J Syst Sci Complex (2014) 27: 327–337

A QUADRATIC OBJECTIVE PENALTY FUNCTION FOR BILEVEL PROGRAMMING*

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DOI: 10.1007/s11424-014-2128-7
Received: 25 June 2012 / Revised: 21 January 2013

Abstract The bilevel programming is applied to solve hierarchical intelligence control problems in such fields as industry, agriculture, transportation, military, and so on. This paper presents a quadratic objective penalty function with two penalty parameters for inequality constrained bilevel programming. Under some conditions, the optimal solution to the bilevel programming defined by the quadratic objective penalty function is proved to be an optimal solution to the original bilevel programming. Moreover, based on the quadratic objective penalty function, an algorithm is developed to find an optimal solution to the original bilevel programming, and its convergence proved under some conditions. Furthermore, under the assumption of convexity at lower level problems, a quadratic objective penalty function without lower level problems is defined and is proved equal to the original bilevel programming.

Keywords Algorithm, bilevel programming, penalty function, quadratic objective.

1 Introduction

In this paper, we consider the following bilevel programming:

$$
\begin{aligned}
\text{(BP)}\quad \min_{x,y}\quad & f_1(x,y)\\
\text{s.t.}\quad & g_i(x,y)\le 0,\ i\in I=\{1,2,\cdots,p\},\\
& y\ \text{ solves the following problem:}\\
\min_{y}\quad & f_2(x,y)\\
\text{s.t.}\quad & h_j(x,y)\le 0,\ j\in J=\{1,2,\cdots,q\},
\end{aligned}
$$

where $f_1,f_2,g_i,h_j: R^n\times R^m\to R,\ i\in I, j\in J$. Its feasible set is denoted by $X=\{(x,y)\in R^n\times R^m \mid g_i(x,y)\le 0,\ i\in I\}$ and $Y=\{(x,y)\in R^n\times R^m \mid h_j(x,y)\le 0,\ j\in J\}$.

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*This research was supported by the National Natural Science Foundation of China under Grant Nos. 11271329 and 10971193.
◇ *This paper was recommended for publication by Editor DAI Yuhong.*

In 1952, the bilevel programming came from the Stackelberg game[1], which is applied to solve hierarchical intelligence control problems. Because there are many such problems in such fields as industry, agriculture, transportation, military, and so on[2–7], it is very important to study the theory of the bilevel programming. Dempe and Colson, et al. reviewed of about 500 papers[8, 9] as to theories and algorithms of bilevel programming. A bilevel programming may be transformed into mathematical program with equilibrium constraints. Luo, Pang, and Ralph[10] gave a system of theories for mathematical programs with equilibrium constraints. In 2002, Dempe published Foundations of Bilevel Programming covering linear bilevel problems, optimality conditions, solution algorithm, and discrete bilevel problems. In his book, Dempe discussed a descent algorithm, a bundle algorithm, penalty methods, a trust region method, and smoothing methods under the assumption of convexity at the lower level problem[11].

It is well-known that it is difficult to solve a nonlinear bilevel programming, since a global solution can hardly be obtained. In recent years, penalty methods become an efficient tool in solving mathematical programming. Penalty function method provides an important approach to solving constrained optimization problems[12]. Its main idea is to transform the constrained optimization problems into a sequence of unconstrained optimization problems by enlarging penalty parameters. The unconstrained optimization problem is defined by a penalty parameter with constrained functions, which is then added to the objective function. Penalty methods were presented by researchers to study solution algorithms or optimal conditions for bilevel programming. For example, Marcotte and Zhu[13] studied exact and inexact penalty methods for the generalized bilevel programming problem. Ye, Zhu, and Zhu[14] gave exact penalization and necessary optimality conditions for generalized bilevel programming problems. Stefan and Michael[15] discussed exact penalization of mathematical programs with equilibrium constraints. Liu, Han, and Zhang[16] studied exact penalty functions for convex bilevel programming problems. Yang, et al. studied lower order penalty methods[17], partial augmented Lagrangian method[18], and a sequential smooth penalization approach[19] to mathematical programs with complementarity constraints. Lü, et al.[20] presented a penalty function method based on Kuhn-Tucker condition to solve linear bilevel programming. Calvete and Galé[21] gave a penalty approach to optimality, obtained an algorithm for bilevel multiplicative problems through cutting plan. Ankhili and Mansouri[22] introduced an exact penalty function for bilevel programs with linear vector optimization at lower level. In above papers, i.e., [13–22], some researchers use the convexity assumption or linear assumption for the lower level problem, while others use penalty methods for the lower level problem which is replaced either by mathematical programs with equilibrium constraints or mathematical programs with complementarity constraints. All in all, the penalty function methods can ease the difficulties in solving bilevel programming.

