Minimizing the Risk of Absolute Ruin Under a Diffusion Approximation Model with Reinsurance and Investment*

BI Xiuchun · ZHANG Shuguang

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Abstract This paper studies the optimization problem with both investment and proportional reinsurance control under the assumption that the surplus process of an insurance entity is represented by a pure diffusion process. The company can buy proportional reinsurance and invest its surplus into a Black-Scholes risky asset and a risk free asset without restrictions. The authors define absolute ruin as that the liminf of the surplus process is negative infinity and propose absolute ruin minimization as the optimization scenario. Applying the HJB method the authors obtain explicit expressions for the minimal absolute ruin function and the associated optimal investment strategy. The authors find that the minimal absolute ruin function here is convex, but not S-shaped investigated by Luo and Taksar (2011). And finally, from behavioral finance point of view, the authors come to the conclusion: It is the restrictions on investment that results in the kink of minimal absolute ruin function.

Keywords Absolute ruin probability, dynamic investment control, HJB equation, proportional reinsurance.

1 Introduction

optimization problem is always playing a powerful role in the field of actuarial science. Some related results in this area can be found in [1–5], etc. In particular, [3] considered the optimization problem under a diffusion approximation risk model with both investment and proportional reinsurance control under investment constrains, where they defined absolute ruin as the event that the liminf of the surplus process is negative infinity and obtained explicit expressions for the S-shaped minimal absolute ruin function and its associated optimal investment-reinsurance strategy by applying the HJB method. We wonder what results in the S-shaped function, and

BI Xiuchun (Corresponding author) · ZHANG Shuguang
Department of Statistic and Finance, University of Science and Technology of China, Hefei 230026, *China.*
Email: xcbi@mail.ustc.edu.cn; sgzhang@ustc.edu.cn.
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whether the minimal absolute ruin probability function is still S-shaped without investment restrictions. We try to consider the problem from behavior finance point of view (see [6] and [7]) and conjecture that if we not impose restrictions on investment the minimum absolute ruin function will not remain at a high level of risk for a long time, just like Figures 2–4 in [3], that is, minimum absolute ruin function is not concave, but convex.

Motivated by [3], [4], and above conjecture, we consider a diffusion approximation model where the surplus is modeled by a Brownian motion with drift:

$$dR_t = \mu_0 dt + \sigma_0 dB_t^{(0)}, \quad R_0 = x.$$

We assume that the company can buy non cheap proportional reinsurance. The dynamics of the surplus with reinsurance is given by

$$dR_t^a = [\mu_0 - (1 - a(t))\lambda]dt + a(t)\sigma_0 dB_t^{(0)}, \quad R_0 = x,$$

where $0 \le a(t) \le 1$ is called the risk exposure, and $\lambda > \mu_0$.

Suppose that the insurer is allowed to invest dynamically its surplus in a financial market consisting of a risky asset and a risk free asset. Now for $t \ge 0$, let $\beta(t)$ is the amount of the surplus invested in a risky asset which is governed by Black-Scholes dynamics:

$$dS_t = S_t(\mu_1 dt + \sigma_1 dB_t^{(1)}),$$

where $\mu_1, \sigma_1 > 0$ are constants, and $\{B_t^{(1)} : t \ge 0\}$ is a standard Brownian motion. $R_t - \beta(t)$ is then invested in the risk-free asset with dynamics:

$$dP_t = rP_t dt,$$

where r is a constant rate for borrowing and lending satisfying $0 < r < \mu_1$.

Under an investment-reinsurance control policy $\pi := \{(a(t), \beta(t))\}_{t \ge 0}$, the dynamics of the surplus becomes

$$dR_t^\pi = [\mu_0 - (1 - a(t))\lambda + rR_t^\pi + (\mu_1 - r)\beta(t)]dt + a(t)\sigma_0 dB_t^{(0)} + \sigma_1\beta(t)dB_t^{(1)} \tag{1}$$

with $R_0 = x$.

