}(0) = X^*(0) \in \overline{\mathbb{M}}$. Since the accumulated energy function $W^h(X)$ is continuous, we conclude

$$W^h(X^*(0)) = \lim_{k\to\infty} W^h(X^{(i_k)}(0)) = 0. \tag{47}$$

Therefore, $X^* \in \mathbb{Z} \subseteq \mathbb{M}$, which is contradictory to $X^* \in \overline{\mathbb{M}}$. In conclusion, system $[A, C; Q|\Lambda]$ is W-detectable. ∎

The following equivalent notions of detectability and exact detectability for stochastic system were introduced in [1, 3], respectively. Here, we generalize these definitions to system $[A, C; Q|\Lambda]$ and get the main result of this part.

Definition 4.5 System $[A, C; Q|\Lambda]$ is said to be detectable, if for each $x_0 \in \mathbb{R}^n$ and the initial distribution $\theta(0) = \theta_0$,

$$E[y^{\mathrm{T}}(t)y(t)] = 0, \quad \forall t \in \mathbb{R}^{+,0} \Rightarrow \lim_{t\to\infty} E[x^{\mathrm{T}}(t)x(t)] = 0. \tag{48}$$

Definition 4.6 System $[A, C; Q|\Lambda]$ is said to be exactly detectable, if for each $x_0 \in \mathbb{R}^n$ and the initial distribution $\theta(0) = \theta_0$,

$$y(t) = 0 \ \text{ a.s.}, \quad 0 \le t \le T, \ \ \forall T > 0 \Rightarrow \lim_{t\to\infty} E\|x(t)\|^2 = 0. \tag{49}$$

Remark 4.7 It is worthwhile to point out that $E[y^{\mathrm{T}}(t)y(t)] = 0, \ \forall t \ge 0$ is equivalent to $y(t) = 0$ a.s. $0 \le t \le T, \ \forall T > 0$. Thus, for system $[A, C; Q|\Lambda]$, detectability in Definition 4.5 is equivalent to exact detectability in Definition 4.6.

Theorem 4.8 *For system $[A, C; Q|\Lambda]$, the following statements are equivalent.*

(a) *System $[A, C; Q|\Lambda]$ is W-detectable in the sense of Definition* 4.1.

(b) *System $[A, C; Q|\Lambda]$ is detectable in the sense of Definition* 4.5.

(c) *System $[A, C; Q|\Lambda]$ is exactly detectable in the sense of Definition* 4.6.

(d) *For each $x_0 \in \mathbb{R}^n$ and the initial distribution $\theta(0) = \theta_0$, there exists some $h > 0$ such that*

$$W^h(X(0)) = 0 \ \Rightarrow \lim_{t\to\infty} \|X(t)\|_E = 0.$$

(e) (*Stochastic PBH Criterion*) *There does not exist nonzero $Z = (Z_1, Z_2, \cdots, Z_N) \in \mathbb{S}_{n+}^N$ such that*

$$\mathcal{L}(Z) = \lambda Z, \ (Q_1 Z_1, Q_2 Z_2, \cdots, Q_N Z_N) = 0, \quad \mathrm{Re}(\lambda) \ge 0. \tag{50}$$

Proof It is obvious that (a) ⇔ (d) follows from Theorem 4.4 and (b) ⇔ (c) follows from Remark 4.7. (c) ⇔ (e) is Theorem 3.2 in [8]. According to Remark 2.7 and Lemma 4.3, we have that

$$\begin{aligned} W^h(X(0)) = 0, \ (\exists h > 0) \Leftrightarrow \ W^t(X(0)) &= \int_0^t E\|y(\tau)\|^2 d\tau = 0 \\ \Leftrightarrow \|Y(t)\|_E &= E\|y(t)\|^2 = 0, \quad \forall t \ge 0. \end{aligned} \tag{51}$$

Comparing (40) with (48), we have (b) ⇔ (d). ∎

Remark 4.9 Compared with some results reported recently in literature, we have the following remarks.