The objective penalty function with an objective penalty parameter was discussed in [23] with its numerical results showing that the objective penalty function algorithm has a better convergence than those of the constrained penalty function algorithm. Hence, we studied an objective penalty function of bilevel programming[24]. In this paper, based on the idea of objective penalty function, we present another objective penalty function which differs from that in [24]. The second item of objective penalty function in [24] does not include any parameter

N and M.

2 Objective Penalty Function for (BP)

In this section, an objective penalty optimization problem of (BP) is defined as

$$F_1(x,y;M) = Q(f_1(x,y)-M) + M^2\sum_{i\in I} Q(g_i(x,y))$$

and

$$F_2(x,y;N) = Q(f_2(x,y)-N) + N^2\sum_{j\in J} Q(h_j(x,y)),$$

where the objective parameter $M, N \in R$, and $Q(t) = \max\{t,0\}^2$. Consider the following bilevel programming problem:

$$\begin{aligned} \mathrm{BP}(M,N) \quad \min_{x,y} \quad & F_1(x,y;M) \\ \text{s.t.} \quad & (x,y)\in R^n\times R^m, \\ & y \ \text{ solves the following problem:} \\ \min_{y} \quad & F_2(x,y;N) \\ \text{s.t.} \quad & (x,y)\in R^n\times R^m. \end{aligned}$$

Now, a theorem is proved as follows.

Theorem 2.1 *Suppose that M, N are two constants and (x_M^*, y_N^*) is an optimal solution to* BP(M,N) *for $M < f_1(x_M^*, y_N^*)$ and $N < f_2(x_M^*, y_N^*)\}$. If $(x_M^*, y_N^*) \in X\cap Y$, then (x_M^*, y_N^*) is an optimal solution to* (BP).

Proof First, we prove that y_N^* is an optimal solution as follows.

$$\begin{aligned} \min \quad & f_2(x_M^*, y) \\ \text{s.t.} \quad & h_j(x_M^*, y)\le 0,\ j\in J=\{1,2,\cdots,q\}. \end{aligned}$$

Under the given conditions, for any $(x_M^*, y)\in Y$, we have

$$0 < Q(f_2(x_M^*, y_N^*)-N) = F_2(x_M^*, y_N^*; N) \le F_2(x_M^*, y; N) = Q(f_2(x_M^*, y)-N).$$

Hence, $f_2(x_M^*, y_N^*) - N \le f_2(x_M^*, y) - N$, that is, $f_2(x_M^*, y_N^*) \le f_2(x_M^*, y)$ and (x_M^*, y_N^*) is a feasible solution to (BP).

Then, let (x,y) be a feasible solution to (BP). We have

$$0 < Q(f_1(x_M^*, y_N^*)-M) = F_1(x_M^*, y_N^*; M) \le F_1(x,y;M) = Q(f_1(x,y)-M).$$

Hence, $f_1(x_M^*, y_N^*) - M \le f_1(x,y) - M$, that is, $f_1(x_M^*, y_N^*) \le f_1(x,y)$ and (x_M^*, y_N^*) is an optimal solution to (BP).

Based on Theorem 2.1, an algorithm differing from the penalty algorithm in [11] is proposed to compute an optimal solution to (BP), which solves a sequential problem BP(M,N) and is called as quadratic objective penalty function algorithm (QOPFA for short).