Here, we assume that $\{B_t^{(0)} : t \ge 0\}$ and $\{B_t^{(1)} : t \ge 0\}$ are two correlated Brownian motions, and denote their correlation coefficient by ρ, i.e., $E[B_t^{(0)} B_t^{(1)}] = \rho t$, which is different from the one in [3] where $\{B_t^{(0)} : t \ge 0\}$ and $\{B_t^{(1)} : t \ge 0\}$ are independent.

We define absolute ruin like that in [3], that is,

$$\mathcal{O}^\pi = \{\omega \in \Omega : \liminf_{t \to \infty} R_t^\pi = -\infty\}$$

and the probability of absolute ruin, called value function, under policy π is

$$V_\pi(x) = P(\mathcal{O}^\pi | R_0 = x) =: P_x(\mathcal{O}^\pi). \tag{2}$$

The objective is to find the optimal value function

$$V(x) = \inf_{\pi \in \Pi} V_\pi(x) \tag{3}$$

with boundary conditions

$$V(-\infty) = 1,\ V(\infty) = 0. \tag{4}$$

and the optimal policy π^* such that

$$V_{\pi^*}(x) = V(x). \tag{5}$$

The resulting optimization problem is an infinite time horizon control problem, which we will solve using dynamic programming techniques (see [8] and [9]).

Techniques and approaches for solving our problems are motivated by [2, 3] and [4].

We suppose that all random variables are defined in a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and that the two standard Brownian motions $B^{(0)}(\cdot)$ and $B^{(1)}(\cdot)$ are adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. A strategy π is said to be admissible if $\pi := \{(a(t), \beta(t))\}_{t\geq 0}$ is $\mathcal{F}_t$-progressively measurable, and satisfies $0 \leq a(t) \leq 1$ and the integrability condition that $\int_0^t \beta(s)ds < \infty$ almost surely for all $t \geq 0$. Denote the set of all admissible strategies by Π. We allow the company to short the risky asset and borrow money for investing long in the risky asset.

The article is structured as follows. In Section 2, after formulating the problem, we give corresponding Hamilton-Jacobi-Bellman equation and the verification theorem. The solution to HJB equation is given in Section 3. In Section 4, we give some conclusions.

2 The Hamilton-Jacobi-Bellman Equation and Verification Theorem

We will give the HJB equation and verification theorem in this section.

Firstly, we consider the case $rx \geq \lambda - \mu_0$. It is obvious that the surplus of the insurance company will never vanish in the case, because the interest obtained by investing all the surplus on the risk-free bond can cover the shortfall between the premiums received and the amount needed to pay to the reinsurance company which covers 100% of each claim. That is, if $\pi = (0, 0)$ then $V^\pi(x) = 0$, and ruin never occurs. So $\pi^* = (a^*, \beta^*) = (0, 0)$ and $V^{\pi^*}(x) = 0$ in this case. Set

$$x^* = \frac{\lambda - \mu_0}{r}, \tag{6}$$

and rewrite boundary conditions (4) as

$$V(-\infty) = 1,\ V(x^*) = 0. \tag{7}$$

For any C^2 function W, write

$$\mathcal{L}^{\pi}(W)(x) = \{[\mu_0 - (1-a)\lambda + rx + \beta(\mu_1 - r)]W'(x) + Q(a,\beta)W''(x)\},$$

where

$$Q(a,\beta) = \frac{1}{2}(a^2\sigma_0^2 + \beta^2\sigma_1^2 + 2a\beta\rho\sigma_0\sigma_1).$$

We start with the associated Hamilton-Jacobi-Bellman (HJB) equation for the optimal value function V on $(-\infty, x^*)$.

