STABILIZATION OF COUPLED PDE-ODE SYSTEMS WITH SPATIALLY VARYING COEFFICIENT*

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Abstract The paper is concerned with the stabilization of a class of coupled PDE-ODE systems with spatially varying coefficient, via state-feedback or output-feedback. The system is more general than that of the related literature due to the presence of the spatially varying coefficient which makes the problem more difficult to solve. By infinite-dimensional backstepping method, both state-feedback and output-feedback stabilizing controllers are explicitly constructed, which guarantee that the closed-loop system is exponentially stable in the sense of certain norm. It is worthwhile pointing out that, in the case of output-feedback, by appropriately choosing the state observer gains, the severe restriction on the ODE sub-system in the existing results is completely removed. A simulation example is presented to illustrate the effectiveness of the proposed method.

Key words Coupled PDE-ODE systems, infinite-dimensional backstepping transformation, spatially varying coefficient, stabilization.

1 Introduction

In this paper, the stabilization is considered for the following coupled systems consisting of an ordinary differential equation (ODE) system and a parabolic partial differential equation (PDE):

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \\ y(t) = CX(t), \\ u_t(x,t) = u_{xx}(x,t) + \lambda(x)y(t), \\ u_x(0,t) = 0, \\ u(D,t) = U(t), \end{cases} \tag{1}$$

where $X(t) \in \boldsymbol{R}^n$ and $u(x,t)$ with the initial values $X(0) = X_0$ and $u(x,0) = u_0(x)$ are the vector state and scalar state of the ODE sub-system and the PDE sub-system, respectively;

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$y(t) \in \boldsymbol{R}$ is the output of the ODE sub-system; $U(t)$ is the scale input to the entire system; $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$; $A \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times 1}$, and $C \in \boldsymbol{R}^{1\times n}$ are known constant matrices and the pair $(A,\, B)$ is stabilizable; $\lambda(x)$ is a continuous function defined on $[0,\, D]$; D is an arbitrary positive constant which denotes the length of the PDE domain.

From coupled equations (1), we see that, the output of the PDE sub-system (i.e., $u(0,t)$) acts as the input of the ODE sub-system, and the output of the ODE sub-system (i.e., $y(t)$), affects the PDE sub-system in $[0,\, D]$ with a specified influence function $\lambda(x)$. Coupled equations (1) can also be viewed as a nontrivial extension of those in [1] and [2] where boundary controller acts on the plant through a diffusion-dominated actuator and the plant does not affect the diffusion equation (i.e., $\lambda(x) \equiv 0$). In fact, the action from the plant to the actuator cannot be avoided or ignored sometimes in practice (i.e., interaction exists in the coupled equations), and if no relevant treatment is offered, the performance of the closed-loop system would become unexpected. Therefore, it is worthy of studying how to design control for the coupled equations (1), and simultaneously to effectively eliminate the negative effect caused by the interaction.

The controls of coupled PDE-ODE systems have attracted continuous attention (see e.g., [3–18] and the references therein), and recently, the stabilization for a system of ODE coupled with parabolic PDE has received investigation (see [3] and [4]). Quite different from System (1), in [3], the input of the ODE sub-system is the Neumann boundary value (i.e., $u_x(0,t)$) of the PDE sub-system, rather than the Dirichlet boundary value (i.e., $u(0,t)$), and the action from the ODE sub-system to the PDE one only takes effect at one end of the PDE domain, rather than inside the domain. This shows that some steps should be taken to prevent PDE sub-system from being affected by the ODE one inside the domain. Moreover, just as System (1), the ODE sub-system affects the PDE one inside the domain in [4], but the action is identical in the whole domain (i.e., $\lambda(x) \equiv 1$), which will exclude many cases from practice. More generally, in this paper, the action from ODE sub-system to PDE one has a spatially varying coefficient $\lambda(x)$ which clearly includes the studied case $\lambda(x) \equiv 1$ in the literature and causes more difficulties in control design and performance analysis.

In this paper, both state-feedback and output-feedback stabilizing controllers are proposed for the coupled System (1). First, by introducing an infinite-dimensional backstepping transformation, the state-feedback controller is constructed explicitly and the original closed-loop system is changed into a stable target system whose stability implies that of the original closed-loop system in the same sense. Then, when only the PDE sub-system output $u(0,t)$ is available for feedback, a state observer is designed by the infinite-dimensional backstepping method. Based on the observation of system states and the state-feedback controller designed, the output-feedback controller is constructed with the help of separation principle, which guarantees the desirable stability of the closed-loop system. It is worthwhile emphasizing that the presence of the spatially varying coefficient $\lambda(x)$ makes the controller parameters (i.e., kernel functions of the infinite-dimensional backstepping transformations) can not be explicitly derived by the method in [4], and hence makes the stabilization of System (1) more difficult to solve. Moreover, the restriction on the system matrix A in [4] is completely removed for the case of output-feedback by choosing appropriate observer gains.