QOPFA Algorithm

Step 1 Choose (x^1, y^1), $M_1, N_1 < 0, a > 1$, and $k = 1$.

Step 2 Solve the following problem:

$$\begin{array}{lll} \mathrm{BP}(M_k, N_k) & \min\limits_{(x,y)} & F_1(x, y; M_k) \\ & \text{s.t.} & (x, y) \in R^n \times R^m, \\ & & y \ \text{ solves the following problem:} \\ & \min\limits_{y} & F_2(x, y; N_k) \\ & \text{s.t.} & (x, y) \in R^n \times R^m. \end{array}$$

Let (x^{k+1}, y^{k+1}) be an optimal solution to $\mathrm{BP}(M_k, N_k)$.

Step 3 If $(x^{k+1}, y^{k+1}) \in X \cap Y$ and $M_k < f_1(x^{k+1}, y^{k+1}), N_k < f_2(x^{k+1}, y^{k+1})$, stop, then (x^{k+1}, y^{k+1}) is an optimal solution to (BP). Otherwise, let $M_{k+1} = aM_k$, $N_{k+1} = aN_k$, $k := k + 1$ and go to Step 2.

Theorem 2.2 *Suppose that $f_1, f_2, g_i (i \in I)$ and $h_j (j \in J)$ are continuous on $R^n \times R^m$, and $\lim_{(x,y)\to\infty} f_1(x, y) = +\infty$. Let $\{(x^k, y^k)\}$ be the sequence generated by the* QOPFA *algorithm.*

(i) *If $\{(x^k, y^k)\}(k = 1, 2, \cdots, \overline{k})$ is a finite sequence (i.e., the* QOPFA *algorithm stops at the $\overline{k}$-th iteration), then $x^{\overline{k}}$ is an optimal solution to* (BP)*.*

(ii) *If $\{(x^k, y^k)\}$ is an infinite sequence and there is some $k' > 1$ such that $M_k < f_1(x^{k+1}, y^{k+1})$, $N_k < f_2(x^{k+1}, y^{k+1})$ for all $k > k'$, then $\{(x^k, y^k)\}$ is bounded and any limit point of it is an optimal solution to* (BP)*. Otherwise, $f_1(x^k, y^k) \to -\infty$ or $f_2(x^k, y^k) \to -\infty$ as $k \to +\infty$.*

Proof (i) If the QOPFA algorithm terminates at the $\overline{k}$-th iteration with the first condition in Step 3 occuring, $(x^{\overline{k}}, y^{\overline{k}})$ is an optimal solution to (BP) by Theorem 1.

(ii) Suppose that $\{(x^k, y^k)\}$ is an infinite sequence and there is some $k' > 1$ such that $M_k < f_1(x^{k+1}, y^{k+1})$ and $N_k < f_2(x^{k+1}, y^{k+1})$ for all $k > k'$. Let (x', y') be a feasible solution to (BP). The bounded sequence $\{(x^k, y^k)\}$ is checked as follows. Since (x^{k+1}, y^{k+1}) is an optimal solution to $\mathrm{BP}(M_k, N_k)$,

$$Q(f_1(x^{k+1}, y^{k+1}) - M_k) \le F_1(x^{k+1}, y^{k+1}; M_k) \le Q(f_1(x', y') - M_k), \quad k = 1, 2, \cdots.$$

So,

$$(f_1(x^{k+1}, x^{k+1}) - M_k)^2 \le (f_1(x', y') - M_k)^2, \quad k = k' + 1, k' + 2, \cdots.$$

