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Optimal Reinsurance Under Distortion Risk Measures and Expected Value Premium Principle for Reinsurer*

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Abstract This paper discusses optimal reinsurance strategy by minimizing insurer's risk under one general risk measure: Distortion risk measure. The authors assume that the reinsurance premium is determined by the expected value premium principle and the retained loss of the insurer is an increasing function of the initial loss. An explicit solution of the insurer's optimal reinsurance problem is obtained. The optimal strategies for some special distortion risk measures, such as value-at-risk (VaR) and tail value-at-risk (TVaR), are also investigated.

Keywords Distortion risk measure, expected value premium principle, optimal reinsurance strategy, TVaR, VaR.

1 Introduction

Recently, the discussion on optimal reinsurance strategy is an interesting topic. For a policy of an insurer, the potential loss suffered by the insurer can be expressed as a non-negative random variable X, and a reasonable assumption on X is that $0 < E[X] < \infty$. To control risk, the insurer cedes a part of its risk, denoted as $f(X) \in [0, X]$, to a reinsurer. The insurer's retained loss can be expressed as $I_f(X) = X - f(X)$. As a compensation for covering the insurer's loss, the reinsurer will receive a reinsurance premium from the insurer, denoted as

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$\mu(f(X))$. Under the reinsurance arrangement, the total payment $T_f(X)$ of the insurer can be expressed as a sum of the retained loss $I_f(X)$ and the paid reinsurance premium $\mu(f(X))$,

$$T_f(X) = I_f(X) + \mu(f(X)).$$

To investigate optimal reinsurance strategy, one needs to discuss the optimization problem under the insurer's risk measure. As the widely used risk measures, value-at-risk (VaR) and tail value-at-risk (TVaR) are applied for the discussion on optimal reinsurance (see [1] and [2]). The VaR of a random variable $X \geq 0$ at a confidence level $1-\alpha$, $0<\alpha<1$, is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{y : P(X > y) \leq \alpha\}.$$

The TVaR of a random variable $X \geq 0$ at a confidence level $1-\alpha$, $0<\alpha<1$, is defined as

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_s(X)ds.$$

It is known that VaR and TVaR belong to one family of risk measures: Distortion risk measures. A distortion function $g(x) : [0,1] \to [0,1]$ is an increasing function satisfying that $g(0) = 0$ and $g(1) = 1$. Associated with distortion function $g(x)$, the distortion risk measure is defined as

$$\rho_g[X] = \int_0^\infty g(P(X > x))dx. \tag{1}$$

For a detailed introduction about distortion risk measure, see [3]. Note that $\mathrm{VaR}_\alpha(X)$ has the associated distortion function $g(x) = \boldsymbol{I}_{\{x>\alpha\}}$, and $\mathrm{TVaR}_\alpha(X)$ has the associated distortion function $g(x) = \min\{\frac{x}{\alpha}, 1\}$. It is known that Wang's transform (WT) risk measure[4] also belongs to the family of distortion risk measures. The distortion risk measure will be used in this paper to measure the insurer's risk.

As for the reinsurance premium principle, we assume that the reinsurance premium is calculated by the expected value premium principle, that is,

$$\mu(f(X)) = (1+\beta)E[f(X)], \tag{2}$$

where $\beta > 0$ is the safety loading factor. Note that the expected value premium principle is widely used (see [1]).

For investigating the optimal reinsurance strategy, three widely used families of the ceded loss function $f(x)$ are summarized as follows:

$$\mathcal{F} := \{f(x) : 0 \leq f(x) \leq x,\ f(x) \text{ is an increasing and convex function}\},$$

$$\mathcal{H} := \{f(x) : 0 \leq f(x) \leq x,\ \text{both } f(x) \text{ and } I_f(x) \text{ are increasing functions}\},$$

$$\mathcal{L} := \{f(x) : 0 \leq f(x) \leq x,\ I_f(x) \text{ is an increasing and left continuous function}\}.$$

Note that when $f \in \mathcal{H}$, both $f(x)$ and $I_f(x)$ are continuous. Thus it is easy to verify that $\mathcal{F} \subsetneqq \mathcal{H} \subsetneqq \mathcal{L}$, see [5] for details. Some papers found that the truncated stop-loss function, which does not belong to families $\mathcal{F}$ and $\mathcal{H}$, is optimal (see [6–9]). Thus it is interesting to consider the optimal reinsurance strategy in the family $\mathcal{L}$.

Many papers on VaR-minimization and TVaR-minimization are focused on families $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{L}$. In the family $\mathcal{F}$, [1] and [2] concerned the VaR-minimization and TVaR-minimization under the expected value premium principle. [2] revisited the VaR-minimization within the family $\mathcal{F}$ under Wang's premium principle. In the family $\mathcal{H}$, [5] obtained closed-form solutions for VaR and TVaR minimization problems under different premium principles, and [10] focused on the distortion risk measure. In the family $\mathcal{L}$, [11] considered a local reinsurance model and derived that the stop-loss strategy is optimal. Under the expected value premium principle, [5] considered the optimal reinsurance strategies in the three families $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{L}$, respectively, and derived optimal reinsurance strategies for VaR-minimization.

This paper will focus on the optimal reinsurance in the family $\mathcal{L}$. The optimization problem can be expressed as follows:

$$\min_{f\in\mathcal{L}}\{\rho_g[T_f(X)]\}, \tag{3}$$

where ρ_g is the distortion risk measure in Equation (1) and the reinsurance premium is calculated by the expected value premium principle in Equation (2).

The paper is organized as follows. Section 2 presents the explicit solution for our optimal reinsurance model. The proofs of our main theorems are given in Section 3. The conclusions are drawn in Section 4. Some proofs are put in appendix.

2 The Main Results

In this section, the optimization problem (3) will be discussed. Before giving the optimal reinsurance treaty, some preparation work is needed.

2.1 Preparation Work

For $f \in \mathcal{L}$, $I_f^{-1}(x) := \inf\{t : I_f(t) > x\}$. By the monotonicity and the left-continuity of $I_f(x)$, it is easy to verify

$$\{X : I_f(X) > t\} = \{X : X > I_f^{-1}(t)\}, \quad \forall \text{ r.v. } X \geq 0.$$

Define $B_X(t) = (1+\beta)S_X(t) - g(S_X(t))$, where $S_X(t) := P(X > t)$ is the survival function of X.

Proposition 2.1 *The distortion risk measure of the insurer's total payment can be represented as*

$$\rho_g[T_f(X)] = (1+\beta)E[X] - \int_{\overline{R}^+} B_X d\nu_f,$$

where the measure ν_f *is defined by* $\nu_f([a,b)) = I_f(b) - I_f(a), [a,b) \in \mathcal{B}_{\overline{R}^+}$.

Proof Given the distortion risk measure ρ_g in (1), for any positive constant c we have

$\rho_g[X+c] = \rho_g[X] + c$ (see [3]). Thus it yields that

$$\begin{aligned}\rho_g[T_f(X)] &= \mu(f(X)) + \rho_g[I_f(X)] \\ &= (1+\beta)E[f(X)] + \rho_g[I_f(X)] \\ &= (1+\beta)E[X] + \rho_g[I_f(X)] - (1+\beta)E[I_f(X)] \\ &= (1+\beta)E[X] - \int_0^\infty \big[(1+\beta)P(I_f(X)>t) - g(P(I_f(X)>t))\big]dt \\ &= (1+\beta)E[X] - \int_0^\infty \big[(1+\beta)S_X(I_f^{-1}(t)) - g(S_X(I_f^{-1}(t)))\big]dt \\ &= (1+\beta)E[X] - \int_0^\infty B_X(I_f^{-1}(t))dt \\ &= (1+\beta)E[X] - \int_{\overline{R}^+} B_X \circ I_f^{-1} d\mu, \end{aligned} \tag{4}$$

where μ denotes the Lebesgue measure. It is known that there exists a unique measure ν_f defined by $\nu_f(B) = \mu(\{x : I_f^{-1}(x) \in B\}), B \in \mathcal{B}_{\overline{R}^+}$, such that

$$\int_{\overline{R}^+} B_X \circ I_f^{-1} d\mu = \int_{\overline{R}^+} B_X d\nu_f, \tag{5}$$

see [12] for details. Using the fact that the two inequalities $I_f(s) > t$ and $s > I_f^{-1}(t)$ are equivalent, we have

$$\nu_f([a,b)) = \mu(\{x : a \le I_f^{-1}(x) < b\}) = \mu([I_f(a), I_f(b))) = I_f(b) - I_f(a),\ \forall [a,b) \subseteq [0,\infty).$$

Then combining (4) and (5), the proposition is proved. ∎

To simplify our discussion, we make the following assumption on the distortion function $g(x)$.

Assumption A Assume that $g(x), x \in [0,1]$ is left-continuous, and its domain has a finite partition $[0,1] = \bigcup_{i=1}^n (\alpha_i, \alpha_{i+1}] \cup \{0\}$, such that for each $i = 1, 2, \cdots, n$, the function $g(x)$ is either concave or convex on $(\alpha_i, \alpha_{i+1}]$.

Some common distortion risk measures, such as VaR, TVaR, and Wang's transform risk measure, satisfy Assumption A. Figure 1 gives the corresponding distortion functions with confidence level $1-\alpha = 0.1$.