The reminder of the paper is organized as follows. Sections 2 and 3 present the state-feedback and output-feedback control design, respectively. Section 4 provides a numerical simulation to illustrate the effectiveness of the proposed method. Section 5 gives the concluding remarks. The paper ends with an appendix which collects useful inequalities and the proofs of important propositions.

2 State-Feedback Control Design

In this section, state-feedback control design for System (1) is presented. Motivated by [4], an infinite-dimensional backstepping transformation is found to change System (1) into a new stable target system whose stability implies the stability of the original closed-loop system in the same sense. However, the presence of the spatially varying coefficient $\lambda(x)$ makes the kernel equations can not be solved by the existing methods.

For System (1), we adopt the following infinite-dimensional backstepping transformation:

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy - \gamma(x)X(t), \tag{2}$$

where kernel functions $k(x,y)$ and $\gamma(x)$ will be determined later. With appropriate kernel functions, System (1) can be changed into the following stable system (see Theorem 1 of [4]) under the above transformation:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t), \\ w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = 0, \\ w(D,t) = 0, \end{cases} \tag{3}$$

with $K \in \boldsymbol{R}^{1\times n}$ such that $A + BK$ is Hurwitz, from which, it is more convenient to analyze the stability of the original closed-loop system. Once the desirable transformation is obtained, by (2) and the fourth equation of (3), we obtain the following controller:

$$U(t) = \int_0^D k(D,y)u(y,t)dy + \gamma(D)X(t). \tag{4}$$

To derive the desirable kernel functions $k(x,y)$ and $\gamma(x)$, a sufficient condition will be found to guarantee that original System (1) with control (4) in loop can be transformed to System (3) under transformation (2), which is given by the following proposition.

Proposition 1 *The sufficient condition, which guarantees that System* (1) *can be changed into System* (3) *under transformation* (2)*, is that kernel functions* $\gamma(x)$ *and* $k(x,y)$ *should respectively satisfy*

$$\begin{cases} \gamma(x)'' - \gamma(x)A - \int_0^x \int_0^{x-y} \gamma(\xi)Bd\xi\lambda(y)Cdy + \lambda(x)C = 0, \\ \gamma(0)' = 0, \\ \gamma(0) = K, \end{cases} \tag{5}$$

(*the above equations are called kernel equations*) *and*

$$k(x,y) = \int_0^{x-y} \gamma(\xi)Bd\xi, \tag{6}$$

where $\gamma(x)' = \frac{d\gamma(x)}{dx}$, $\gamma(x)'' = \frac{d\gamma(x)'}{dx}$.

Proof See Section B of Appendix in the paper. ∎

From the above two equations, we see that once the desirable $\gamma(x)$ is specified from (5), the desirable $k(x,y)$ will be obtained from (6). However, Equation (5) can not be solved explicitly by the methods in [4] due to the presence of the spatially varying coefficient $\lambda(x)$

which makes the equations being non-homogeneous integro-differential equations with spatially varying coefficient. By the method of successive approximation, the explicit solution of (5) is obtained in the form of the infinite series.

Proposition 2 *The kernel Equation* (5) *has the following unique solution:*

$$\gamma(x) = \sum_{i=0}^{+\infty} \gamma_i(x), \tag{7}$$

with $\gamma_0(x) = K - \int_0^x \int_0^\eta \lambda(\xi) C d\xi d\eta$,

$$\gamma_{i+1}(x) = \int_0^x \int_0^\eta \gamma_i(\xi) A d\xi d\eta + \frac{1}{2}\int_0^x \int_0^{x-y} (x-y-\xi)^2 \gamma_i(\xi) B d\xi \lambda(y) C dy,\ i = 0, 1, \cdots. \tag{8}$$

Moreover, there exists a positive constant M_1 *such that* $\gamma(x)$ *and* $k(x,y)$ *have the following properties:*

$$\begin{cases} \sup\limits_{x\in[0,\,D]} \|\gamma(x)\| \le M_1, & \sup\limits_{x\in[0,\,D]} \|\gamma(x)'\| \le M_1, \\ \sup\limits_{x\in[0,\,D], y\in[0,\,D]} |k(x,y)| \le M_1, & \sup\limits_{x\in[0,\,D], y\in[0,\,D]} |k_x(x,y)| \le M_1, \end{cases} \tag{9}$$

where $\|\cdot\|$ *denotes the Euclidean norm for column vectors, or the corresponding induced norm for row vectors or matrices.*

Proof See Section C of Appendix in the paper. ▮

It is necessary to point out that an inverse backstepping transformation exists for (2):

$$u(x,t) = w(x,t) + \int_0^x l(x,y) w(y,t) dy + \beta(x) X(t), \tag{10}$$

(which will be used in the stability analysis of the closed-loop system) where

$$\begin{cases} \beta(x) \ = K\,[I \quad 0] \exp\left(\begin{bmatrix} 0 & A+BK \\ I & 0 \end{bmatrix} x\right) [I \quad 0]^{\mathrm{T}} \\ \qquad\qquad + \displaystyle\int_0^x [0 \quad -\lambda(\tau) C] \exp\left(\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} (x-\tau)\right) [I \quad 0]^{\mathrm{T}}\, d\tau, \\ l(x,y) = \displaystyle\int_0^{x-y} \beta(\xi) B d\xi. \end{cases}$$