We have $f_1(x^{k+1}, y^{k+1}) \le f_1(x', y'), k = k' + 1, k' + 2, \cdots$. Hence, the sequence $\{x^{k+1}, y^{k+1}\}$ is bounded, since $\lim_{(x,y)\to\infty} f_1(x, y) = +\infty$. Without loss of generality, we assume $(x^k, y^k) \to (\overline{x}, \overline{y})$. And, for any given feasible solution (x, y) to (BP), we have

$$0 < (f_1(x^{k+1}, y^{k+1}) - M_k)^2 + M_k^2 \sum_{i\in I} Q(g_i(x^{k+1}, y^{k+1})) \le (f_1(x, y) - M_k)^2, \quad \forall k > k',$$

that is,

$$\sum_{i\in I} Q(g_i(x^{k+1}, y^{k+1})) \le \frac{1}{M_k^2}(f_1(x, y) - f_1(x^{k+1}, y^{k+1}))(f_1(x^{k+1}, y^{k+1}) + f_1(x, y) - 2M_k), \forall k > k'.$$

It is clear that $M_k \to -\infty$ as $k \to +\infty$. By letting $k \to +\infty$ in the above equation, we obtain $\sum_{i\in I} Q(g_i(\overline{x},\overline{y})) = 0$. Hence, $(\overline{x},\overline{y}) \in X$ and $f_1(\overline{x},\overline{y}) \leq f_1(x,y)$. On the other hand, for any $(\overline{x},y) \in Y$, we have

$$0 < (f_2(x^{k+1},y^{k+1}) - N_k)^2 + N_k^2 \sum_{j\in J} Q(h_j(x^{k+1},y^{k+1})) \leq (f_2(\overline{x},y) - N_k)^2, \quad \forall k > k',$$

that is,

$$\sum_{j\in J} Q(h_j(x^{k+1},y^{k+1})) \leq \frac{1}{N_k^2}(f_2(\overline{x},y) - f_2(x^{k+1},y^{k+1}))(f_2(x^{k+1},y^{k+1}) + f_2(\overline{x},y) - 2N_k), \forall k > k'.$$

It is clear that $N_k \to -\infty$ as $k \to +\infty$. By letting $k \to +\infty$ in the above equation, we obtain $\sum_{j\in J} Q(h_j(\overline{x},\overline{y})) = 0$. Hence, $(\overline{x},\overline{y}) \in Y$ and $f_2(\overline{x},\overline{y}) \leq f_2(\overline{x},y)$. Therefore, $(\overline{x},\overline{y})$ is an optimal solution to (BP).

Example 1 Consider the problem:

$$\begin{array}{lll} \text{(MP1)} & \min\limits_{x,y} & f_1(x,y) = (x-2)^2 + (y-2)^2 \\ & \text{s.t.} & -x \leq 0, -y \leq 0, \\ & & y \ \text{ solves the following problem:} \\ & \min\limits_{y} & f_2(x,y) = x^2 + y^2 \\ & \text{s.t.} & -x \leq 0, -y \leq 0. \end{array}$$

It is clear that $(2,0)$ is an optimal solution to (MP1). Let an objective penalty function problem of (MP1) as follows.

$$\begin{array}{lll} \text{MP1}(M,N) & \min\limits_{x,y} & F_1(x,y;M) = \max\{(x-2)^2 + (y-2)^2 - M, 0\}^2 \\ & & \qquad +M^2(\{\max\{-x,0\}^2 + \max\{-y,0\}^2) \\ & \text{s.t.} & (x,y) \in R^1 \times R^1, \\ & & y \ \text{ solves the following problem:} \\ & \min\limits_{y} & F_2(x,y;N) = \max\{x^2 + y^2 - N, 0\}^2 \\ & & \qquad +N^2(\{\max\{-x,0\}^2 + \max\{-y,0\}^2) \\ & \text{s.t.} & (x,y) \in R^n \times R^m. \end{array}$$

So when $M, N < 0$, $(2,0)$ is obviously an optimal solution to MP1(M,N).

3 Objective Penalty Function with the Convexity to the Lower Level Problem

In the section, suppose that $f_2(x,y)$ and $h_j(x,y)(j \in J)$ with respect to $y \in R^m$ are convex, f_2 and $h_j(j \in J)$ are continuous differentiable on $R^n \times R^m$. Because $Q(t) = \max\{t,0\}^2$ is convex and monotone increasing on R^1, $F_2(x,y;N)$ with respect to $y \in R^m$ is convex and differentiable[25].