In the following, we will discuss the reinsurance optimization problem (3) under Assumption A. The right-continuity of $S_X(t)$ and the left-continuity of $g(x), x \ge 0$ guarantee that the function $B_X(t), t \ge 0$ is right-continuous. It is obvious that $B_X(t-), t \ge 0$ is left-continuous.

A point $x_0 \in E \subseteq R$ is called a local maximum point if there exists a neighborhood $U_E(x_0)$ of x_0 in E, such that for any $x \in U_E(x_0)$ we have $f(x) \le f(x_0)$. And the value $f(x_0)$ is called the local maximum value (see [13, p214]).

For a real-valued function f, we denote the set of all its local maximum values on the set B as $\mathrm{loc}(f(x), x \in B)$. And for simplicity, sometimes we write (a,∞) as $(a,\infty]$.

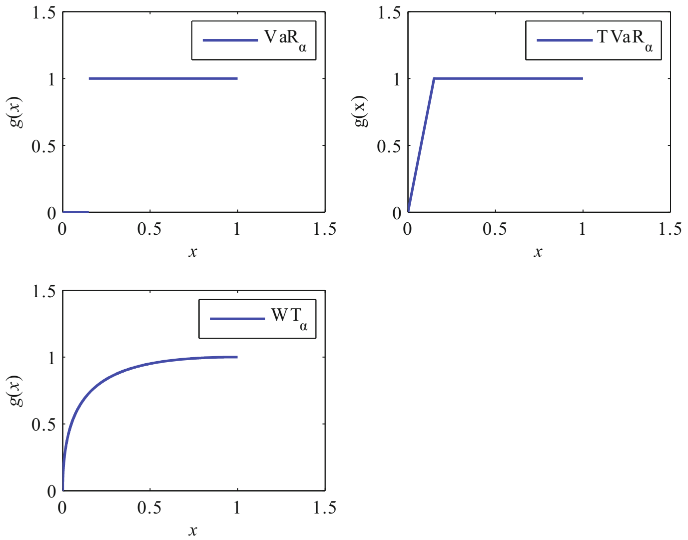

Figure 1 The associated distortion functions for VaR, TVaR, and Wang's transform risk measure

Lemma 2.2 *Suppose that the function* $g(x), x \in [0,1]$ *satisfies Assumption A. Then there exists a finite partition* $[0,\infty) = \bigcup_{i=1}^{d}[\gamma_i, \gamma_{i+1})$*, such that for each* $i = 1, 2, \cdots, d$*, the function* $B_X(t)$ *is monotonic on* $[\gamma_i, \gamma_{i+1})$ *and the function* $B_X(t-)$ *is monotonic on* $(\gamma_i, \gamma_{i+1}]$*.*

Proof For simplicity, we write $B(x) = (1+\beta)x - g(x)$. Based on Assumption A, $g(x)$ is concave or convex on $(\alpha_i, \alpha_{i+1}]$, thus $B(x)$ is convex or concave on $(\alpha_i, \alpha_{i+1}]$. From the properties of convex and concave functions, we know there exists one $\alpha_i \le \beta_i \le \alpha_{i+1}$ such that $B(x)$ is monotonic on $(\alpha_i, \beta_i]$ and $(\beta_i, \alpha_{i+1}]$ respectively. Therefore, there exists a finite partition $(0,1] = \bigcup_{i=1}^{n}(\alpha_i, \beta_i] \cup (\beta_i, \alpha_{i+1}]$ such that the function $B(x)$ is monotonic on each subinterval.

Define $S_X^{-1}(a) = \inf\{x \ge 0 : S_X(x) \le a\}$. Then $S_X(t) \le a$ if and only if $t \ge S_X^{-1}(a)$ (Cui, et al.[10]), which means that $S_X(t) \in (a, b]$ if and only if $t \in [S_X^{-1}(b), S_X^{-1}(a))$. Then we have

$$[0,\infty) = \bigcup_{i=1}^{n}([S_X^{-1}(\alpha_{i+1}), S_X^{-1}(\beta_i)) \cup [S_X^{-1}(\beta_i), S_X^{-1}(\alpha_i))) \cup [S_X^{-1}(0), \infty).$$

Due to the facts that $S_X(t), t \ge 0$ is decreasing and $B_X(t) = B(S_X(t)), t \ge 0$, the function $B_X(t)$ is monotonic on the subintervals $[S_X^{-1}(\beta_i), S_X^{-1}(\alpha_i))$ and $[S_X^{-1}(\alpha_{i+1}), S_X^{-1}(\beta_i))$, and $B_X(t) \equiv 0, t \in [S_X^{-1}(0), \infty)$. Thus there exists a finite partition, denoted as $[0,\infty) = \bigcup_{i=1}^{d}[\gamma_i, \gamma_{i+1})$, such that $B_X(t)$ is monotonic on $[\gamma_i, \gamma_{i+1})$. Finally, $B_X(t-)$ is monotonic on $(\gamma_i, \gamma_{i+1}]$ for each $i = 1, 2, \cdots, d$. ∎

From Lemma 2.2, the two functions $B_X(t), t \ge 0$ and $B_X(t-), t \ge 0$ have finitely many local maximum values. Thus the function $\max\{B_X(t), B_X(t-)\}, t \ge 0$ also has finitely many local maximum values. We define

$$M_{1,0} = \max\{y : y \in \mathrm{loc}(\max\{B_X(t), B_X(t-)\}, t \in (0,\infty))\}$$

and

$$M_1 = \max\{B_X(0), M_{1,0}\}.$$

Write

$$r_1 = \begin{cases} \sup\{t : \max\{B_X(t), B_X(t-)\} = M_{1,0}\}, & \text{if } M_{1,0} = M_1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Then we define

$$m_1 = \begin{cases} \inf\{0 \le t \le r_1 : \max\{B_X(s), B_X(s-)\} = M_1,\ \forall s \in [t, r_1]\}, & \text{if } M_{1,0} = M_1, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Note that $\max\{B_X(t), B_X(t-)\} = M_1, t \in (m_1, r_1)$, and $m_1 = 0$ if $r_1 = 0$.

Similarly, for $i \ge 2$ we write

$$M_i = \max\{y : y \in \mathrm{loc}(\max\{B_X(t), B_X(t-)\}, t \in (r_{i-1}, \infty))\},$$
$$r_i = \sup\{t : \max\{B_X(t), B_X(t-)\} = M_i\},$$

and

$$m_i = \inf\{t \le r_i : \max\{B_X(s), B_X(s-)\} = M_i \text{ for each } s \in [t, r_i]\}. \tag{8}$$

Repeat the above procedure until there exists no positive local maximum point. The number of the M_i is denoted as s. From Lemma 2.2, the number s is finite. For simplicity, we define $m_0 = 0, m_{s+1} = \infty$. Then

$$m_0 = 0 \le m_1 < \cdots < m_s < m_{s+1} = \infty, \quad M_1 > \cdots > M_s > 0.$$

Lemma 2.3 *If $m_1 > 0$, we have*

$$\max\{B_X(r_1), B_X(r_1-)\} = M_1, \quad \max\{B_X(m_1), B_X(m_1-)\} = M_1;$$

Otherwise, $B_X(0) = M_1$ follows. Moreover, for $i = 2, 3, \cdots, s$,

$$\max\{B_X(r_i), B_X(r_i-)\} = M_i, \quad \max\{B_X(m_i), B_X(m_i-)\} = M_i.$$

The proof of the lemma will be given in appendix.

In the next, we define

$$\overline{B}_X(t) = \begin{cases} B_X(t-), & \text{if } t = m_i \text{ for some } i = 1, 2, \cdots, s, \\ & \qquad\qquad \text{where } m_i > 0 \text{ and } B_X(m_i-) > B_X(m_i), \\ B_X(t), & \text{otherwise.} \end{cases} \tag{9}$$

It is obvious that $\overline{B}_X(t) \ge B_X(t)$ and $\overline{B}_X(t) = B_X(t)$ for $t \ne m_i, i \le s$. By Lemma 2.3, $\overline{B}_X(m_i) = M_i, i = 1, 2, \cdots, s$, are local maximum values of $\overline{B}_X(t)$.

Based on $m_i, i = 1, 2, \cdots, s+1$, and the function $\overline{B}_X$, a sequence $m_i^*, i = 0, 1, \cdots, s$, can be defined. Let $m_0^* = 0$, and for $1 \le i \le s$,

$$m_i^* = \inf\{t :\ t \ge m_i \text{ and } B_X(t) \le \overline{B}_X(m_{i+1})\}. \tag{10}$$

Here we define $\inf \emptyset = \infty$.

Remark 2.4 1) When $m_i > 0$, from the definitions of m_i and m^*_{i-1}, there exists a $\delta > 0$ such that $B_X(t) \le \overline{B}_X(m_i), t \in (m_i - \delta, m_i]$. Then $m^*_{i-1} \le m_i - \delta < m_i$ and $m_i - m^*_{i-1} > 0$ follows. Note that $m_i > 0$ for $i \ge 2$, then we have $m^*_{i-1} < m_i$ for each $i = 2, 3, \cdots, s$. And

$$m_0 = m^*_0 = 0 \le m_1 \le m^*_1 < m_2 \le \cdots < m_{s-1} \le m^*_{s-1} < m_s \le m^*_s \le m_{s+1} = \infty.$$

2) When $m_i > 0$, the point m_i is one local maximum point of $\max\{B_X(t), B_X(t-)\}$. Note that we define $M_1 = B_X(0)$ instead of $\max\{B_X(0), B_X(0-)\}$ when $B_X(0) > M_{1,0}$. Thus, if $B_X(0-) > B_X(0)$, we obtain that $M_1 = B_X(0)$ is not the local maximum value. In this case, m_1 equals to 0.