From the above two equations, it can be verified that kernel functions $\beta(x)$ and $l(x,y)$ have the following properties:

$$\begin{cases} \sup\limits_{x\in[0,\,D]} \|\beta(x)\| \le M_2, & \sup\limits_{x\in[0,\,D]} \|\beta(x)'\| \le M_2, \\ \sup\limits_{x\in[0,\,D], y\in[0,\,D]} |l(x,y)| \le D\|B\| M_2, & \sup\limits_{x\in[0,\,D], y\in[0,\,D]} |l_x(x,y)| \le \|B\| M_2, \end{cases} \tag{11}$$

where $M_2 = \exp\left(D \max\{1,\, \|A+BK\|\}\right) \left(\|K\| + nD\|C\| \max_{x\in[0,\,D]} |\lambda(x)|\right)$.

Remark 1 The inverse backstepping transformation (10) guarantees that target system (3) can be changed into the original closed-loop System (1) and (4). This can be verified by substituting (10) into (1) and using (3). Then, kernel functions $l(x,y)$ and $\beta(x)$ should respectively satisfy

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = 0, \\ l(x,x) = 0, \\ l_y(x,0) = -\beta(x) B, \end{cases}$$

and

$$\begin{cases} \beta(x)'' - \beta(x)(A+BK) + \lambda(x)C = 0, \\ \beta(0)' = 0, \\ \beta(0) = K. \end{cases}$$

In view of the proof of Theorem 1 in [4], target system (3) is exponentially stable in the sense of norm $\left(\|X(t)\|^2 + \int_0^D w(x,t)^2 dx + \int_0^D w_x(x,t)^2 dx\right)^{\frac{1}{2}}$, from which, it can be concluded the stability of the closed-loop system in the same sense. This is summarized in the following theorem.

Theorem 1 *For any initial condition X_0 and $u_0(x)$ satisfying $\int_0^D u_0(x)^2 dx < +\infty$ and $\int_0^D \left(\frac{du_0(x)}{dx}\right)^2 dx < +\infty$, the closed-loop system consisting of* (1) *and* (4) *is exponentially stable in the sense of norm* $\left(\|X(t)\|^2 + \int_0^D u(x,t)^2 dx + \int_0^D u_x(x,t)^2 dx\right)^{\frac{1}{2}}$.

Proof From (10), we have

$$\begin{cases} u(x,t)^2 \le 3w(x,t)^2 + 3\int_0^x l(x,y)^2 dy \int_0^x w(y,t)^2 dy + 3\|\beta(x)\|^2\|X(t)\|^2, \\ u_x(x,t)^2 \le 3w_x(x,t)^2 + 3\int_0^x l_x(x,y)^2 dy \int_0^x w(y,t)^2 dy + 3\|\beta(x)'\|^2\|X(t)\|^2. \end{cases}$$

Integrating both sides of the above inequalities over $[0,\ D]$ and noting $0 \le x \le D$, there hold

$$\begin{cases} \int_0^D u(x,t)^2 dx \le 3\left(1 + \int_0^D \int_0^x l(x,y)^2 dy dx\right) \int_0^D w(x,t)^2 dx \\ \qquad +3\int_0^D \|\beta(x)\|^2 dx \|X(t)\|^2, \\ \int_0^D u_x(x,t)^2 dx \le 3\int_0^D w_x(x,t)^2 dx + 3\int_0^D \int_0^x l_x(x,y)^2 dy dx \int_0^D w(x,t)^2 dx \\ \qquad +3\int_0^D \|\beta(x)'\|^2 dx \|X(t)\|^2, \end{cases} \tag{12}$$

which, together with (11), yields

$$\begin{aligned} &\|X(t)\|^2 + \int_0^D u(x,t)^2 dx + \int_0^D u_x(x,t)^2 dx \\ &\le 3\left(1 + \int_0^D \int_0^x \left(l(x,y)^2 + l_x(x,y)^2\right) dy dx\right) \int_0^D w(x,t)^2 dx + 3\int_0^D w_x(x,t)^2 dx \\ &\quad + \left(1 + 3\int_0^D \left(\|\beta(x)\|^2 + \|\beta(x)'\|^2\right) dx\right) \|X(t)\|^2 \\ &\le \frac{1}{\underline{\delta}} E(t), \end{aligned} \tag{13}$$

where $\underline{\delta} = 1/\max\{3 + 3D^2M_2^2\|B\|^2(D^2+1),\ 1 + 6DM_2^2\}$ and $E(t) = \|X(t)\|^2 + \int_0^D w(x,t)^2 dx + \int_0^D w_x(x,t)^2 dx$.