Consider the following nonlinear optimization problem:

$$\begin{aligned} \mathrm{BP1}(M,N) \quad \min_{x,y} \quad & F_1(x,y;M) \\ \text{s.t.} \quad & \nabla_y F_2(x,y;N) = 0, \end{aligned}$$

where

$$\nabla_y F_2(x,y;N) = \max\{f_2(x,y) - N, 0\}\nabla_y f_2(x,y) + N^2 \sum_{j\in J} \max\{h_j(x,y), 0\}\nabla_y h_j(x,y).$$

Theorem 3.1 *Suppose that M and N are given constants and $f_2(x,y)$ and $h_j(x,y)(j \in J)$ with respect to $y \in R^m$ are convex. Then, (x_M^*, y_N^*) is an optimal solution to* BP1(M,N) *if and only if (x_M^*, y_N^*) is an optimal solution to* BP(M,N).

Proof Suppose that (x_M^*, y_N^*) is an optimal solution to BP1(M,N). According to the assumption, we have that $F_2(x,y;N)$ with respect to $y \in R^m$ is convex. From

$$\nabla_y F_2(x_M^*, y_N^*; N) = 0,$$

we know that y_N^* is an optimal solution to $\min_{y\in R^m} F_2(x_M^*, y; N)$. Hence, (x_M^*, y_N^*) is a feasible solution to BP(M,N). Let (x,y) be a feasible solution to BP(M,N). It is clear that $\nabla_y F_2(x,y;N) = 0$. Then, (x,y) is a feasible solution to BP1(M,N). So, $F_1(x_M^*, y_N^*; M) \le F_1(x,y;M)$, which means that (x_M^*, y_N^*) is an optimal solution to BP(M,N).

Now, suppose that (x_M^*, y_N^*) is an optimal solution to BP(M,N). According to the assumption, we have that $F_2(x,y;N)$ with respect to $y \in R^m$ is convex. From

$$\nabla_y F_2(x_M^*, y_N^*; N) = 0,$$

(x_M^*, y_N^*) is a feasible solution to BP1(M,N). Let (x,y) be a feasible solution to BP1(M,N). We know that y is an optimal solution to $\min_{y\in R^m} F_2(x,y;N)$. Then, (x,y) is a feasible solution to BP(M,N). From $F_1(x_M^*, y_N^*; M) \le F_1(x,y;M)$, (x_M^*, y_N^*) is an optimal solution to BP1(M,N).

Define penalty function

$$\begin{aligned} \mathrm{BP2}(M,N) \quad \min_{x,y} \quad & F(x,y;M,N) = F_1(x,y;M) + M^2 \sum_{j\in J} \max\{h_j(x,y), 0\}^2 \\ & \qquad + M^2 \|\nabla_y F_2(x,y;N)\|^2 \\ \text{s.t.} \quad & (x,y) \in R^n \times R^m. \end{aligned}$$

From Theorems 2.1 and 3.1, it is easily known that the following results holds.

Theorem 3.2 *Suppose that M and N are constants. Suppose that $f_2(x,y)$ and $h_j(x,y)(j \in J)$ with respect to $y \in R^m$ are convex, and (x_M^*, y_N^*) is an optimal solution to* BP2(M,N) *with $M < f_1(x_M^*, y_N^*)$ and $N < f_2(x_M^*, y_N^*)$. If $(x_M^*, y_N^*) \in X \cap Y$ and $\nabla_y F_2(x_M^*, y_N^*; N) = 0$, then (x_M^*, y_N^*) is an optimal solution to* BP1(M,N), BP(M,N), *and* (BP), *respectively.*

Example 2 Consider the problem:

$$\begin{aligned}
(\mathrm{MP2})\quad & \min_{x,y} && f_1(x,y) = (x-2)^2 + (y-2)^2 \\
& \text{s.t.} && -x \le 0, -y \le 0, \\
& && y \text{ solves the following problem:} \\
& \min_{y} && f_2(x,y) = x^2 + y^2 \\
& \text{s.t.} && -x \le 0, -y \le 0.
\end{aligned}$$