Theorem 2.1 *Assume that V defined by* (3) *is twice continuously differentiable on* $(-\infty, x^*)$. *Then V satisfies the following Hamilton-Jacobi-Bellman equation:*

$$\inf_{0\le a\le 1, \beta\in R} \mathcal{L}^{\pi}(V)(x) = 0,$$

or equivalently,

$$0 = \inf_{0\le a\le 1,\beta\in R}\{[\mu_0 - (1-a)\lambda + rx + \beta(\mu_1 - r)]V'(x) + Q(a,\beta)V''(x)\} \tag{8}$$

with boundary conditions (7).

The proof of Theorem 2.1 is standard (see Chapter IV in [8] or [5]).

Notice that, for Brownian motions $B_t^{(0)}$ and $B_t^{(1)}$ with correlation coefficient ρ, there exists another Brownian motion $B_t^{(2)}$, which is independent of $B_t^{(0)}$, such that

$$B_t^{(1)} = \rho B_t^{(0)} + \sqrt{1-\rho^2}B_t^{(2)}.$$

Hence (1) becomes

$$dR_t^{\pi} = [\mu_0 - (1-a)\lambda + rR_t^{\pi} + (\mu_1 - r)\beta]dt + (a\sigma_0 + \beta\rho\sigma_1)dB_t^{(0)} + \beta\sigma_1\sqrt{1-\rho^2}dB_t^{(2)}.$$

Thus, it is easy to prove that the Lemmas 2.1–2.4, and Lemma 3.1 in [3] are applicable in our risk system, so is the Theorem 3.1. The following verification theorem, which is parallel to Theorem 3.1 in [3], is essential in solving the associated stochastic control problem.

Theorem 2.2 *Let $W \in C^2$ be a decreasing solution on $(-\infty, x^*)$ to HJB equation* (8) *subject to the boundary conditions* (7). *Then the value function V given by* (3) *coincides with W on $(-\infty, x^*)$. Furthermore, let $\pi^* = (a^*(x), \beta^*(x))$ fulfill*

$$0 = \{[\mu_0 - (1-a^*)\lambda + rx + \beta^*(\mu_1 - r)]W'(x) + Q(a^*,\beta^*)W''(x)\}, \tag{9}$$

for all $-\infty < x < x^*$. Then the policy $\pi^*(\cdot)$ of the following feedback form $\pi^*(s) = (a^*(R_s^{\pi^*}), \beta^*(R_s^{\pi^*}))$, where $R_s^{\pi^*}$ is the solution to (1.1), *is the optimal policy. That is, $W(x) = V(x) = V_{\pi^*}(x)$.*

3 A Solution to the HJB Equation

Set

$$\mathcal{D} = \{x : -\infty < x < x^*\}. \tag{10}$$

In this section we seek a decreasing C^2 solution on $\mathcal{D}$ to HJB equation (8) with boundary condition (7) automatically extending this solution to $(-\infty, \infty)$ by setting $V(x) = 0$ for $x \geq x^*$ and the corresponding minimizing function $\pi^* = (a^*, \beta^*)$.

Differentiating (8) with respect to a and β, and letting the derivatives equal zero, we can get

$$a(x) = -\frac{\lambda V'(x) + \beta\rho\sigma_0\sigma_1 V''(x)}{\sigma_0^2 V''(x)}, \tag{11}$$

and

$$\beta(x) = -\frac{(\mu_1 - r)V'(x) + a\rho\sigma_0\sigma_1 V''(x)}{\sigma_1^2 V''(x)}. \tag{12}$$

Solve (11) and (12), and we can get

$$a(x) = -\frac{\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r)}{\sigma_0^2\sigma_1(1-\rho^2)}\frac{V'(x)}{V''(x)} =: -A\frac{V'(x)}{V''(x)}, \tag{13}$$

and

$$\beta(x) = \frac{\rho\lambda\sigma_1 - \sigma_0(\mu_1 - r)}{\sigma_0\sigma_1^2(1-\rho^2)}\frac{V'(x)}{V''(x)} =: B\frac{V'(x)}{V''(x)}, \tag{14}$$

where

$$A := \frac{\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r)}{\sigma_0^2\sigma_1(1-\rho^2)}, \quad B := \frac{\rho\lambda\sigma_1 - \sigma_0(\mu_1 - r)}{\sigma_0\sigma_1^2(1-\rho^2)}. \tag{15}$$

The expressions above will be used to find the minimizers of the HJB equation.