(a) If the matrix group $C \equiv 0$, system $[A, C; Q|\Lambda]$ comes down to continuous-time Markov jump linear system described in [5], i.e.,

$$\begin{cases} dx(t) = A(\theta(t))x(t)dt, & x(0) = x_0, \\ y(t) = Q(\theta(t))x(t), & t \in \mathbb{R}^{+,0}. \end{cases} \tag{52}$$

W-observability and W-detectability criteria in this paper still hold for System (52).

(b) If the state space of Markov process be $\overline{N} = \{1\}$, system $[A, C; Q|\Lambda]$ is equivalent to continuous-time stochastic system described in [7] and the main results of this paper still hold.

5 An Illustrative Numerical Example

In this section, we consider one simple continuous-time stochastic Markov jump system to illustrate the some results above.

Example 5.1 Consider the following discrete-time stochastic Markov jump system $[A, C; Q|\Lambda]$ with $n = 2$, the finite state space $\overline{N} = \{1, 2\}$, $\zeta = n^2 N = 8$ and

$$\Lambda = \begin{bmatrix} -1 & 1 \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

According to the equation (10), the matrix $\mathscr{A}$ can be computed as follows

$$\mathscr{A} = \begin{bmatrix} -2 & -1 & -1 & 1 & \frac{2}{5} & 0 & 0 & 0 \\ 1 & -2 & 0 & -2 & 0 & \frac{2}{5} & 0 & 0 \\ 1 & 0 & -2 & -2 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & \frac{2}{5} \\ 1 & 0 & 0 & 0 & -\frac{7}{5} & -2 & -2 & 4 \\ 0 & 1 & 0 & 0 & 2 & -\frac{7}{5} & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 & 0 & -\frac{7}{5} & -2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 2 & -\frac{7}{5} \end{bmatrix}.$$

It is not hard to get the spectrum set $\sigma(\mathscr{A}) = \{0.5, -1, -2.4, -1.521, -2.312 \pm 4.033\mathrm{i}, -1.783 \pm 2.628\} \nsubseteq \mathbb{C}^-$. Thus system $[A, C; Q|\Lambda]$ is unstable in mean square sense.

On the other hand, consider the matrix group $O(k) = (O_1(k), O_2(k))$ and the observable matrix $\mathscr{O}_i$ $(i = 1, 2)$. According to $O(k+1) = \mathcal{L}^*(O(k))$ with $O(0) = Q^{\mathrm{T}}Q$. Therefore, $O_i(k)$ satisfies the following equation

$$O_1(k+1) = A_1^{\mathrm{T}} O_1(k) + O_1(k) A_1 + C_1^{\mathrm{T}} O_1(k) C_1 + \lambda_{11} O_1(k) + \lambda_{12} O_2(k),$$
$$O_2(k+1) = A_2^{\mathrm{T}} O_2(k) + O_2(k) A_2 + C_2^{\mathrm{T}} O_2(k) C_1 + \lambda_{21} O_1(k) + \lambda_{22} O_2(k).$$

Then, we can get a series of matrix group $O(k) = (O_1(k), O_2(k))$ as follows recursively,

$$O_1(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad O_1(1) = \begin{bmatrix} -3 & -2 \\ -2 & 3 \end{bmatrix}, \quad O_1(2) = \begin{bmatrix} 1.4 & 10 \\ 10 & 4.6 \end{bmatrix}, \quad \cdots,$$
$$O_2(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad O_2(1) = \begin{bmatrix} -0.6 & 0 \\ 0 & 2.6 \end{bmatrix}, \quad O_2(2) = \begin{bmatrix} -0.36 & 5.6 \\ 5.6 & -4.84 \end{bmatrix}, \quad \cdots$$

Since $\text{rank}[O_1^{\mathrm{T}}(0)\vdots O_1^{\mathrm{T}}(1)]^{\mathrm{T}} = 2$ and $\text{rank}[O_1(0)] = 2$, it is obvious that the observable matrix $\mathscr{O}_i$ $(i = 1, 2)$ has full column rank. Thus, system $[A, C; Q|\Lambda]$ is observable based on Proposition 3.5, which indicates that $[A, C; Q|\Lambda]$ is also detectable by Remark 4.2.