Let objective penalty function problem of (MP2) as follows.

$$\begin{aligned}
\mathrm{MP2}-1(M,N)\quad \min_{x,y}\quad & F_1(x,y;M) = \max\{(x-2)^2 + (y-2)^2 - M, 0\}^2 \\
& \qquad + M^2(\{\max\{-x,0\}^2 + \max\{-y,0\}^2) \\
\text{s.t.}\quad & (x,y) \in R^1 \times R^1, \\
& \nabla_y F_2(x,y;N) = 4y \max\{x^2+y^2-N,0\} - 2N^2 \max\{-y,0\} = 0.
\end{aligned}$$

When $M, N < 0$, $(2,0)$ is an optimal solution to $\mathrm{MP2}-1(M,N)$. Let objective penalty function problem of (MP2) as follows.

$$\begin{aligned}
\mathrm{MP2}-2(M,N)\quad \min_{x,y}\quad & F(x,y;M,N) = \max\{(x-2)^2 + (y-2)^2 - M, 0\}^2 \\
& \qquad + M^2(\{\max\{-x,0\}^2 + \max\{-y,0\}^2) \\
& \qquad + M^2(4y \max\{x^2+y^2-N,0\} - 2N^2 \max\{-y,0\})^2 \\
\text{s.t.}\quad & (x,y) \in R^1 \times R^1.
\end{aligned}$$

When $M, N < 0$, $(2,0)$ is an optimal solution to $\mathrm{MP2}-2(M,N)$.

Let

$$Z(N) = \{(x,y) \in X \cap Y | \nabla_y F_2(x,y;M) = 0\}$$

and

$$E(x,y;N) = \sum_{i \in I} Q(g_i(x,y)) + \sum_{j \in J} Q(h_j(x,y)) + \|\nabla_y F_2(x,y;N)\|^2.$$

We have $F(x,y;M,N) = Q(f_1(x,y) - M) + M^2 E(x,y;N)$.

Lemma 3.3 *Suppose that $f_2(x,y)$ and $h_j(x,y) (j \in J)$ with respect to $y \in R^m$ are convex for every $x \in R^n$. For any given $N < f_2(x,y)$, $E(x,y;N) = 0$ or $(x,y) \in Z(N)$, if and only if (x,y) is a feasible solution to* (BP).

Proof It is obvious that $E(x,y;N) = 0$ is equal to $(x,y) \in Z(N)$. If for any given $N < f_2(x,y)$, $(x,y) \in Z(N)$, then $\nabla_y F_2(x,y;N) = 0$. According to the assumption, it is easy for us to know that $F_2(x,y;N)$ with respect to $y \in R^m$ is convex for every $x \in R^n$. Hence, y is an optimal solution to $\min_{y \in R^m} F_2(x,y;N)$. From $(x,y) \in Y$, we have

$$0 < Q(f_2(x,y) - N) = F_2(x,y;N) \le F_2(x,z;N) = Q(f_2(x,z) - N), \quad \forall (x,z) \in Y.$$

So,

$$f_2(x, y) \leq f_2(x, z), \quad \forall (x, z) \in Y.$$

Therefore, y is an optimal solution to $\min_y f_2(x, y)$ s.t. $(x, y) \in Y$, and (x, y) is a feasible solution to (BP).

We prove when (x, y) is a feasible solution to (BP), $E(x, y; N) = 0$. Since y is an optimal solution to $\min_{y \in R^m} F_2(x, y; N)$, we have $\nabla_x F_2(x, y; N) = 0$ and $(x, y) \in X \cap Y$. So, $E(x, y; N) = 0$.

From Lemma 3.1 we have the following result.