We seek the solution in two cases:

1) $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) > 0$;

2) $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) \leq 0$.

3.1 The Case $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) > 0$

In this case, we have $a(x) > 0$ when $V''(x) > 0$; and $a(x) < 0$ when $V''(x) < 0$. So define sets

$$\mathcal{D}_1 = \{-\infty < x < x^* : V''(x) > 0, 0 < a(x) < 1\},$$

$$\mathcal{D}_2 = \{-\infty < x < x^* : V''(x) > 0, a(x) \geq 1\},$$

$$\mathcal{D}_3 = \{-\infty < x < x^* : V''(x) < 0\},$$

which form a partition of $\mathcal{D}$. Notice that if $V(x)$ solves HJB equation (8), then the minimizers in the equation are $a^*(x) = a(x)$ on $\mathcal{D}_1$; and $a^*(x) = 1$ on $\mathcal{D}_2 \cup \mathcal{D}_3$. Thus we have the following lemma:

Lemma 3.1 *Suppose V is a decreasing and twice continuously differentiable function on $\mathcal{D}$, then the following conclusions are true.*

(i) *If V is a solution to*

$$(\mu_0 - \lambda + rx)\frac{V'(x)}{V''(x)} - C\left(\frac{V'(x)}{V''(x)}\right)^2 = 0, \tag{16}$$

where

$$C := \frac{1}{2\sigma_0^2\sigma_1^2(1-\rho^2)}[\lambda^2\sigma_1^2 + \sigma_0^2(\mu_1 - r)^2 - 2\rho\lambda\sigma_0\sigma_1(\mu_1 - r)] > 0. \tag{17}$$

Then V solves HJB equation (8) on $\mathcal{D}_1$ and the converse of this statement also holds.

(ii) *If V is a solution to*

$$\frac{1}{2}\sigma_0^2(1-\rho^2) + \left(\mu_0 + rx - \frac{\rho\sigma_0(\mu_1 - r)}{\sigma_1}\right)\frac{V'(x)}{V''(x)} - \frac{(\mu_1 - r)^2}{2\sigma_1^2}\left(\frac{V'(x)}{V''(x)}\right)^2 = 0, \tag{18}$$

then V solves HJB equation (8) on $\mathcal{D}_2 \cup \mathcal{D}_3$ and vice-versa.

Proof (i) Recall HJB equation (8) and

$$\mathcal{L}^{\pi}(V)(x) = \left\{[\mu_0 - (1-a)\lambda + rx + \beta(\mu_1 - r)]V'(x) + \frac{1}{2}Q(a,\beta)V''(x)\right\}, \tag{19}$$

which is a quadratic function of a and β. Since $A > 0$, $V'(x) < 0$ and $V''(x) > 0$, hence $a(x)$ defined in (13) is positive on $\mathcal{D}_1$, and $\pi(x) = (a(x), \beta(x))$ minimizes the right hand side of equation (19). And we can get $\mathcal{L}^{\pi(x)}(V)(x) = 0$ from (16). So V, which satisfies (16) solves HJB equation (8) on $\mathcal{D}_1$. Conversely, suppose $V(x)$ solves the HJB equation (8), then $\pi^*(x) = \pi(x) = (a(x), \beta(x))$ is the minimizer on $\mathcal{D}_1$. Hence $\mathcal{L}^{\pi(x)}(V)(x) = 0$, from which we get (16).