From (2), by the similar way in deriving (12), we have

$$\begin{cases} \int_0^D w(x,t)^2dx \leq 3\Big(1+\int_0^D\int_0^x k(x,y)^2dydx\Big)\int_0^D u(x,t)^2dx + 3\int_0^D \|\gamma(x)\|^2dx\|X(t)\|^2, \\ \int_0^D w_x(x,t)^2dx \leq 3\int_0^D u_x(x,t)^2dx + 3\int_0^D\int_0^x k_x(x,y)^2dydx\int_0^D u(x,t)^2dx \\ \qquad\qquad +3\int_0^D \|\gamma(x)'\|^2dx\|X(t)\|^2, \end{cases}$$

which, together with (9), yields

$$\begin{aligned} E(t) &\leq 3\Big(1+\int_0^D\int_0^x \big(k(x,y)^2+k_x(x,y)^2\big)\,dydx\Big)\int_0^D u(x,t)^2dx + 3\int_0^D u_x(x,t)^2dx \\ &\quad +\Big(1+3\int_0^D \big(\|\gamma(x)\|^2+\|\gamma(x)'\|^2\big)\,dx\Big)\|X(t)\|^2 \\ &\leq \overline{\delta}\Big(\|X(t)\|^2+\int_0^D u(x,t)^2dx+\int_0^D u_x(x,t)^2dx\Big), \end{aligned}$$

where $\overline{\delta}=\max\big\{3+6D^2M_1^2,\,1+6DM_1^2\big\}$. This and (13) imply that

$$\begin{aligned} &\underline{\delta}\Big(\|X(t)\|^2+\int_0^D u(x,t)^2dx+\int_0^D u_x(x,t)^2dx\Big) \\ &\leq E(t) \\ &\leq \overline{\delta}\Big(\|X(t)\|^2+\int_0^D u(x,t)^2dx+\int_0^D u_x(x,t)^2dx\Big). \end{aligned} \tag{14}$$

By the aforementioned exponential stability of the target system (3), there exists a positive constant ε_1 such that $E(t)\leq E(0)\mathrm{e}^{-\varepsilon_1 t}$. Hence, by (14), there holds

$$\begin{aligned} &\|X(t)\|^2+\int_0^D u(x,t)^2dx+\int_0^D u_x(x,t)^2dx \\ &\leq \frac{\overline{\delta}}{\underline{\delta}}\Big(\|X_0\|^2+\int_0^D u_0(x)^2dx+\int_0^D\Big(\frac{du_0(x)}{dx}\Big)^2dx\Big)\mathrm{e}^{-\varepsilon_1 t}, \end{aligned}$$

which implies the original System (1) with controller (4) in loop is exponentially stable in the sense of norm $\big(\|X(t)\|^2+\int_0^D u(x,t)^2dx+\int_0^D u_x(x,t)^2dx\big)^{\frac{1}{2}}$. This completes the proof. ∎

3 Output-Feedback Control Design

In this section, output-feedback control design is presented for System (1) when only the output $u(0,t)$ of the system is available for feedback. Specifically, a state observer is first constructed by infinite-dimensional backstepping method, based on which observation for the states of the system are obtained. Then, an output-feedback controller for System (1) is constructed by using the separation principle, which ensures the desirable stability of the closed-loop system. It is necessary to point out that, by choosing appropriate observer gains, the restriction on the system matrix A in [4] is completely removed.

The state observer is constructed as follows:

$$\begin{cases} \dot{\widehat{X}}(t) = A\widehat{X}(t) + Bu(0,t) + L(u(0,t) - \widehat{u}(0,t)), \\ \widehat{y}(t) = C\widehat{X}(t), \\ \widehat{u}_t(x,t) = \widehat{u}_{xx}(x,t) + \lambda(x)\widehat{y}(t) + p(x)(u(0,t) - \widehat{u}(0,t)), \\ \widehat{u}_x(0,t) = 0, \\ \widehat{u}(D,t) = U(t), \end{cases} \tag{15}$$

where $\widehat{X}(t) \in \boldsymbol{R}^n$ and $\widehat{u}(x,t)$ with their initial values $\widehat{X}_0$ and $\widehat{u}_0(x)$ are the vector state and scalar state, respectively; $\widehat{y}(t)$ is the output of the ODE sub-system; $L \in \boldsymbol{R}^n$ such that the pair $(A,\, L)$ is stabilizable and $p(x) : [0,\, D] \to \boldsymbol{R}$ will be determined later. Once the desirable $p(x)$ is specified, we obtain the reconstruction of the unobservable states of System (1) with the following observation errors:

$$\widetilde{X}(t) = X(t) - \widehat{X}(t), \quad \widetilde{u}(x,t) = u(x,t) - \widehat{u}(x,t),$$

which satisfy the following equations (called error system):

$$\begin{cases} \dot{\widetilde{X}}(t) = A\widetilde{X}(t) - L\widetilde{u}(0,t), \\ \widetilde{u}_t(x,t) = \widetilde{u}_{xx}(x,t) + \lambda(x)C\widetilde{X}(t) - p(x)\widetilde{u}(0,t), \\ \widetilde{u}_x(0,t) = 0, \\ \widetilde{u}(D,t) = 0. \end{cases} \tag{16}$$