Each subinterval $(m_i, m_{i+1}]$ can be partitioned into (m_i, m^*_i) and $[m^*_i, m_{i+1}]$, and the properties of $B_X(t)$ on each subinterval will be given in the following lemma. The proof of the lemma will be given in appendix.

Lemma 2.5 *For each* $0 \le i \le s$, $B_X(t) > \overline{B}_X(m_{i+1}), t \in (m_i, m^*_i)$ *and* $B_X(t) \le \overline{B}_X(m_{i+1}), t \in [m^*_i, m_{i+1}]$, *and the function* $B_X(t)$ *is decreasing on* (m_i, m^*_i).

Example 2.6 (VaR) For $\mathrm{VaR}_\alpha(X)$, we have the associated distortion function $g(x) = \boldsymbol{I}_{\{x>\alpha\}}$. Thus

$$B_X(t) = \begin{cases} (1+\beta)S_X(t) - 1, & \text{if } t < \mathrm{VaR}_\alpha(X), \\ (1+\beta)S_X(t), & \text{otherwise.} \end{cases} \tag{11}$$

It is obvious that $B_X(t)$ is decreasing on the intervals $[0, \mathrm{VaR}_\alpha(X))$ and $[\mathrm{VaR}_\alpha(X), \infty)$ respectively. Suppose that $X \ge 0$ has a continuous and strictly increasing distribution on $(0, \infty)$ with a possible jump at 0.

(a) If $\alpha < S_X(0) - \frac{1}{1+\beta}$, we have $s = 2$, $M_1 = (1+\beta)S_X(0) - 1$, $m_1 = 0$,

$$M_2 = B_X(\mathrm{VaR}_\alpha(X)) = (1+\beta)\alpha \ \text{ and } \ m_2 = \mathrm{VaR}_\alpha(X).$$

By Equation (10), we have

$$m^*_1 = \inf\{t \ge m_1 : \ (1+\beta)S_X(t) - 1 \le (1+\beta)\alpha\} = \mathrm{VaR}_q(X),$$

where $q = \alpha + \frac{1}{1+\beta}$. Similarly, we obtain $m^*_2 = \infty$.

(b) If $\alpha \ge S_X(0) - \frac{1}{1+\beta}$, we have $s = 1$. Thus $M_1 = B_X(\mathrm{VaR}_\alpha(X)) = (1+\beta)\alpha$, $m_1 = \mathrm{VaR}_\alpha(X)$, and $m^*_1 = \infty$.

Example 2.7 (TVaR) For $\mathrm{TVaR}_\alpha(X)$, we have the associated distortion function $g(x) = \min\{\frac{x}{\alpha}, 1\}$, and

$$B_X(t) = \begin{cases} (1+\beta)S_X(t) - 1, & \text{if } t < \mathrm{VaR}_\alpha(X), \\ (1+\beta)S_X(t) - \dfrac{1}{\alpha}S_X(t), & \text{otherwise.} \end{cases} \tag{12}$$

(a) If $\alpha \le \frac{1}{1+\beta}$, the function $B_X(t)$ is decreasing on $[0, \mathrm{VaR}_\alpha(X))$ and increasing on $[\mathrm{VaR}_\alpha(X), \infty)$. By the definitions of M_i, m_i, and m^*_i, we obtain $s = 1$, $M_1 = B_X(0)$, $m_1 = 0$, $m_2 = \infty$, and $m^*_1 = \mathrm{VaR}_{\frac{1}{1+\beta}}(X)$.

(b) If $\alpha > \frac{1}{1+\beta}$, $B_X(t)$ is decreasing on $[0,\infty)$. Moreover, we can get $s=1$, $M_1 = B_X(0)$, $m_1 = 0$, $m_2 = \infty$, and $m_1^* = \infty$.

2.2 The Solution of the Reinsurance Optimization Problem (3)

For the given non-negative random variable X, the corresponding $m_i, m_i^*, i = 0, 1, \cdots, s$, have been defined in the subsection above. Write

$$f^*(x) = \sum_{i=0}^{s} (x - m_i^*)_+ \boldsymbol{I}_{(m_i^*, m_{i+1}]}(x) + (x - m_s^*)_+. \tag{13}$$

And the associated retained loss function can be expressed as

$$I_{f^*}(x) = \sum_{i=1}^{s} \min\{x, m_i^*\} \boldsymbol{I}_{(m_i, m_{i+1}]}(x). \tag{14}$$

According to $\nu_{f^*}([a,b)) = I_{f^*}(b) - I_{f^*}(a)$, we have

$$\begin{aligned}
&\nu_{f^*}([m_{i-1}^*, m_i)) = 0,\ \nu_{f^*}(\{m_i\}) = m_i - m_{i-1}^*, \quad i = 1, 2, \cdots, s,\\
&\nu_{f^*}(B) = \mu(B), \quad \forall B \subseteq \bigcup_{i=1}^{s} (m_i, m_i^*),\\
&\nu_{f^*}([m_s^*, \infty)) = 0.
\end{aligned}$$

Then for any Borel set $A \subseteq [0,\infty)$, the measure $\nu_{f^*}(A)$ can be expressed as

$$\nu_{f^*}(A) = \mu\left(A \cap \bigcup_{i=1}^{s} (m_i, m_i^*]\right) + \sum_{i=1}^{s} (m_i - m_{i-1}^*) \boldsymbol{I}_{\{m_i \in A\}}. \tag{15}$$

It can be directly verified that $0 \le f^*(x) \le x, x \ge 0$, and $I_{f^*}(x)$ is an increasing and left-continuous function. Thus $f^*(x)$ belongs to the family $\mathcal{L}$. As stated, $f^*(x)$ and $I_{f^*}(x)$ can be determined by the points m_i and $m_i^*, i = 0, 1, \cdots, s$. To make the definitions of $f^*(x)$ and $I_{f^*}(x)$ clearer, we give Figure 2 to show the relationships between the points m_i, m_i^* and the two functions $f^*(x), I_{f^*}(x)$.

Theorem 2.8 *Suppose that the distortion function $g(x)$ satisfies Assumption A. Then the infimum of the optimization problem* (3) *can be stated as*

$$\inf_{f \in \mathcal{L}} \rho_g[T_f(X)] = (1+\beta)E[X] - \int_{\overline{R}^+} \overline{B}_X d\nu_{f^*}.$$

If $B_X(m_i) = \overline{B}_X(m_i)$ for all $i = 1, 2, \cdots, s$, then f^* defined in Equation (13) *is one solution of the optimization problem* (3)*, and*

$$\begin{aligned}
\rho_g[T_{f^*}(X)] = &\sum_{i=1}^{s} \int_{m_{i-1}^*}^{m_i} [(1+\beta)(S_X(t) - S_X(m_i)) + g(S_X(m_i))] dt\\
&+ \sum_{i=1}^{s} \int_{m_i}^{m_i^*} g(S_X(t)) dt;
\end{aligned} \tag{16}$$

Otherwise, the optimization problem (3) *has no solution.*

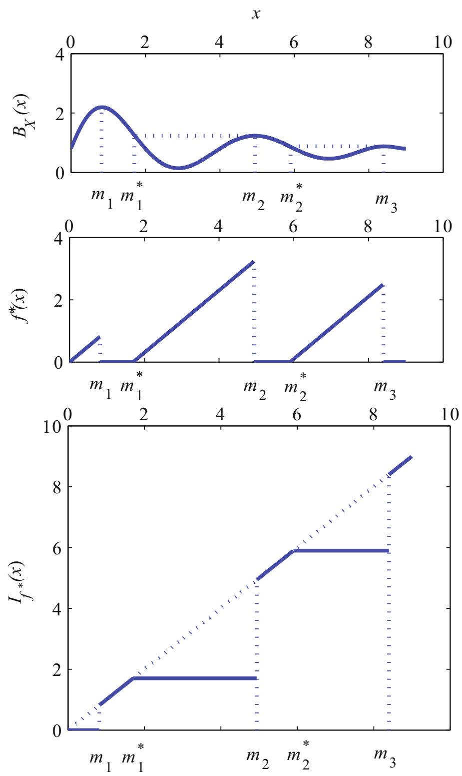

Figure 2 The relationships between the points m_i, m_i^* and the two functions $f^*(x), I_{f^*}(x)$

In case $B_X(m_i) = \overline{B}_X(m_i)$ for all $i = 1, 2, \cdots, s$, we can explain the equation (16) as follows. On the layer $(m_{i-1}^*, m_i]$, the ceded loss part is greater than zero and $B_X(t) \le B_X(m_i)$, thus

$$(1+\beta)(S_X(t) - S_X(m_i)) + g(S_X(m_i)) \le g(S_X(t)); \tag{17}$$

On the layer $(m_i, m_i^*]$, from the definition of $f^*(x)$, we know that there is no ceding on such a layer, and $g(S_X(t))$ is used to measure the risk of the insurer's total payment. Thus the existence of reinsurance will reduce the insurer's risk measured by the distortion risk measure ρ_g.