Theorem 3.4 *Suppose that $f_2(x, y)$ and $h_j(x, y)(j \in J)$ with respect to $y \in R^m$ are convex for every $x \in R^n$. Let (x^*, y^*) be an optimal solution to $\min_{(x,y) \in R^n \times R^m} E(x, y; N)$ with $N < f_2(x^*, y^*)$. If $E(x^*, y^*; N) > 0$, then any $(x, y) \in R^n \times R^m$ is not a feasible solution to* (BP).

Based on Theorems 3.2 and 3.4, we propose an algorithm to solves the sequential problem BP2(M, N), which is called as quadratic convex objective function penalty algorithm (QCOPFA for short).

QCOPFA Algorithm

Step 0 Let $M_1 < 0, a > 1$, $k = 1$ and $N < \min_{(x,y) \in R^n \times R^m} f_2(x, y)$.

Step 1 Solve $\min_{(x,y) \in R^n \times R^m} E(x, y; N)$, then get an optimal solution (x^1, y^1). If $E(x^1, y^1; N) > 0$, stop and there is no feasible solution to (BP). Otherwise, go to Step 2.

Step 2 Solve the following problem:

$$\begin{aligned} \text{BP2}(M_k, N) \quad \min_{x,y} \quad & F(x, y; M_k, N) = Q(f_1(x, y) - M_k) + M_k^2 E(x, y; N) \\ \text{s.t.} \quad & (x, y) \in R^n \times R^m. \end{aligned}$$

Let (x^{k+1}, y^{k+1}) be an optimal solution to BP2(M_k, N).

Step 3 If $(x^{k+1}, y^{k+1}) \in Z(N)$ and $M_k < f_1(x^{k+1}, y^{k+1})$, stop and (x^{k+1}, y^{k+1}) is an optimal solution to (BP). Otherwise, let $M_{k+1} = aM_k$, $k := k + 1$ and go to Step 2.

In the QCOPFA algorithm, it is assumed that we can always get $N < \min_{(x,y) \in R^n \times R^m} f_2(x, y)$.

Theorem 3.5 *Suppose that $f_1, f_2, g_i(i \in I)$, and $h_j(j \in J)$ are continuous on $R^n \times R^m$, and $\lim_{(x,y) \to \infty} f_1(x, y) = +\infty$. Let $\{(x^k, y^k)\}$ be the sequence generated by the* QCOPFA *algorithm.*

(i) *If $\{(x^k, y^k)\}(k = 1, 2, \cdots, \overline{k})$ is a finite sequence (i.e., the* QCOPFA *algorithm stops at the $\overline{k}$-th iteration), then $x^{\overline{k}}$ is an optimal solution to* (BP).

(ii) *If $\{(x^k, y^k)\}$ is an infinite sequence and there is some $k' > 1$ such that $M_k < f_1(x^{k+1}, y^{k+1})$ for all $k > k'$, then $\{(x^k, y^k)\}$ is bounded and any limit point of it is an optimal solution to* (BP). *Otherwise, $f_1(x^k, y^k) \to -\infty$ as $k \to +\infty$.*

Proof (i) If the QCOPFA algorithm terminates at the $\overline{k}$-th iteration with the first condition in Step 3 occuring, $x^{\overline{k}}$ is an optimal solution to (BP) by Theorem 3.1.

(ii) Suppose that $\{(x^k, y^k)\}$ is an infinite sequence and there is some $k' > 1$ such that $M_k < f_1(x^{k+1}, y^{k+1})$ and $N < f_2(x^{k+1}, y^{k+1})$ for all $k > k'$. Let (x', y') be a feasible solution to (BP). The bounded sequence $\{(x^k, y^k)\}$ is checked as follows. Since (x^k, y^k) is an optimal solution to BP2(M_k, N),

$$Q(f_1(x^{k+1}, y^{k+1}) - M_k) \le F(x^{k+1}, y^{k+1}; M_k, N) \le Q(f_1(x', y') - M_k), \quad k = 1, 2, \cdots.$$

So,

$$(f_1(x^{k+1}, x^{k+1}) - M_k)^2 \le (f_1(x', y') - M_k)^2, \quad k = k' + 1, k' + 2, \cdots.$$