Next, we will search for the desirable $p(x)$ which guarantees that the above error system is stable in the sense of certain norm. For this, we introduce the following transformation:

$$\widetilde{w}(x,t) = \widetilde{u}(x,t) - q(x)\widetilde{X}(t), \tag{17}$$

where

$$\begin{aligned} q(x) = {} & K_1 \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x \right) \begin{bmatrix} I & 0 \end{bmatrix}^{\mathrm{T}} \\ & + \int_0^x \begin{bmatrix} 0 & -\lambda(\tau)C \end{bmatrix} \exp\left(\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} (x-\tau) \right) \begin{bmatrix} I & 0 \end{bmatrix}^{\mathrm{T}} d\tau, \end{aligned} \tag{18}$$

with $K_1 \in \boldsymbol{R}^{1\times n}$ such that $A - LK_1$ is Hurwitz. From the above equation, we can see that $q(x)$ satisfies:

$$\begin{cases} q(x)'' - q(x)A + \lambda(x)C = 0, \\ q(0)' = 0, \\ q(0) = K_1, \end{cases} \tag{19}$$

and

$$\sup\nolimits_{x\in[0,\,D]} \|q(x)\| \le M_3, \qquad \sup\nolimits_{x\in[0,\,D]} \|q(x)'\| \le M_3, \tag{20}$$

where $M_3 = \exp\left(D\max\{1,\, \|A\|\}\right)\left(\|K_1\| + nD\|C\| \max_{x\in[0,\,D]} |\lambda(x)|\right)$, which will be useful in the later stability analysis of the error system.

Under transformation (17) and by choosing appropriate $p(x)$, error system (16) can be changed into a new system which is given by the following proposition.

Proposition 3 *By choosing $p(x) = q(x)L$, System* (16) *can be changed into the following target system under transformation* (17)*:*

$$\begin{cases} \dot{\widetilde{X}}(t) = (A - LK_1)\widetilde{X}(t) - L\widetilde{w}(0,t), \\ \widetilde{w}_t(x,t) = \widetilde{w}_{xx}(x,t), \\ \widetilde{w}_x(0,t) = 0, \\ \widetilde{w}(D,t) = -q(D)\widetilde{X}(t). \end{cases} \tag{21}$$

Proof See Section D of Appendix in the paper. ∎

It can be proven that System (21) is exponentially stable in the sense of certain norm, which implies the stability of the original observer system (16) in the same sense (i.e., the states of observer system (15) converge to the actual states of System (1) in certain sense). This is summarized in the following theorem.

Theorem 2 *For any initial condition $\widetilde{X}(0)$ and $\widetilde{u}(x,0)$ satisfying $\int_0^D \widetilde{u}(x,0)^2 dx < +\infty$ and $\int_0^D \widetilde{u}_x(x,0)^2 dx < +\infty$, observer* (15) *with gains L chosen such that the pair* $(A,\ L)$ *is stabilizable and $p(x) = q(x)L$ guarantees that error system* (16) *is exponentially stable in the sense of the following norm:*

$$\left(\|\widetilde{X}(t)\|^2 + \int_0^D \widetilde{u}(x,t)^2 dx + \int_0^D \widetilde{u}_x(x,t)^2 dx\right)^{\frac{1}{2}}.$$

Proof We will first prove the stability of target system (21), and then show that of original error system (16). For this, we choose the following Lyapunov function:

$$\widetilde{V}(t) = \widetilde{X}(t)^{\mathrm{T}} P \widetilde{X}(t) + \frac{\gamma}{2}\int_0^D \widetilde{w}(x,t)^2 dx + \frac{1}{2}\int_0^D \widetilde{w}_x(x,t)^2 dx, \tag{22}$$

where γ is a to-be-specified positive constant, $P = P^{\mathrm{T}} > 0$ satisfies the following Lyapunov equation:

$$(A - LK_1)^{\mathrm{T}} P + P(A - LK_1) = -Q,$$

for some to-be-specified $Q = Q^{\mathrm{T}} > 0$.

By computing the time derivative of $\widetilde{V}(t)$ along the solutions of System (21) and using the integration by parts, we have

$$\begin{aligned} \dot{\widetilde{V}}(t) &= \dot{\widetilde{X}}(t)^{\mathrm{T}} P \widetilde{X}(t) + \widetilde{X}(t)^{\mathrm{T}} P \dot{\widetilde{X}}(t) + \gamma \int_0^D \widetilde{w}(x,t)\widetilde{w}_{xx}(x,t)dx + \int_0^D \widetilde{w}_x(x,t)\widetilde{w}_{xt}(x,t)dx \\ &= -\widetilde{X}(t)^{\mathrm{T}} Q \widetilde{X}(t) - 2\widetilde{X}(t)^{\mathrm{T}} PL\widetilde{w}(0,t) + \gamma\widetilde{w}(D,t)\widetilde{w}_x(D,t) - \gamma\int_0^D \widetilde{w}_x(x,t)^2 dx \\ &\quad + \widetilde{w}_x(D,t)\widetilde{w}_t(D,t) - \int_0^D \widetilde{w}_{xx}(x,t)^2 dx. \end{aligned} \tag{23}$$