(ii) Firstly, we find that $\mathcal{L}^{\pi}(V)(x)$, as a quadratic function of a, gets its minimum value at $a^*(x) = 1$ on $\mathcal{D}_2$, since its coefficient of quadratic term is positive and the symmetry axis of its imagine $a(x) \geq 1$. In this case,

$$\beta^*(x) = -\frac{(\mu_1 - r)V'(x)}{\sigma_1^2 V''(x)} + \frac{\rho\sigma_0}{\sigma_1}.$$

Substituting $\pi^*(x) = (a^*(x), \beta^*(x))$ into Equation (19), and combining with (18), one can get that $V(x)$ solves the HJB equation (8).

Secondly, on $\mathcal{D}_3$, $a(x) < 0$ under the conditions of $V''(x) < 0$ and $A > 0$. Hence $\mathcal{L}^{\pi}(V)(x)$, as a quadratic function of a, achieves its minimum value at $a^*(x) = 1$ on $\mathcal{D}_3$, when $\beta^*(x) = -\frac{(\mu_1 - r)V'(x)}{\sigma_1^2 V''(x)} + \frac{\rho\sigma_0}{\sigma_1}$. By substituting them into Equation (19) and combining with (18) one can derive that $V(x)$ solves the HJB equation (8).

Conversely, if $V(x)$ is a solution of the HJB equation (8), we find that a minimizer of $\mathcal{L}^{\pi}(V)(x)$ is

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \left(1, -\frac{(\mu_1 - r)V'(x)}{\sigma_1^2 V''(x)} + \frac{\rho\sigma_0}{\sigma_1}\right)$$

on $\mathcal{D}_2 \cup \mathcal{D}_3$. So $\mathcal{L}^{\pi^*(x)}(V)(x) = 0$, which is (18). Now the proof is finished. ∎

We proceed with solving HJB equation (8) and derive explicit expressions for $V(x)$ in each of the cases of a associated with $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$.

Firstly, from Lemma 3.1 (i), for any $x \in \mathcal{D}_1$ we get

$$\frac{V''(x)}{V'(x)} = -\frac{C}{\lambda - \mu_0 - rx} =: f(x) < 0. \tag{20}$$

The corresponding $a(x)$ is given by

$$a(x) = \frac{A}{C}(\lambda - \mu_0 - rx) > 0, \tag{21}$$

and the condition $a(x) \in (0,1)$ is necessary from Lemma 3.1 (i), which implies

$$x > x_0 := \frac{\lambda - \mu_0}{r} - \frac{C}{rA}. \tag{22}$$

So $\mathcal{D}_1 = (x_0, x^*)$. Then we get the solution of (20) with boundary conditions $V(x^*) = 0$ is

$$V_1(x) = c_1(\lambda - \mu_0 - rx)^{\frac{C}{r}+1}, \quad x_0 < x < x^*, \tag{23}$$

which is a solution to HJB equation (8) on (x_0, x^*), where c_1 is a constant to be determined later. In this case, we can obtain

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \left(\frac{A(\lambda - \mu_0 - rx)}{C}, \frac{B(\lambda - \mu_0 - rx)}{C}\right). \tag{24}$$

Secondly, for $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) > 0$, we find an expression for the solution to (8) on $\mathcal{D}_2$, when

$$a^*(x) = 1, \quad \beta^*(x) = -\frac{(\mu_1 - r)V'(x)}{\sigma_1^2 V''(x)} - \frac{\rho\sigma_0}{\sigma_1}. \tag{25}$$

If $V(x)$ solves (18) on $\mathcal{D}_2$, then V satisfies

$$\frac{V'(x)}{V''(x)} = \frac{(\mu_0 + rx - \rho\sigma_0\theta) - \sqrt{\Delta}}{\theta^2} =: \frac{1}{g(x)} < 0, \tag{26}$$

where

$$\theta := \frac{\mu_1 - r}{\sigma_1} > 0, \quad \Delta := (\mu_0 + rx - \rho\sigma_0\theta)^2 + \sigma_0^2(1 - \rho^2)\theta^2.$$

Thus, $a(x) = -Ag(x) \geq 1$ is equivalent to $x \leq x_0$ (see appendix for details), which means $\mathcal{D}_2 = \{-\infty < x \leq x_0\}$. Hence the solution to (26), with boundary condition $V(-\infty) = 1$, is

$$V_2(x) = 1 - c_2 \int_{-\infty}^{x} \exp\left\{-\int_{u}^{x_0} g(t)dt\right\} du, \quad -\infty < x \leq x_0, \tag{27}$$

where c_2 is a constant to be determined later. In addition, the method for solving equation (26) is motivated in [5].