According to Theorem 2.8, when $\overline{B}_X(m_i) > B_X(m_i)$ for some $i \le s$, we have that for each $f \in \mathcal{L}$,

$$\rho_g[T_f(X)] > (1+\beta)E(X) - \int_{\overline{R}^+} \overline{B}_X d\nu_{f^*}.$$

The difference between $\rho_g[T_{f^*}(X)]$ and $\inf_{f\in\mathcal{L}} \rho_g[T_f(X)]$ can be expressed as follows:

$$\begin{aligned}\rho_g[T_{f^*}(X)] - \inf_{f\in\mathcal{L}} \rho_g[T_f(X)] &= \int_{\overline{R}^+} \overline{B}_X d\nu_{f^*} - \int_{\overline{R}^+} B_X d\nu_{f^*} \\ &= \sum_{i=1}^{s} (\overline{B}_X(m_i) - B_X(m_i))(m_i - m_{i-1}^*) \\ &> 0.\end{aligned}$$

Remark 2.9 In [10], under the expected value premium principle, the insurer's minimum risk measure can be expressed as $\int_0^\infty \min\{(1+\beta)S_X(t), g(S_X(t))\}dt$. In the next, we compare our minimum risk measure in Equation (16) with the above risk measure.

For fixed $i \in \{1, 2, \cdots, s\}$, consider the interval $[m_{i-1}^*, m_i]$ and the interval (m_i, m_i^*) separately. First we focus on the interval $[m_{i-1}^*, m_i]$. Since $B_X(m_i) > 0$, we have

$$(1+\beta)(S_X(t) - S_X(m_i)) + g(S_X(m_i)) < (1+\beta)S_X(t). \tag{18}$$

Meanwhile, using the fact that $B_X(t) \leq B_X(m_i), t \in [m_{i-1}^*, m_i]$, we get that

$$(1+\beta)(S_X(t) - S_X(m_i)) + g(S_X(m_i)) \leq g(S_X(t)). \tag{19}$$

Combining Equations (18) with (19), we have

$$(1+\beta)(S_X(t) - S_X(m_i)) + g(S_X(m_i)) < \min\{(1+\beta)S_X(t), g(S_X(t))\}, \quad t \in [m_{i-1}^*, m_i].$$

In the following, we consider the interval (m_i, m_i^*). Since $B_X(t) > B_X(m_{i+1}) \geq 0$, we have $(1+\beta)S_X(t) - g(S_X(t)) > 0$ and

$$\min\{(1+\beta)S_X(t), g(S_X(t))\} = g(S_X(t))$$

follows.

Based on the above consideration, we conclude that

$$\begin{aligned}\rho_g[T_{f^*}(X)] &= \sum_{i=1}^{s} \int_{m_i}^{m_i^*} g(S_X(t))dt \\ &\quad + \sum_{i=1}^{s} \int_{m_{i-1}^*}^{m_i} [(1+\beta)(S_X(t) - S_X(m_i)) + g(S_X(m_i))]dt \\ &\leq \sum_{i=1}^{s} \int_{m_i}^{m_i^*} \min\{(1+\beta)S_X(t), g(S_X(t))\}dt \\ &\quad + \sum_{i=1}^{s} \int_{m_{i-1}^*}^{m_i} \min\{(1+\beta)S_X(t), g(S_X(t))\}dt \\ &= \int_0^\infty \min\{(1+\beta)S_X(t), g(S_X(t))\}dt. \end{aligned} \tag{20}$$

The equation (20) states that the minimum risk of the insurer's risk measure according to Theorem 2.8 is smaller than or equal to the one according to Cui, et al.[10]. The reason is that the family $\mathcal{H}$ considered in [10] is smaller than our family $\mathcal{L}$.

The next two examples give the optimal reinsurance strategies for VaR-optimization and TVaR-optimization.

Example 2.10 (VaR optimization) Consider the optimal problem in Example 2.6.

1) If $\alpha < S_X(0) - \frac{1}{1+\beta}$, the optimal ceded loss function and the optimal retained function can be expressed as

$$f^*(x) = (x - \mathrm{VaR}_q(X))_+ \boldsymbol{I}_{\{x \le \mathrm{VaR}_\alpha(X)\}}$$

and

$$I_{f^*}(x) = \min\{x, \mathrm{VaR}_q(X)\} \boldsymbol{I}_{\{x \le \mathrm{VaR}_\alpha(X)\}} + x \boldsymbol{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}.$$

Here $q = \alpha + \frac{1}{1+\beta}$.

2) If $\alpha \ge S_X(0) - \frac{1}{1+\beta}$, the optimal ceded loss function $f^*(x)$ and the optimal retained function $I_{f^*}(x)$ can be expressed as

$$f^*(x) = x \boldsymbol{I}_{\{x \le \mathrm{VaR}_\alpha(X)\}} \quad \text{and} \quad I_{f^*}(x) = x \boldsymbol{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}.$$

Example 2.11 (TVaR optimization) Consider the optimal problem in Example 2.7.

1) If $\alpha \le \frac{1}{1+\beta}$, the optimal ceded loss function and the optimal retained function can be expressed as

$$f^*(x) = (x - \mathrm{VaR}_{\frac{1}{1+\beta}}(X))_+ \quad \text{and} \quad I_{f^*}(x) = \min\{x, \mathrm{VaR}_{\frac{1}{1+\beta}}(X)\}.$$

2) If $\alpha > \frac{1}{1+\beta}$, the optimal ceded loss function is $f^*(x) = 0$, and the optimal retained function is $I_{f^*}(x) = x$.

3 Proofs

This section will give the proof of Theorem 2.8. First we need to give some lemmas.

Lemma 3.1 *Given* $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$ *and* $0 \le c_1 \le c_2 \le \cdots \le c_n$, *we have*

$$\max_{\omega_1, \cdots, \omega_n} \left\{ \sum_{i=1}^n \omega_i x_i : w_1 \le c_1, \cdots, \sum_{i=1}^j \omega_i \le c_j, \omega_j \ge 0, j = 1, 2, \cdots, n \right\}$$
$$= c_1 x_1 + \sum_{i=2}^n (c_i - c_{i-1}) x_i.$$

The above lemma is easy to verify and we omit its proof here.

Lemma 3.2 *For* $\overline{B}_X(t)$ *defined in Equation* (9), *we have*

$$\max_{f \in \mathcal{L}} \left\{ \int_{\overline{R}_+} \overline{B}_X d\nu_f \right\} = \int_{\overline{R}_+} \overline{B}_X d\nu_{f^*}. \tag{21}$$

Proof Based on the definition of ν_{f^*} in Equation (15), we can conclude

$$\int_{\overline{R}_+} \overline{B}_X d\nu_{f^*} = \sum_{i=1}^s \left\{ \overline{B}_X(m_i)(m_i - m_{i-1}^*) + \int_{m_i}^{m_i^*} \overline{B}_X(t) dt \right\}. \tag{22}$$

On the other hand, for each ceded loss function $f \in \mathcal{L}$, from the definition of $L-S$ integral, we have

$$\int_{\overline{R}_+} \overline{B}_X d\nu_f = \lim_{n\to\infty}\sum_{k=0}^{n2^n}\frac{k}{2^n}\nu_f\left(\left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\}\right) - \lim_{m\to\infty}\sum_{k=0}^{m2^m}\frac{k}{2^m}\nu_f\left(\left\{t: \frac{k}{2^m} \le \overline{B}_X^-(t) < \frac{k+1}{2^m}\right\}\right), \tag{23}$$

where $\overline{B}_X^+(t) := \max\{\overline{B}_X(t), 0\}$ and $\overline{B}_X^-(t) := \max\{-\overline{B}_X(t), 0\}$.

Denote $\lfloor x \rfloor$ to be the largest integer less than or equal to x. Based on Lemma 2.5, for each $1 \le i \le s$, we have

$$0 \le \frac{\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}{2^n} \le \overline{B}_X(t) < \frac{\lfloor \overline{B}_X(m_i)2^n \rfloor + 1}{2^n}, \quad t \in (m_i, m_i^*). \tag{24}$$

Since $\overline{B}_X(t)$ is positive and decreasing on (m_i, m_i^*), then for each $k = 0, 1, \cdots, n2^n$, there exists an interval $\langle a_{i,k}, a_{i,k+1}\rangle$ such that

$$\langle a_{i,k}, a_{i,k+1}\rangle = \left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\} \cap (m_i, m_i^*), \tag{25}$$

where "$\langle$" denotes "(" or "[" according to that the left-end point of the above interval is close or open, similarly for "$\rangle$".