We have $f_1(x^{k+1}, y^{k+1}) \le f_1(x', y'), k = k' + 1, k' + 2, \cdots$. Hence, the sequence $\{x^{k+1}, y^{k+1}\}$ is bounded, since $\lim_{(x,y)\to\infty} f_1(x, y) = +\infty$. Without loss of generality, we assume $(x^k, y^k) \to (\overline{x}, \overline{y})$. And, for any given feasible solution (x, y) to (BP), we have

$$0 < (f_1(x^{k+1}, y^{k+1}) - M_k)^2 + M_k^2 E(x^{k+1}, y^{k+1}; N) \le (f_1(x, y) - M_k)^2, \quad \forall k > k'.$$

That is,

$$E(x^{k+1}, y^{k+1}; N) \le \frac{1}{M_k^2}(f_1(x, y) - f_1(x^{k+1}, y^{k+1}))(f_1(x^{k+1}, y^{k+1}) + f_1(x, y) - 2M_k), \ \forall k > k'.$$

It is clear that $M_k \to -\infty$ as $k \to +\infty$. By letting $k \to +\infty$ in the above equation, we obtain $E(\overline{x}, \overline{y}) = 0$. Hence, $(\overline{x}, \overline{y}) \in X$ and $f_1(\overline{x}, \overline{y}) \le f_1(x, y)$. Therefore, from Theorem 3.2, $(\overline{x}, \overline{y})$ is an optimal solution to (BP).

Example 3[26] Consider the problem:

$$\begin{aligned}
\text{(MP3)} \quad \min_{x,y} \quad & f_1(x, y) = (x_1 - 30)^2 + (x_2 - 20)^2 - 20y_1 + 20y_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \le 30, x_1 + x_2 \ge 20, 0 \le x_1 \le 15, 0 \le x_2 \le 15, \\
& y \ \text{ solves the following problem:} \\
\min_{y} \quad & f_2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \\
\text{s.t.} \quad & 0 \le y_1 \le 15, 0 \le y_2 \le 15.
\end{aligned}$$

Let objective penalty function problem of (MP3) as follows.

$$\begin{aligned}
\text{MP3}(M, N) \quad \min_{x,y} \quad F(x, y; M, N) = & \max\{(x_1 - 30)^2 + (x_2 - 20)^2 - 20y_1 + 20y_2 - M, 0\}^2 \\
& + M^2(\{\max\{x_1 + 2x_2 - 30, 0\}^2 + \max\{20 - x_1 - x_2, 0\}^2) \\
& + M^2(\max\{-x_1, 0\}^2 + M^2 \max\{x_1 - 15, 0\}^2) \\
& + M^2(\max\{-x_2, 0\}^2 + M^2 \max\{x_2 - 15, 0\}^2) \\
& + M^2[4(y_1 - x_1) \max\{(x_1 - y_1)^2 + (x_2 - y_2)^2 - N, 0\} \\
& - 2N^2 \max\{-y_1, 0\} + 2N^2 \max\{y_1 - 15, 0\}]^2 \\
& + M^2[4(y_2 - x_2) \max\{(x_1 - y_1)^2 + (x_2 - y_2)^2 - N, 0\} \\
& - 2N^2 \max\{-y_2, 0\} + 2N^2 \max\{y_2 - 15, 0\}]^2 \\
\text{s.t.} \quad (x, y) \in R^2 \times R^2, &
\end{aligned}$$

where $M, N < 0$. According to the QCOPFA algorithm, we obtain that $(x, y) = (15.000026, 7.499954, 15.000007, 7.499954)$ is an approximate solution to MP3(M, N) at fourth iteration by

Matlab, which it is the same as the smoothing SQP algorithm in [26]. But, since $F(x, y; M, N)$ is always not convex, it is very difficult to find out a global solution to the subproblem BP2(M_k, N) in the QCOPFA algorithm.

The objective penalty function algorithm presented in this paper has some interesting perspectives in solving objective and constrained functions which are not convex, and is worthy of further study.

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