In this case, it can be shown that $a(x) \geq 1$ on $(-\infty, x_0]$. Thus, (27) solves (8) on $(-\infty, x_0]$, and (25) becomes

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \left(1, -\frac{\mu_1 - r}{\sigma_1^2 g(x)} - \frac{\rho\sigma_0}{\sigma_1}\right). \tag{28}$$

We learn from the above that $\mathcal{D}_3 = \emptyset$.

In order to determining the free constants c_1 and c_2, we apply a smooth fit at x_0 by setting

$$V_1(x_0) = V_2(x_0), \qquad V_1'(x_0) = V_2'(x_0).$$

Solving the above equations for c_1 and c_2 results in

$$c_1 = \frac{c_2}{(C+r)(\lambda - \mu_0 - rx_0)^{\frac{C}{r}}}, \tag{29}$$

$$c_2 = \left(\int_{-\infty}^{x_0} \exp\left\{-\int_u^{x_0} g(t)dt\right\} du + \frac{\lambda - \mu_0 - rx_0}{C+r}\right)^{-1}. \tag{30}$$

Summarizing the above results, we obtain the following theorem.

Theorem 3.2 *If $\lambda\sigma_1 > \rho\sigma_0(\mu_1 - r)$, then the minimum absolute ruin function $V(x)$ is a decreasing C^2 function on $(-\infty, x^*)$ given by*

$$V(x) = \begin{cases} 1 - c_2 \displaystyle\int_{-\infty}^{x} \exp\left\{-\int_u^{x_0} g(t)dt\right\} du, & -\infty < x \leq x_0, \\ c_1(\lambda - \mu_0 - rx)^{\frac{C}{r}+1}, & x_0 < x < x^*, \\ 0, & x \geq x^*. \end{cases}$$

The corresponding minimizing function $\pi^*(x)$ is given by

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \begin{cases} \left(1, -\dfrac{\mu_1 - r}{\sigma_1^2 g(x)} - \dfrac{\rho\sigma_0}{\sigma_1}\right), & -\infty < x \leq x_0, \\ \left(\dfrac{A(\lambda - \mu_0 - rx)}{C}, \dfrac{B(\lambda - \mu_0 - rx)}{C}\right), & x_0 < x < x^*, \\ (0, 0), & x \geq x^*, \end{cases}$$

where x_0 is defined by (22), *c_1 and c_2 are defined by* (29) *and* (30), *respectively, A, B are defined by* (15) *and C by* (17).

Remark 3.3 It is not difficult to check that V is a C^2 function on $(-\infty, x^*)$. Further, from Lemma 3.1, V solves the HJB equation (8), and the results can be obtained by the verification theorem.

Remark 3.4 As we can see from the above results the minimum absolute ruin function $V(x)$ is convex, but not S-shaped here.

3.2 The Case $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) \le 0$

In this case, we have $a(x) \le 0$ when $V''(x) > 0$; and $a(x) \ge 0$ when $V''(x) < 0$. So we define sets

$$\begin{aligned}
\mathcal{D}_{V4} &= \{-\infty < x < x^* : V''(x) > 0\}, \\
\mathcal{D}_{V5} &= \{-\infty < x < x^* : V''(x) < 0, a(x) \ge 1\}, \\
\mathcal{D}_{V6} &= \{-\infty < x < x^* : V''(x) < 0, 1/2 \le a(x) < 1\}, \\
\mathcal{D}_{V7} &= \{-\infty < x < x^* : V''(x) < 0, 0 \le a(x) < 1/2\},
\end{aligned}$$

which form a partition of the interval $(-\infty, x^*)$. Notice that if $V(x)$ solves HJB equation (8), then the minimizers in the equation are $a^*(x) = 0$ on $\mathcal{D}_{V4} \cup \mathcal{D}_{V5} \cup \mathcal{D}_{V6}$; and $a^*(x) = 1$ on $\mathcal{D}_{V7}$. Thus we have the following lemma:

Lemma 3.5 *Suppose $V(x) \in C^2$ is a decreasing function on $\mathcal{D}$.*

(i) *If $V(x) \in C^2$ is a solution to*

$$(\mu_0 - \lambda + rx)\frac{V'(x)}{V''(x)} - \frac{(\mu_1 - r)^2}{2\sigma_1^2}\left(\frac{V'(x)}{V''(x)}\right)^2 = 0, \tag{31}$$

then V solves HJB equation (8) *on $\mathcal{D}_{V4} \cup \mathcal{D}_{V5} \cup \mathcal{D}_{V6}$, and vice-versa.*

(ii) *If $V(x) \in C^2$ is a solution to*

$$\frac{1}{2}\sigma_0^2(1 - \rho^2) + \left(\mu_0 + rx - \frac{\rho\sigma_0(\mu_1 - r)}{\sigma_1}\right)\frac{V'(x)}{V''(x)} - \frac{(\mu_1 - r)^2}{2\sigma_1^2}\left(\frac{V'(x)}{V''(x)}\right)^2 = 0, \tag{32}$$

then V solves HJB equation (8) *on $\mathcal{D}_{V7}$. Conversely, if $V(x) \in C^2$ is a decreasing solution to the HJB equation* (8)*, then $V(x)$ solves* (32) *on $\mathcal{D}_{V7}$.*

The proof of Lemma 3.2 is similar to that of Lemma 3.1.

Next, we will identify the regions of $\mathcal{D}_{V4}$, $\mathcal{D}_{V5}$, $\mathcal{D}_{V6}$ and $\mathcal{D}_{V7}$, and derive explicit expressions for $V(x)$ in each of the cases of a associated with $\mathcal{D}_{Vi}$, $i = 4, 5, 6, 7$.

Suppose $V(x)$ solves HJB equation (8) on $\mathcal{D}_{V4}$, when

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \left(0, -\frac{\mu_1 - r}{\sigma_1^2}\frac{V'(x)}{V''(x)}\right). \tag{33}$$

We get by (31) that

$$\frac{V''(x)}{V'(x)} = \frac{(\mu_1 - r)^2}{2\sigma_1^2}\frac{1}{rx - (\lambda - \mu_0)} =: h(x) < 0, \tag{34}$$

for every $x \in \mathcal{D}$. In this case, $a(x) = -A\frac{V'(x)}{V''(x)} \le 0$, since $A \le 0$ under the condition $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) \le 0$. Hence $\mathcal{D}_{V4} = \mathcal{D}$. Then $\mathcal{D}_{V5} = \mathcal{D}_{V6} = \mathcal{D}_{V7} = \emptyset$.

We proceed to solve (34) with boundary conditions (7) to get

$$V_3(x) = c_3 \int_x^{x^*} \exp\left\{-\int_u^{x^*} h(t)dt\right\} du, \ -\infty < x \le x^*, \tag{35}$$

where

$$c_3 = \left(\int_{-\infty}^{x^*} \exp\left\{-\int_u^{x^*} h(t)dt\right\} du\right)^{-1}. \tag{36}$$

Then (33) becomes

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \left(0, 2\frac{\lambda - \mu_0 - rx}{\mu_1 - r}\right).$$

Summarizing the above results, we give the following theorem.