From (21), we see that $\widetilde{w}(D,t) = -q(D)\widetilde{X}(t)$, and hence $\widetilde{w}_t(D,t) = -q(D)(A - LK_1)\widetilde{X}(t) + q(D)L\widetilde{w}(0,t)$. Then by (23) and Young's Inequality, we get

$$\dot{\widetilde{V}}(t) = -\widetilde{X}(t)^{\mathrm{T}} Q \widetilde{X}(t) - 2\widetilde{X}(t)^{\mathrm{T}} PL\widetilde{w}(0,t) - \gamma q(D)\widetilde{X}(t)\widetilde{w}_x(D,t) - \gamma\int_0^D \widetilde{w}_x(x,t)^2 dx$$

$$
\begin{aligned}
&\quad -q(D)(A-LK_1)\widetilde{X}(t)\widetilde{w}_x(D,t)+q(D)L\widetilde{w}(0,t)\widetilde{w}_x(D,t)-\int_0^D \widetilde{w}_{xx}(x,t)^2dx\\
&\le -\lambda_{\min}(Q)\|\widetilde{X}(t)\|^2+\frac{\gamma}{8(1+4D^2)}\widetilde{w}(0,t)^2+\frac{8(1+4D^2)}{\gamma}\|PL\|^2\|\widetilde{X}(t)\|^2\\
&\quad +\frac{\gamma}{8(1+4D^2)}\widetilde{w}(0,t)^2+\frac{2(1+4D^2)}{\gamma}|q(D)L|^2\widetilde{w}_x(D,t)^2+\frac{1}{\gamma}\widetilde{w}_x(D,t)^2\\
&\quad +\frac{\gamma^3}{4}\|q(D)\|^2\|\widetilde{X}(t)\|^2+\frac{1}{\gamma}\widetilde{w}_x(D,t)^2+\frac{\gamma}{4}\|q(D)(A-LK_1)\|^2\|\widetilde{X}(t)\|^2\\
&\quad -\gamma\int_0^D \widetilde{w}_x(x,t)^2dx-\int_0^D \widetilde{w}_{xx}(x,t)^2dx\\
&= -\left(\lambda_{\min}(Q)-\frac{8(1+4D^2)}{\gamma}\|PL\|^2-\frac{\gamma^3}{4}\|q(D)\|^2-\frac{\gamma}{4}\|q(D)(A-LK_1)\|^2\right)\|\widetilde{X}(t)\|^2\\
&\quad +\frac{\gamma}{4(1+4D^2)}\widetilde{w}(0,t)^2-\gamma\int_0^D \widetilde{w}_x(x,t)^2dx\\
&\quad +\frac{2+2(1+4D^2)|q(D)L|^2}{\gamma}\widetilde{w}_x(D,t)^2-\int_0^D \widetilde{w}_{xx}(x,t)^2dx,
\end{aligned}
\tag{24}
$$

where $\lambda_{\min}(Q)$ denotes the minimum eigenvalue of Q.

By Agmon's Inequality (i.e., Lemma A.3 in Section A of Appendix in the paper) and completing the square, we obtain

$$
\begin{aligned}
\widetilde{w}(0,t)^2 &\le \widetilde{w}(D,t)^2+2\sqrt{\int_0^D \widetilde{w}(x,t)^2dx\int_0^D \widetilde{w}_x(x,t)^2dx}\\
&\le \widetilde{w}(D,t)^2+\int_0^D \widetilde{w}(x,t)^2dx+\int_0^D \widetilde{w}_x(x,t)^2dx.
\end{aligned}
$$

Then, by Poincaré's Inequality (i.e., Lemma A.2 in section A of Appendix in the paper), there holds

$$
\widetilde{w}(0,t)^2 \le (1+2D)\widetilde{w}(D,t)^2+(1+4D^2)\int_0^D \widetilde{w}_x(x,t)^2dx. \tag{25}
$$

Moreover, noting that $\widetilde{w}_x(0,t)=0$, from Poincaré's Inequality and Agmon's Inequality, we have

$$
\int_0^D \widetilde{w}_x(x,t)^2dx \le 4D^2\int_0^D \widetilde{w}_{xx}(x,t)^2dx, \tag{26}
$$

$$
\begin{aligned}
\widetilde{w}_x(D,t)^2 &\le 2\sqrt{\int_0^D \widetilde{w}_x(x,t)^2dx\int_0^D \widetilde{w}_{xx}(x,t)^2dx}\\
&\le 4D\int_0^D \widetilde{w}_{xx}(x,t)^2dx.
\end{aligned}
\tag{27}
$$