Based on Equations (24) and (25), we can partition the interval (m_i, m_i^*) as

$$(m_i, m_i^*) = \bigcup_{k=\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}^{\lfloor \overline{B}_X(m_i)2^n \rfloor} \langle a_{i,k}, a_{i,k+1}\rangle. \tag{26}$$

For the right-hand side of Equation (23) and any positive integer $n > \overline{B}_X(m_1)$, we have

$$\begin{aligned}
&\sum_{k=0}^{n2^n}\frac{k}{2^n}\nu_f\left(\left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\}\right) - \sum_{k=0}^{m2^m}\frac{k}{2^m}\nu_f\left(\left\{t: \frac{k}{2^m} \le \overline{B}_X^-(t) < \frac{k+1}{2^m}\right\}\right)\\
&\le \sum_{k=0}^{n2^n}\frac{k}{2^n}\nu_f\left(\left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\}\right)\\
&= \sum_{i=1}^{s}\left\{\sum_{k=0}^{n2^n}\frac{k}{2^n}\nu_f\left(\left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\} \cap [m_{i-1}^*, m_i]\right)\right.\\
&\quad \left.+\sum_{k=0}^{n2^n}\frac{k}{2^n}\nu_f\left(\left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\} \cap (m_i, m_i^*)\right)\right\}\\
&\quad +\sum_{k=0}^{n2^n}\frac{k}{2^n}\nu_f\left(\left\{t: \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\} \cap [m_s^*, \infty)\right)\\
&= \sum_{i=1}^{s}\left\{I_{i,1} + I_{i,2}\right\} + I_3.
\end{aligned} \tag{27}$$

Consider $I_{i,1}, I_{i,2}$, and I_3, respectively. Since $\overline{B}_X(t) \le \overline{B}_X(m_i), t \in [m_{i-1}^*, m_i]$, we can derive that

$$I_{i,1} \le \overline{B}_X(m_i)\nu_f([m_{i-1}^*, m_i]). \tag{28}$$

For the second term $I_{i,2}$, Equation (26) leads to

$$I_{i,2} = \sum_{k=\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}^{\lfloor \overline{B}_X(m_i)2^n \rfloor} \frac{k}{2^n}\nu_f(\langle a_{i,k}, a_{i,k+1}\rangle). \tag{29}$$

For the last term I_3 of Equation (27), since $\overline{B}_X(t) \le 0$ on $[m_s^*, \infty)$, we have

$$I_3 = 0. \tag{30}$$

Then Equations (27), (28), (29), and (30) yield that

$$\begin{aligned}
&\sum_{k=0}^{n2^n} \frac{k}{2^n}\nu_f\left(\left\{t : \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n}\right\}\right) - \sum_{k=0}^{m2^m} \frac{k}{2^m}\nu_f\left(\left\{t : \frac{k}{2^m} \le \overline{B}_X^-(t) < \frac{k+1}{2^m}\right\}\right) \\
&\le \sum_{i=1}^{s} \Big\{\overline{B}_X(m_i)\nu_f([m_{i-1}^*, m_i]) + \sum_{k=\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}^{\lfloor \overline{B}_X(m_i)2^n \rfloor} \frac{k}{2^n}\nu_f(\langle a_{i,k}, a_{i,k+1}\rangle)\Big\} \\
&\le \sum_{i=1}^{s} \Big\{\overline{B}_X(m_i)\nu_f([m_{i-1}^*, m_i]) + \sum_{k=\lfloor \overline{B}_X(m_i^*-)2^n \rfloor+1}^{\lfloor \overline{B}_X(m_i)2^n \rfloor} \frac{k}{2^n}\nu_f(\langle a_{i,k}, a_{i,k+1}\rangle) \\
&\qquad + \overline{B}_X(m_i^*-)\nu_f(\langle a_{i,\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}, a_{i,\lfloor \overline{B}_X(m_i^*-)2^n \rfloor+1}\rangle)\Big\} \\
&= \sum_{i=1}^{s} \Big\{\overline{B}_X(m_i)\nu_f([m_{i-1}^*, m_i]) + \frac{\lfloor \overline{B}_X(m_i)2^n \rfloor}{2^n}\nu_f(\langle m_i, a_{i,\lfloor \overline{B}_X(m_i)2^n \rfloor+1}\rangle) \\
&\qquad + \frac{\lfloor \overline{B}_X(m_i)2^n \rfloor - 1}{2^n}\nu_f(\langle a_{i,\lfloor \overline{B}_X(m_i)2^n \rfloor+1}, a_{i,\lfloor \overline{B}_X(m_i)2^n \rfloor+2}\rangle) + \cdots \\
&\qquad + \frac{\lfloor \overline{B}_X(m_i^*-)2^n \rfloor + 1}{2^n}\nu_f(\langle a_{i,\lfloor \overline{B}_X(m_i^*-)2^n \rfloor-1}, a_{i,\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}\rangle) \\
&\qquad + \overline{B}_X(m_i^*-)\nu_f(\langle a_{i,\lfloor \overline{B}_X(m_i^*-)2^n \rfloor}, a_{i,\lfloor \overline{B}_X(m_i^*-)2^n \rfloor+1}\rangle)\Big\} \\
&:= I_4.
\end{aligned} \tag{31}$$

Based on Lemma 2.5, we obtain

$$\begin{aligned}
\overline{B}_X(m_1) &\ge \frac{\lfloor \overline{B}_X(m_1)2^n \rfloor}{2^n} \ge \frac{\lfloor \overline{B}_X(m_1)2^n \rfloor - 1}{2^n} \ge \frac{\lfloor \overline{B}_X(m_1)2^n \rfloor - 2}{2^n} \ge \cdots \\
&\ge \frac{\lfloor \overline{B}_X(m_1^*-)2^n \rfloor + 2}{2^n} \ge \frac{\lfloor \overline{B}_X(m_1^*-)2^n \rfloor + 1}{2^n} \ge \overline{B}_X(m_1^*-) \\
&\ge \overline{B}_X(m_2) \ge \frac{\lfloor \overline{B}_X(m_2)2^n \rfloor}{2^n} \ge \frac{\lfloor \overline{B}_X(m_2)2^n \rfloor - 1}{2^n} \ge \cdots .
\end{aligned}$$

Since $\nu_f([a,b]) = I_f(b+) - I_f(a) \le I_f(b+) \le b$ for any $a < b$, we have the following inequalities

$$\begin{aligned}
&\nu_f([0,m_1]) \le m_1,\\
&\nu_f([0,m_1]) + \nu_f((m_1, a_{1,\lfloor \overline{B}_X(m_1)2^n\rfloor+1}\rangle) \le a_{1,\lfloor \overline{B}_X(m_1)2^n\rfloor+1},\\
&\nu_f([0,m_1]) + \nu_f((m_1, a_{1,\lfloor \overline{B}_X(m_1)2^n\rfloor+1}\rangle) + \nu_f(\langle a_{1,\lfloor \overline{B}_X(m_1)2^n\rfloor+1}, a_{1,\lfloor \overline{B}_X(m_1)2^n\rfloor+2}\rangle)\\
&\le a_{1,\lfloor \overline{B}_X(m_1)2^n\rfloor+2},\\
&\cdots.
\end{aligned}$$

Then it follows from Lemma 3.1 and Equation (31) that

$$\begin{aligned}
I_4 \le \sum_{i=1}^{s} \Bigg\{ &\overline{B}_X(m_i)(m_i - m_{i-1}^*) + \sum_{k=\lfloor \overline{B}_X(m_i^*-)2^n\rfloor+1}^{\lfloor \overline{B}_X(m_i)2^n\rfloor} \frac{k}{2^n}(a_{i,k+1} - a_{i,k})\\
&+\overline{B}_X(m_i^*-)(a_{i,\lfloor \overline{B}_X(m_i^*-)2^n\rfloor+1} - a_{i,\lfloor \overline{B}_X(m_i^*-)2^n\rfloor}) \Bigg\}. \qquad (32)
\end{aligned}$$

On the other hand, from Equation (22), we can get

$$\begin{aligned}
\int_{\overline{R}_+} \overline{B}_X d\nu_{f^*} &= \sum_{i=1}^{s} \left\{ \overline{B}_X(m_i)(m_i - m_{i-1}^*) + \int_{m_i}^{m_i^*} \overline{B}_X(t)dt \right\}\\
&\ge \sum_{i=1}^{s} \Bigg\{ \overline{B}_X(m_i)(m_i - m_{i-1}^*)\\
&\qquad + \sum_{k=\lfloor \overline{B}_X(m_i^*-)2^n\rfloor+1}^{\lfloor \overline{B}_X(m_i)2^n\rfloor} \frac{k}{2^n}\mu\left(\left\{ t : \frac{k}{2^n} \le \overline{B}_X^+(t) < \frac{k+1}{2^n} \right\} \cap (m_i, m_i^*) \right)\\
&\qquad +\overline{B}_X(m_i^*-)(a_{i,\lfloor \overline{B}_X(m_i^*-)2^n\rfloor+1} - a_{i,\lfloor \overline{B}_X(m_i^*-)2^n\rfloor}) \Bigg\}\\
&= \sum_{i=1}^{s} \Bigg\{ \overline{B}_X(m_i)(m_i - m_{i-1}^*) + \sum_{k=\lfloor \overline{B}_X(m_i^*-)2^n\rfloor+1}^{\lfloor \overline{B}_X(m_i)2^n\rfloor} \frac{k}{2^n}(a_{i,k+1} - a_{i,k})\\
&\qquad +\overline{B}_X(m_i^*-)(a_{i,\lfloor \overline{B}_X(m_i^*-)2^n\rfloor+1} - a_{i,\lfloor \overline{B}_X(m_i^*-)2^n\rfloor}) \Bigg\}. \qquad (33)
\end{aligned}$$

Combining Equations (23), (31), (32), and (33), we obtain

$$\int_{\overline{R}_+} \overline{B}_X d\nu_f \le \int_{\overline{R}_+} \overline{B}_X d\nu_{f^*}.$$

Note that $f^* \in \mathcal{L}$ and f is any ceded loss function belonging to the family $\mathcal{L}$, then Equation (21) holds and Lemma 3.2 is proved. ∎

Similar to Lemma 3.2, we can obtain a general corollary. Its proof will be given in the appendix.