Theorem 3.6 *If $\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r) \le 0$, then the minimum absolute ruin function $V(x)$ is a decreasing C^2 function on $(-\infty, x^*)$ given by*

$$V(x) = \begin{cases} c_3 \int_x^{x^*} \exp\left\{-\int_u^{x^*} h(t)dt\right\} du, & -\infty < x < x^*, \\ 0, & x \ge x^*. \end{cases}$$

The corresponding minimizing function $\pi^*(x)$ is given by

$$\pi^*(x) = (a^*(x), \beta^*(x)) = \begin{cases} \left(0, 2\frac{\lambda - \mu_0 - rx}{\mu_1 - r}\right), & -\infty < x < x^*, \\ (0, 0), & x \ge x^*, \end{cases}$$

where c_3 is defined by (36).

Remark 3.7 We can see from Theorem 3.2 that under another conditions of given parameters, the minimum absolute ruin function $V(x)$ is still convex, but not S-shaped.

4 Conclusion

In conclusion, we find that the minimum absolute ruin function is convex, not a S-shaped just like that in [3]. Why? For this phenomenon, we can explain it from behavior finance point of view (see [6] and [7]). The (Cumulated) Prospect Theory, developed by Kahneman and Tversky, demonstrates that people make decisions based on gains and losses (relative to some reference point) in wealth. The investors are risk averse when the final wealth is above the reference point; in contrast, they are risk seeking when the final wealth is below the reference point. The insurers are risky seeking when the surplus is below their reference points, and they will short risk asset in large amounts, or borrow from the bank in large amounts for investing long in the risky asset. After doing so, the minimum absolute ruin function will not remain at a high level of risk for a long time, that is, minimum absolute ruin function is not concave, but convex when the surplus is below a critical level. Thus, it is the restrictions on investment that results in the kink of minimum absolute ruin function in [3].

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Appendix

Recall x_0 defined by (22)

$$x_0 := \frac{\lambda - \mu_0}{r} - \frac{C}{rA},$$

where

$$A := \frac{\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r)}{\sigma_0^2\sigma_1(1-\rho^2)} > 0,$$

and

$$C := \frac{1}{2\sigma_0^2\sigma_1^2(1-\rho^2)}[\lambda^2\sigma_1^2 + \sigma_0^2(\mu_1 - r)^2 - 2\rho\lambda\sigma_0\sigma_1(\mu_1 - r)] > 0.$$

Then

$$\mu_0 + rx_0 = \frac{\lambda A - C}{A} = \frac{\lambda^2\sigma_1^2 - \sigma_0^2(\mu_1 - r)^2}{2\sigma_1(\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r))}. \tag{A.1}$$

And rewrite (30),

$$\frac{V'(x)}{V''(x)} = \frac{(\mu_0 + rx - \rho\sigma_0(\mu_1 - r)/\sigma_1) - \sqrt{\Delta}}{\frac{(\mu_1 - r)^2}{\sigma_1^2}} < 0,$$

where

$$\Delta = \left(\mu_0 + rx - \frac{\rho\sigma_0(\mu_1 - r)}{\sigma_1}\right)^2 + \sigma_0^2(1-\rho^2)\frac{(\mu_1 - r)^2}{\sigma_1^2}.$$

Thus,

$$a(x) = -A\frac{V'(x)}{V''(x)} \geq 1$$

is equivalent to

$$\frac{\mu_0 + rx - \frac{\rho\sigma_0(\mu_1 - r)}{\sigma_1} - \sqrt{\Delta}}{\frac{(\mu_0 - r)^2}{\sigma_1^2}} \leq -\frac{1}{A}, \tag{A.2}$$

or

$$\mu_0 + rx \leq \frac{\lambda^2\sigma_1^2 - \sigma_0^2(\mu_1 - r)^2}{2\sigma_1(\lambda\sigma_1 - \rho\sigma_0(\mu_1 - r))} = \mu_0 + rx_0.$$

Now, we see that

$$a(x) = -A\frac{V'(x)}{V''(x)} \geq 1$$

is equivalent to $x \leq x_0$.