Substituting (25) and (27) into (24) yields

$$
\begin{aligned}
\dot{\widetilde{V}}(t) &\le -\left(\lambda_{\min}(Q)-\frac{8(1+4D^2)}{\gamma}\|PL\|^2-\frac{\gamma^3}{4}\|q(D)\|^2-\frac{\gamma}{4}\|q(D)(A-LK_1)\|^2\right)\|\widetilde{X}(t)\|^2\\
&\quad +\frac{\gamma(1+2D)}{4(1+4D^2)}\widetilde{w}(D,t)^2-\frac{3\gamma}{4}\int_0^D \widetilde{w}_x(x,t)^2dx\\
&\quad -\left(1-\frac{2+2(1+4D^2)|q(D)L|^2}{\gamma}\times 4D\right)\int_0^D \widetilde{w}_{xx}(x,t)^2dx.
\end{aligned}
$$

Choosing

$$\gamma > 8D(1+(1+4D^2)M_3^2\|L\|^2), \tag{28}$$

and by (26) and Poincaré's Inequality while noting $\widetilde{w}(D,t)=-q(D)\widetilde{X}(t)$, we have

$$\begin{aligned}
\dot{\widetilde{V}}(t) \le & -\Big(\lambda_{\min}(Q)-\frac{8(1+4D^2)}{\gamma}\|PL\|^2-\frac{\gamma^3}{4}\|q(D)\|^2-\frac{\gamma}{4}\|q(D)(A-LK_1)\|^2\Big)\|\widetilde{X}(t)\|^2 \\
& +\frac{\gamma(1+2D)}{4(1+4D^2)}\widetilde{w}(D,t)^2+\frac{3\gamma}{8D}\widetilde{w}(D,t)^2-\frac{3\gamma}{16D^2}\int_0^D \widetilde{w}(x,t)^2dx \\
& -\frac{1}{4D^2}\Big(1-\frac{2+2(1+4D^2)|q(D)L|^2}{\gamma}\times 4D\Big)\int_0^D \widetilde{w}_x(x,t)^2dx \\
\le & -\Big(\lambda_{\min}(Q)-\frac{8(1+4D^2)}{\gamma}\|PL\|^2-\Big(\frac{\gamma^3}{4}+\frac{\gamma(1+2D)}{4(1+4D^2)}+\frac{3\gamma}{8D}\Big)\|q(D)\|^2 \\
& -\frac{\gamma}{4}\|q(D)(A-LK_1)\|^2\Big)\|\widetilde{X}(t)\|^2-\frac{3\gamma}{16D^2}\int_0^D \widetilde{w}(x,t)^2dx \\
& -\frac{1}{4D^2}\Big(1-\frac{2+2(1+4D^2)|q(D)L|^2}{\gamma}\times 4D\Big)\int_0^D \widetilde{w}_x(x,t)^2dx.
\end{aligned}$$

To make $\dot{\widetilde{V}}$ non-positive, we choose $\lambda_{\min}(Q)>\eta=\frac{8(1+4D^2)}{\gamma}\|PL\|^2+\Big(\frac{\gamma^3}{4}+\frac{\gamma(1+2D)}{4(1+4D^2)}+\frac{3\gamma}{8D}\Big)M_3^2+\frac{\gamma}{4}\|(A-LK_1)\|^2M_3^2$. Then, by (28) and noting $\widetilde{X}(t)^{\mathrm{T}}P\widetilde{X}(t)\le\lambda_{\max}(P)\|\widetilde{X}(t)\|^2$, we have

$$\begin{aligned}
\dot{\widetilde{V}}(t) \le & -\frac{\lambda_{\min}(Q)-\eta}{\lambda_{\max}(P)}\widetilde{X}(t)^{\mathrm{T}}P\widetilde{X}(t)-\frac{3\gamma}{16D^2}\int_0^D \widetilde{w}(x,t)^2dx \\
& -\frac{1}{4D^2}\Big(1-\frac{2+2(1+4D^2)|q(D)L|^2}{\gamma}\times 4D\Big)\int_0^D \widetilde{w}_x(x,t)^2dx \\
\le & -\varepsilon_2\widetilde{V}(t),
\end{aligned} \tag{29}$$

where $\varepsilon_2=\min\Big\{\frac{\lambda_{\min}(Q)-\eta}{\lambda_{\max}(P)},\ \frac{3}{8D^2},\ \frac{1}{2D^2}\Big(1-\frac{2+2(1+4D^2)|q(D)L|^2}{\gamma}\times 4D\Big)\Big\}$, $\lambda_{\max}(P)$ denotes the maximum eigenvalue of P. Then, we have

$$\widetilde{V}(t)\le\widetilde{V}(0)\mathrm{e}^{-\varepsilon_2 t}. \tag{30}$$

We are now ready to prove the stability of error system (16). First, from transformation (17) and by completing the square, we have

$$\begin{cases}
\int_0^D \widetilde{w}(x,t)^2dx \le 2\int_0^D \widetilde{u}(x,t)^2dx+2\int_0^D \|q(x)\|^2dx\|\widetilde{X}(t)\|^2, \\
\int_0^D \widetilde{w}_x(x,t)^2dx \le 2\int_0^D \widetilde{u}_x(x,t)^2dx+2\int_0^D \|q(x)'\|^2dx\|\widetilde{X}(t)\|^2, \\
\int_0^D \widetilde{u}(x,t)^2dx \le 2\int_0^D \widetilde{w}(x,t)^2dx+2\int_0^D \|q(x)\|^2dx\|\widetilde{X}(t)\|^2, \\
\int_0^D \widetilde{u}_x(x,t)^2dx \le 2\int_0^D \widetilde{w}_x(x,t)^2dx+2\int_0^D \|q(x)'\|^2dx\|\widetilde{X}(t)\|^2.
\end{cases}$$