Corollary 3.3 *For a function $h(t)$ defined on the interval $[0,d], 0<d<\infty$, assume that there exists a finite partition $[0,d)=\bigcup_{i=0}^{s_h}[m_{h,i},m_{h,i+1})$ satisfying that*

$$0=m_{h,0}\le m_{h,1}<m_{h,2}<\cdots\le m_{h,s_{h+1}}=d$$

and

$$h(m_{h,1})\ge h(m_{h,2})\ge\cdots\ge h(m_{h,s_h})\ge h(m_{h,s_{h+1}})\ge 0.$$

Furthermore, for each $0\le i\le s_h$ there exists one $m^*_{h,i}$ satisfying $m^*_{h,0}=0$ and $m_{h,i}\le m^*_{h,i}\le m_{h,i+1}$ for $1\le i\le s_h$, such that

$$h(t)\ge h(m_{h,i+1}),\ t\in(m_{h,i},m^*_{h,i}),\quad h(t)\le h(m_{h,i+1}),\ t\in[m^*_{h,i},m_{h,i+1}]$$

and $h(t)$ is decreasing on $[m_{h,i},m^*_{h,i})$. Then

$$\max_{f\in\mathcal{L}}\left\{\int_{[0,d]}h(t)d\nu_f(t)\right\}=\sum_{i=1}^{s_h}\left[h(m_{h,i})(m_{h,i}-m^*_{h,i-1})+\int_{m_{h,i}}^{m^*_{h,i}}h(t)dt\right]$$
$$+h(m_{h,s_{h+1}})(m_{h,s_{h+1}}-m^*_{h,s_h}). \tag{34}$$

Remark 3.4 For the function $h(t)$ defined on the interval $[0,d], 0<d\le\infty$ with $h(d)=0$, if it satisfies the conditions of Corollary 3.3, we have

$$\max_{f\in\mathcal{L}}\left\{\int_{[0,d)}h(t)d\nu_f(t)\right\}=\sum_{i=1}^{s_h}\left[h(m_{h,i})(m_{h,i}-m^*_{h,i-1})+\int_{m_{h,i}}^{m^*_{h,i}}h(t)dt\right]. \tag{35}$$

Before giving the proof of Theorem 2.8, we discuss two examples for $h(t)$ which will be used in the proof of Theorem 2.8.

Example 3.5 For given $1\le i\le s+1$, define a function on $[0,d]=[0,m^*_{i-1}]$ by

$$h(t)=\overline{B}_X(t)\boldsymbol{I}_{[0,m^*_{i-1})}(t)+B_X(m^*_{i-1}-)\boldsymbol{I}_{\{m^*_{i-1}\}}(t). \tag{36}$$

Then

$$\max_{l\in\mathcal{L}}\left\{\int_{[0,m^*_{i-1}]}h(t)d\nu_l\right\}=\sum_{k=1}^{i-1}\overline{B}_X(m_k)(m_k-m^*_{k-1})+\sum_{k=1}^{i-1}\int_{m_k}^{m^*_k}\overline{B}_X(t)dt. \tag{37}$$

Proof We will check that $h(t)$ satisfies all the conditions in Corollary 3.3.

From the definition of $h(t)$, we know $h(t)=\overline{B}_X(t)$ on $[0,m^*_{i-1})$, and $h(m^*_{i-1})=B_X(m^*_{i-1}-)$. Thus $s_h=i-1, m_{h,s_h+1}=m^*_{i-1}$, and for each $0\le k\le s_h$, $m_{h,k}=m_k, m^*_{h,k}=m^*_k$, then $[0,m^*_{i-1})=[0,m_{h,s_h+1})=\bigcup_{k=0}^{s_h}[m_{h,k},m_{h,k+1})$.

Based on the properties of $m_k, m^*_k, k=0,1,\cdots,i-1$, and the facts that $m_{h,k}=m_k, m^*_{h,k}=m^*_k, 0\le k\le s_h$, we have

$$0=m_{h,0}\le m_{h,1}\le m^*_{h,1}<m_{h,2}\le m^*_{h,2}<\cdots<m_{h,s_h}\le m^*_{h,s_h}=m_{h,s_h+1}=d$$

and

$$h(m_{h,1})>h(m_{h,2})>\cdots>h(m_{h,s_h})\ge h(m_{h,s_h+1})=B_X(m^*_{i-1}-)\ge 0.$$

Moreover, for each $0 \le k \le s_h$, from Lemma 2.5 we know $h(t), t \in [m_{h,k}, m^*_{h,k})$ is decreasing and greater than $h(m_{h,k+1})$, and $h(t) \le h(m_{h,k+1}), t \in [m^*_{h,k}, m_{h,k+1}]$. Thus $h(t)$ in Equation (36) satisfies all the conditions in Corollary 3.3. Then Equation (37) follows from Corollary 3.3. ∎

Example 3.6 For the given $1 \le i \le s$, define

$$h(t) = \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}\boldsymbol{I}_{\{t=m_i\}} + \overline{B}_X(t)\boldsymbol{I}_{\{t>m_i\}} \tag{38}$$

on $[0, \infty)$. Then we have

$$\max_{l\in\mathcal{L}}\left\{\int_{[0,\infty)} h(t)d\nu_l\right\} = \max\{B_X(m_i), \overline{B}_X(m_{i+1})\} \times m_i + \sum_{k=i}^{s-1}\overline{B}_X(m_{k+1})(m_{k+1} - m^*_k) + \sum_{k=i}^{s}\int_{m_k}^{m^*_k}\overline{B}_X(t)dt. \tag{39}$$

Proof We will check that $h(t)$ satisfies all the conditions in Remark 3.4. We know that $h(t) = 0, t \in [0, m_i)$, $h(m_i) = \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}$, $h(t) = \overline{B}_X(t), t \in (m_i, \infty)$, and $h(\infty) = \overline{B}_X(\infty) = 0$.

From the definitions of m_i and m^*_i, we have $s_h = s - i + 1$, $m_{h,k} = m_{i+k-1}, 1 \le k \le s_h + 1$, and $m^*_{h,k} = m^*_{i+k-1}, 1 \le k \le s_h$. Thus we can get

$$0 = m_{h,0} \le m_{h,1} \le m^*_{h,1} < m_{h,2} \le m^*_{h,2} < \cdots < m_{h,s_h} \le m^*_{h,s_h} \le m_{h,s_h+1} = \infty$$

and

$$h(m_{h,1}) > h(m_{h,2}) > \cdots > h(m_{h,s_h}) > h(m_{h,s_h+1}) = \overline{B}_X(\infty) = 0.$$

In the next, we consider the intervals $[0, m_i)$ and $[m_i, \infty)$ separately. On the interval $[0, m_i)$, it is easy to check that $h(t) \equiv 0 \le h(m_i)$. On the interval $[m_i, \infty)$, for each $i \le k \le s$, the function $h(t), t \in [m_k, m^*_k)$ is decreasing and greater than $h(m_{k+1})$, and $h(t) \le h(m_{k+1}), t \in [m^*_k, m_{k+1}]$. Then applying Remark 3.4, we conclude that Equation (39) holds.

Proof of Theorem 2.8 1) Suppose that $B_X(m_i) = \overline{B}_X(m_i)$ for all $i = 1, 2, \cdots, s$. Then $B_X(t) = \overline{B}_X(t), t \ge 0$, and hence Equation (21) can be rewritten as

$$\max_{f\in\mathcal{L}}\left\{\int_{\overline{R}_+} B_X d\nu_f\right\} = \int_{\overline{R}_+} B_X d\nu_{f^*}.$$

Thus from Proposition 2.1, we have

$$\min_{f\in\mathcal{L}} \rho_g[T_f(X)] = \rho_g[T_{f^*}(X)].$$

We can get Equation (16) by direct verification.

2) Suppose that $\overline{B}_X(m_i) > B_X(m_i)$ for some $1 \le i \le s$. Based on the definition of $\overline{B}_X(t)$, for those $m_i, i = 1, 2, \cdots, s$ satisfying $\overline{B}_X(m_i) > B_X(m_i)$, we have that $m_i > 0$ and $\overline{B}_X(m_i) = B_X(m_i-)$. In the following, we will prove

$$\inf_{f\in\mathcal{L}} \rho_g[T_f(X)] = (1+\beta)E[X] - \int_{\overline{R}^+} \overline{B}_X d\nu_{f^*}. \tag{40}$$

Its proof will be finished by proving that there exists a sequence $\{f_n \in \mathcal{L}\}$ such that

$$\lim_{n\to\infty}\int_{\overline{R}_+} B_X d\nu_{f_n} = \int_{\overline{R}_+} \overline{B}_X d\nu_{f^*}, \tag{41}$$

and for $f \in \mathcal{L}$,

$$\int_{[0,\infty)} B_X(t)d\nu_f < \int_{[0,\infty)} \overline{B}_X(t)d\nu_{f^*}. \tag{42}$$

(a) We will prove Equation (41) in this part.