6 Conclusion

In this paper, the notions of observability and detectability for continuous-time stochastic Markov jump systems have been introduced, which unify various observability and detectability in the same framework such as W-observability and exact observability[2]; W-detectability, detectability[1] and exact detectability[3]. Moreover, some efficient criteria for observability and detectability and interesting properties have also been proposed.

References

[1] Damm T, On detectability of stochastic systems, *Automatica*, 2007, **43**(5): 928–933.

[2] Zhang W H and Chen B S, On stabilizability and exact observability of stochastic systems with their applications, *Automatica*, 2004, **40**(1): 87–94.

[3] Zhang W H, Zhang H S, and Chen B S, Generalized Lyapunov equation approach to state-dependent stochastic stabilization/detectability criterion, *IEEE Trans. Autom. Control*, 2008, **53**(7): 1630–1642.

[4] Costa E F and do Val J B R, On the detectability and observability of discrete-time Markov jump linear systems, *Systems & Control Letters*, 2001, **44**(2): 135–145.

[5] Costa E F and do Val J B R, On the observability and detectability of continuous-time Markov jump linear systems, *SIAM Jour. Cont. Opt.*, 2002, **41**(4): 1295–1314.

[6] Li Z Y, Wang Y, Zhou B, and Duan G R, Detectability and observability of discrete-time stochastic systems and their applications, *Automatica*, 2009, **45**(5): 1340–1346.

[7] Li Z Y, Wang Y, Zhou B, and Duan G R, On unified concepts of detectability and observability for continuous-time stochastic systems, *Appl. Math. Comp.*, 2010, **217**(2): 521–536.

[8] Ni Y H, Zhang W H, and Fang H T, On the observability and detectability of linear stochastic systems with Markov jumps and multiplicative noise, *Journal of Systems Science and Complexity*, 2010, **23**(1): 100–113.

[9] Zhang W H and Tan C, On detectability and observability of discrete-time stochastic Markov jump systems with state-dependent noise, *Asian J. Control*, 2013, **15**(5): 1366–1375.

[10] Costa O L V and Fragoso M D, Stability results for discrete-time linear systems with Markovian jumping parameters, *J. Math. Anal. Appl.*, 1993, **179**(1): 154–178.

[11] Ji Y D and Chizeck H J, Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Autom. Control*, 1990, **35**(7): 777–788.

[12] Feng X, Loparo K A, Ji Y D, and Chizeck H J, Stochastic stability properties of jump linear systems, *IEEE Trans. Autom. Control*, 1992, **37**(1): 38–53.

[13] Feng J E, Lam J, Xu S, and Shu Z, Stabilization of stochastic systems with optimal decay rate, *IET Control Theory Appl.*, 2008, **2**(1): 1–6.

[14] Hager W W and Horowitz L L, Convergence and stability properties of the discrete Riccati oprator equation and the associated optimal control and filtering problems, *SIAM J. Control Optim.*, 1976, **14**(2): 295–312.

[15] Chen B S and Zhang W H, Stochastic H_2/H_∞ control with state-dependent noise, *IEEE Trans. Autom. Control*, 2004, **49**(1): 45–57.

[16] Huang Y L, Zhang W H, and Zhang H S, Infinite horizon linear quadratic optimal control for discrete-time stochastic systems, *Asian J. Control*, 2008, **10**(5): 608–615.

[17] Zhang W H and Chen B S, H-representation and applications to generalized Lyapunov equations and linear stochastic systems, *IEEE Trans. Autom. Control*, 2012, **57**(12): 3009–3022.

[18] Anderson B D O and Moore J B, Detectability and stabilizability of time-varying discrete-time linear systems, *SIAM J. Control Optim.*, 1981, **19**(1): 20–32.

[19] Ugrinovskii V A, Observability of linear stochastic uncertain systems, *IEEE Trans. Autom. Control*, 2003, **48**(12): 2264–2269.

[20] Dragan V and Morozan T, Observability and detectability of a class of discrete-time stochastic linear systems, *IMA J. Math. Control Info.*, 2006, **23**(3): 371–394.