As stated, for those $m_i, i = 1, 2, \cdots, s$ satisfying $\overline{B}_X(m_i) > B_X(m_i)$, we have $m_i > 0$ and $\overline{B}_X(m_i) = B_X(m_i-)$. Then there exists a sequence $0 < \delta_i^{(n)} < m_i - m_{i-1}^*$ satisfying $\lim_{n\to\infty}\delta_i^{(n)} = 0$, such that $\lim_{n\to\infty} B_X(m_i - \delta_i^{(n)}) = B_X(m_i-) = \overline{B}_X(m_i)$. For those $m_i, i = 1, 2, \cdots, s$ satisfying $\overline{B}_X(m_i) = B_X(m_i)$, we define $\delta_i^{(n)} \equiv 0$. Thus $\lim_{n\to\infty} B_X(m_i - \delta_i^{(n)}) = B_X(m_i) = \overline{B}_X(m_i)$. Therefore, $\lim_{n\to\infty} B_X(m_i - \delta_i^{(n)}) = \overline{B}_X(m_i)$ holds for all $m_i, i = 1, 2, \cdots, s$.

Based on the sequence $\{\delta_i^{(n)}\}$, we define a function as follows,

$$\begin{aligned} f_n(x) = &\sum_{i=0}^{s-1}\{(x-m_i^*)_+ \boldsymbol{I}_{(m_i^*, m_{i+1}-\delta_i^{(n)}]}(x) + (x - m_{i+1} + \delta_i^{(n)})\boldsymbol{I}_{(m_{i+1}-\delta_i^{(n)}, m_{i+1}]}(x)\} \\ &+(x-m_s^*)_+ \in \mathcal{L}. \end{aligned} \tag{43}$$

The function f_n deduces a measure ν_{f_n} satisfying that

$$\nu_{f_n}(\{m_i\}) = \delta_i^{(n)}, \quad \nu_{f_n}(\{m_i - \delta_i^{(n)}\}) = m_i - m_{i-1}^* - \delta_i^{(n)},$$

and $\nu_{f_n}(B) = \nu_{f^*}(B)$ for any $B \in \overline{R}_+/\{m_i, m_i - \delta_i^{(n)};\ i = 1, 2, \cdots, s\}$. Thus,

$$\begin{aligned} &\int_{\overline{R}_+} \overline{B}_X d\nu_{f^*} - \int_{\overline{R}_+} B_X d\nu_{f_n} \\ =& \sum_{i=1}^{s}\left\{\overline{B}_X(m_i)(m_i - m_{i-1}^*) + \int_{m_i}^{m_i^*} \overline{B}_X d\mu\right\} \\ &- \sum_{i=1}^{s}\left\{\overline{B}_X(m_i)\delta_i^{(n)} + B_X(m_i - \delta_i^{(n)})(m_i - m_{i-1}^* - \delta_i^{(n)}) + \int_{m_i}^{m_i^*} \overline{B}_X d\mu\right\} \\ =& \sum_{i=1}^{s}\{(\overline{B}_X(m_i) - B_X(m_i - \delta_i^{(n)}))(m_i - m_{i-1}^* - \delta_i^{(n)}) + \delta_i^{(n)}(\overline{B}_X(m_i) - B_X(m_i))\} \\ \to& 0 \end{aligned}$$

as $n \to \infty$. Equation (41) is proved.

(b) We will prove Equation (42) in this part.

Choosing i such that $B_X(m_i) < \overline{B}_X(m_i)$, we define

$$\widehat{B}_X(t) = \overline{B}_X(t)\boldsymbol{I}_{\{t\neq m_i\}} + B_X(t)\boldsymbol{I}_{\{t=m_i\}}.$$

It is obvious that $\widehat{B}_X(t) \geq B_X(t), t \in [0, \infty)$, therefore

$$\int_{[0,\infty)} B_X(t)d\nu_f \leq \int_{[0,\infty)} \widehat{B}_X(t)d\nu_f \tag{44}$$

holds for each $f \in \mathcal{L}$. Thus, if we can prove the following inequality

$$\int_{[0,\infty)} \widehat{B}_X(t)d\nu_f < \int_{[0,\infty)} \overline{B}_X(t)d\nu_{f^*}, \tag{45}$$

Equation (42) can be obtained. In the following we will prove Equation (45).

On the interval $[0, m^*_{i-1}]$, we define

$$\nu_{f_i}(B) = \nu_f(B) + \boldsymbol{I}_{\{m^*_{i-1} \in B\}}(m^*_{i-1} - \nu_f[0, m^*_{i-1}]), \quad B \in \mathcal{B}_{\overline{R}^+}.$$

It is easy to prove that the associated f_i belongs to $\mathcal{L}$. Thus, for $h(t)$ defined in Example 3.5, we have

$$\int_{[0,m^*_{i-1}]} h(t)d\nu_{f_i} = \int_{[0,m^*_{i-1}]} \overline{B}_X(t)d\nu_f + B_X(m^*_{i-1}-)(m^*_{i-1} - \nu_f([0, m^*_{i-1}])). \tag{46}$$

From Equations (37) and (46), we have

$$\begin{aligned}\int_{[0,m^*_{i-1}]} \overline{B}_X(t)d\nu_f \leq \sum_{k=1}^{i-1} &\left\{ \overline{B}_X(m_k)(m_k - m^*_{k-1}) + \int_{m_k}^{m^*_k} \overline{B}_X(t)dt \right\} \\ &-B_X(m^*_{i-1}-)(m^*_{i-1} - \nu_f([0, m^*_{i-1}])).\end{aligned} \tag{47}$$

Next consider the interval (m_i, ∞). For $h(t)$ defined in Example 3.6, we have

$$\int_{[0,\infty)} h(t)d\nu_f = \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}\nu_f([0, m_i]) + \int_{(m_i,\infty)} B_X(t)d\nu_f. \tag{48}$$

Based on Equations (39) and (48), we can obtain

$$\begin{aligned}\int_{(m_i,\infty)} \overline{B}_X(t)d\nu_f &\leq \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0, m_i])) \\ &+\sum_{k=i}^{s-1} \overline{B}_X(m_{k+1})(m_{k+1} - m^*_k) + \sum_{k=i}^{s} \int_{m_k}^{m^*_k} \overline{B}_X(t)dt.\end{aligned} \tag{49}$$

Then combining Equations (47) and (49), we have

$$\begin{aligned}&\int_{[0,\infty)} \widehat{B}_X(t)d\nu_f \\ &= \int_{[0,m^*_{i-1}]} \overline{B}_X(t)d\nu_f + \int_{(m^*_{i-1},m_i]} B_X(t)d\nu_f + \int_{(m_i,\infty)} \overline{B}_X(t)d\nu_f \\ &\leq \sum_{k=1}^{i-1} \overline{B}_X(m_k)(m_k - m^*_{k-1}) + \sum_{k=1}^{i-1} \int_{m_k}^{m^*_k} \overline{B}_X(t)dt - B_X(m^*_{i-1}-)(m^*_{i-1} - \nu_f([0, m^*_{i-1}])) \\ &\quad + \int_{(m^*_{i-1},m_i]} B_X(t)d\nu_f + \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0, m_i])) \\ &\quad + \sum_{k=i}^{s-1} \overline{B}_X(m_{k+1})(m_{k+1} - m^*_k) + \sum_{k=i}^{s} \int_{m_k}^{m^*_k} \overline{B}_X(t)dt.\end{aligned} \tag{50}$$

Using $\overline{B}_X(m_i) \ge B_X(m_{i-1}^*-)$, Equations (22) and (50) imply that

$$\begin{aligned}
&\int_{[0,\infty)} \widehat{B}_X(t)d\nu_f - \int_{[0,\infty)} \overline{B}_X(t)d\nu_{f^*} \\
&\le -\overline{B}_X(m_i)(m_i - m_{i-1}^*) - B_X(m_{i-1}^*-)(m_{i-1}^* - \nu_f([0,m_{i-1}^*])) \\
&\quad + \int_{(m_{i-1}^*,m_i]} B_X(t)d\nu_f + \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0,m_i])) \\
&\le -\overline{B}_X(m_i)(m_i - \nu_f([0,m_{i-1}^*])) + \int_{(m_{i-1}^*,m_i]} B_X(t)d\nu_f \\
&\quad + \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0,m_i])). \qquad (51)
\end{aligned}$$

From Lemma 2.5, we have $\overline{B}_X(m_i) > \overline{B}_X(m_{i+1})$ and $\overline{B}_X(m_i) \ge B_X(t)$ for any $t \in [m_{i-1}^*, m_i]$. Meanwhile, from the definition of m_i, we know there exists a $0 < \delta < m_i - m_{i-1}^*$ such that $\overline{B}_X(m_i) > B_X(t)$ holds for any $t \in (m_i - \delta, m_i]$. Therefore, from Equation (51), we obtain

$$\begin{aligned}
&-\overline{B}_X(m_i)(m_i - \nu_f([0,m_{i-1}^*])) + \int_{(m_{i-1}^*,m_i]} B_X(t)d\nu_f \\
&+ \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0,m_i])) \\
= &-\overline{B}_X(m_i)(m_i - \nu_f([0,m_i-\delta])) + \int_{(m_i-\delta,m_i]} B_X(t)d\nu_f \\
&+ \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0,m_i])) \\
&+\left\{ - \overline{B}_X(m_i)\nu_f((m_{i-1}^*, m_i-\delta]) + \int_{(m_{i-1}^*,m_i-\delta]} B_X(t)d\nu_f \right\} \\
\le &-\overline{B}_X(m_i)(m_i - \nu_f([0,m_i-\delta])) + \int_{(m_i-\delta,m_i]} B_X(t)d\nu_f \\
&+ \max\{B_X(m_i), \overline{B}_X(m_{i+1})\}(m_i - \nu_f([0,m_i])) \\
= &\int_{(m_i-\delta,m_i]} (B_X(t) - \overline{B}_X(m_i))d\nu_f \\
&+(\max\{B_X(m_i), \overline{B}_X(m_{i+1})\} - \overline{B}_X(m_i))(m_i - \nu_f([0,m_i])) \\
:= &II. \qquad (52)
\end{aligned}$$

Note that one of $\nu_f((m_i-\delta, m_i])$ and $m_i - \nu_f([0,m_i])$ must be greater than 0, otherwise we will have $\nu_f([0,m_i-\delta]) = m_i$ and $f(m_i-\delta-) = m_i - \delta - \nu_f([0,m_i-\delta]) = -\delta < 0$, contradictory to the fact $f \in \mathcal{L}$. Moreover, $B_X(t) < \overline{B}_X(m_i), t \in (m_i-\delta, m_i]$ and $\max\{B_X(m_i), \overline{B}_X(m_{i+1})\} < \overline{B}_X(m_i)$. Thus $II < 0$ follows. Then from Equations (51) and (52), we can obtain

$$\int_{[0,\infty)} \widehat{B}_X(t)d\nu_f - \int_{[0,\infty)} \overline{B}_X(t)d\nu_{f^*} \le II < 0.$$

Thus the inequality equation (45) is proved. ∎

4 Conclusions

This paper discussed the optimal reinsurance treaty for the insurer when the insurer's risk is measured by distortion risk measure and the reinsurance premium is calculated by the expected premium principle. Under the assumption that the retained loss is increasing with the initial loss, explicit solutions for the optimal reinsurance problems were obtained. The optimal reinsurance strategies for the two special risk measures, VaR and TVaR, were also presented as special cases of the distortion risk measures.

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5 Appendix

Proof of Lemma 2.3 First we consider the case that $i = 1$ with $m_1 = 0$. When $r_1 = 0$, it is easily to get $B_X(0) = M_1$. Thus, in the following we will focus on the case $r_1 > 0$. From the definitions of m_1 and r_1, we know $\max\{B_X(t), B_X(t-)\} = M_1, t \in (m_1, r_1)$. Then we can find a sequence $m_1 < x_n < r_1, n = 1, 2, \cdots, \infty$ of the continuity points of $B_X(t)$, such that $x_n \searrow m_1$ as $n \to \infty$ and $\max\{B_X(x_n), B_X(x_n-)\} = M_1$. By the continuity of the function $B_X(t)$ at $t = x_n$, we have $B_X(x_n) = M_1$. Then based on the right continuity of $B_X(t)$, we have

$B_X(0) = B_X(m_1) = \lim_{x_n \searrow m_1} B_X(x_n) = M_1$.

Next we will discuss the case $i = 1$ with $m_1 > 0$ and $i = 2, 3, \cdots, s$.

1) First, we will prove that $\max\{B_X(r_i), B_X(r_i-)\} = M_i$.

From the definition of M_i, we know that

$$B_X(r_i) \le M_i \text{ and } B_X(r_i-) \le M_i. \tag{53}$$

If $\max\{B_X(t), B_X(t-)\} = M_i$ has finitely many solutions, r_i can be rewritten as $r_i = \max\{t : \max\{B_X(t), B_X(t-)\} = M_i\}$, then $\max\{B_X(r_i), B_X(r_i-)\} = M_i$ is obvious. In the following, we assume that the equation $\max\{B_X(t), B_X(t-)\} = M_i$ has infinitely many solutions. In this case, we can find a sequence $\{t_n < r_i\}_{n=1}^{\infty}$, such that $t_n \nearrow r_i$ as $n \to \infty$ and $\max\{B_X(t_n), B_X(t_n-)\} = M_i$. We will prove $\max\{B_X(r_i), B_X(r_i-)\} = M_i$ by considering the following two cases:

(i) Case 1: There exists an infinite subsequence $\{t_n^{(1)}\}_{n=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$, such that $t_n^{(1)} \nearrow r_i$ and $B_X(t_n^{(1)}) = M_i$. Thus $B_X(r_i-) = \lim_{t_n^{(1)} \nearrow r_i} B_X(t_n^{(1)}) = M_i$. Then we can conclude that $\max\{B_X(r_i), B_X(r_i-)\} = M_i$ by Equation (53).

(ii) Case 2: There exists an infinite subsequence $\{t_n^{(2)}\}_{n=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$, which satisfies $t_n^{(2)} \nearrow r_i$ and $B_X(t_n^{(2)}-) = M_i$. Similar to the proof in (i), we can derive that $\max\{B_X(r_i), B_X(r_i-)\} = M_i$.

2) Second, we will prove that $\max\{B_X(m_i), B_X(m_i-)\} = M_i$.

When $m_i = r_i$, the equation $\max\{B_X(m_i), B_X(m_i-)\} = M_i$ is trivial. In the following, we consider the case $m_i < r_i$. By the definitions of m_i and M_i, we have $\max\{B_X(m_i), B_X(m_i-)\} \le M_i$. From the definition of m_i, $\max\{B_X(t), B_X(t-)\} = M_i, t \in (m_i, r_i]$ follows. Similar to the proof for the case $m_1 = 0$, we can find a sequence $\{m_i < y_n \le r_i\}_{n=1}^{\infty}$ of the continuity points of $B_X(t)$, such that $y_n \searrow m_i$ when $n \to \infty$ and $B_X(y_n) = M_i$. Then, based on the right continuity of $B_X(t)$, we have $B_X(m_i) = \lim_{y_n \searrow m_i} B_X(y_n) = M_i$. Therefore $\max\{B_X(m_i), B_X(m_i-)\} = M_i$ follows. ∎

Proof of Lemma 2.5 Let $0 \le i \le s$. Based on the definitions of m_i, m_i^* and the right-continuity of $B_X(t)$, we can verify that $B_X(t) > \overline{B}_X(m_{i+1}), t \in (m_i, m_i^*)$ and $B_X(t) \le \overline{B}_X(m_{i+1}), t \in [m_i^*, m_{i+1}]$. In the following, we will prove that $B_X(t)$ is decreasing on (m_i, m_i^*).

Based on Lemma 2.2, we know there exists a finite partition of (m_i, m_i^*) such that $B_X(t)$ is monotonic on each subinterval. If $B_X(t)$ is not decreasing on (m_i, m_i^*), there exists an interval $(\eta_1, \theta_1) \in (m_i, m_i^*)$ such that $B_X(t)$ is increasing on (η_1, θ_1) and $B_X(t)$ is not a constant on the interval. Define

$$\eta_2 = \sup\{s \le m_i^* : B_X(t) \text{ is increasing on } (\eta_1, s)\}.$$

1) For the case that $\eta_2 = m_i^*$, the function $B_X(t-)$ is increasing on $(\eta_1, m_i^*]$ and

$$B_X(m_i^*-) = \lim_{s \nearrow m_i^*} B_X(s) \ge B_X(t), \quad t \in (\eta_1, m_i^*).$$

Therefore, for any $t \in (\eta_1, m_i^*)$ we have

$$\max\{B_X(m_i^*), B_X(m_i^*-)\} \ge B_X(m_i^*-) \ge \max\{B_X(t), B_X(t-)\}.$$

Meanwhile,

$$\begin{aligned}\max\{B_X(m_i^*), B_X(m_i^*-)\} &\geq B_X(m_i^*-)\\ &> \overline{B}_X(m_{i+1})\\ &= \max\{B_X(m_{i+1}), B_X(m_{i+1}-)\}\\ &\geq \max\{B_X(t), B_X(t-)\}\end{aligned}$$

for any $t \in (m_i^*, m_{i+1})$. Then $\max\{B_X(m_i^*), B_X(m_i^*-)\}$ is a local maximum value bigger than $\max\{B_X(m_{i+1}), B_X(m_{i+1}-)\}$, contradictory to the definition of m_{i+1}.

2) For the case $\eta_2 < m_i^*$, we define $\eta_3 = \sup\{s \leq m_i^* : B_X(t)$ is decreasing on $[\eta_2, s)\}$. Based on Lemma 2.2, we have $\eta_3 > \eta_2$. Note that $B_X(t), t \in (\eta_1, \eta_2)$ is increasing and $B_X(t), t \in [\eta_2, \eta_3)$ is decreasing, then $\max\{B_X(\eta_2), B_X(\eta_2-)\}$ is a local maximum value of $\max\{B_X(t), B_X(t-)\}$. Based on the fact that $B_X(t) > \max\{B_X(m_{i+1}), B_X(m_{i+1}-)\}, t \in (m_i, m_i^*)$, we know

$$\max\{B_X(\eta_2), B_X(\eta_2-)\} > \max\{B_X(m_{i+1}), B_X(m_{i+1}-)\},$$

contradictory to the definition of m_{i+1}.

From the consideration above, the function $B_X(t)$ is decreasing on (m_i, m_i^*). The proof of Lemma 2.5 is completed. ∎