Then, we conclude that

$$\underline{\theta}\Big(\|\widetilde{X}(t)\|^2+\int_0^D \widetilde{u}(x,t)^2dx+\int_0^D \widetilde{u}_x(x,t)^2dx\Big)$$

$$
\begin{aligned}
&\leq \widetilde{V}(t)\\
&\leq \overline{\theta}\bigg(\|\widetilde{X}(t)\|^2 + \int_0^D \widetilde{u}(x,t)^2 dx + \int_0^D \widetilde{u}_x(x,t)^2 dx\bigg),
\end{aligned}
\tag{31}
$$

where

$$
\underline{\theta} = \frac{\gamma\lambda_{\min}(P)}{\max\left\{\gamma\left(1 + 4DM_3^2\right), 4\lambda_{\min}(P), 4\gamma\lambda_{\min}(P)\right\}}, \quad \overline{\theta} = \max\Big\{\lambda_{\max}(P) + DM_3^2(1+\gamma), \gamma, 1\Big\}.
$$

This, together with (30), yields

$$
\begin{aligned}
&\|\widetilde{X}(t)\|^2 + \int_0^D \widetilde{u}(x,t)^2 dx + \int_0^D \widetilde{u}_x(x,t)^2 dx\\
\leq\ & \frac{\overline{\theta}}{\underline{\theta}}\bigg(\|\widetilde{X}(0)\|^2 + \int_0^D \widetilde{u}(x,0)^2 dx + \int_0^D \widetilde{u}_x(x,0)^2 dx\bigg)\mathrm{e}^{-\varepsilon_2 t},
\end{aligned}
\tag{32}
$$

which implies the desirable stability of System (16). This completes the proof. ∎

It is worthwhile emphasizing that, by choosing appropriate observer gains and backstepping transformation, the original error system is changed into a stable target system which is different from that of [4], and hence the restriction on matrix A in the literature is completely removed. In fact, the state observer designed in [4] is applicable under certain restriction on the eigenvalues of matrix A (see Theorems 2 and 3 in [4]). This implies that the output-feedback controller in the literature is effective only for specified system.

We are now in a position to design the output-feedback controller for System (1). In state-feedback controller (4), by respectively replacing $X(t)$ and $u(x,t)$ with their observations $\widehat{X}(t)$ and $\widehat{u}(x,t)$, the output-feedback controller is described as follows:

$$
U(t) = \int_0^D k(D,y)\widehat{u}(y,t)dy + \gamma(D)\widehat{X}(t). \tag{33}
$$

To prove the stability of System (1) with the above controller in the loop, we introduce the following infinite-dimensional backstepping transformation:

$$
\widehat{w}(x,t) = \widehat{u}(x,t) - \int_0^x k(x,y)\widehat{u}(y,t)dy - \gamma(x)\widehat{X}(t), \tag{34}
$$

where $k(x,y)$ and $\gamma(x)$ are the same as (6) and (7). Under the above transformation, System (15) can be changed into the other target system, from which, it is more convenient to prove the stability of the closed-loop System (1), (15), and (33).

Proposition 4 *Under infinite-dimensional backstepping transformation* (34)*, System* (15) *with controller* (33) *in loop can be changed into the following target system:*

$$
\begin{cases}
\dot{\widehat{X}}(t) = (A+BK)\widehat{X}(t) + B\widehat{w}(0,t) + (B+L)(\widetilde{w}(0,t) + K_1\widetilde{X}(t)),\\
\widehat{w}_t(x,t) = \widehat{w}_{xx}(x,t) + M(x)(\widetilde{w}(0,t) + K_1\widetilde{X}(t)),\\
\widehat{w}_x(0,t) = 0,\\
\widehat{w}(D,t) = 0,
\end{cases}
\tag{35}
$$

where $M(x) = p(x) - \int_0^x k(x,y)p(y)dy - \gamma(x)(B+L)$.

Proof See Section E of Appendix in the paper. ∎

We next show that the above target system is exponentially stable in the sense of certain norm, which implies the stability of the closed-loop system in the same sense.

Theorem 3 *For any initial condition X_0, $u_0(x)$, $\widehat{X}(0)$ and $\widehat{u}(x,0)$ satisfying $\int_0^D \big(u_0(x)^2 + \widehat{u}_0(x)^2\big)dx < +\infty$ and $\int_0^D \big(\frac{du_0(x)}{dx}\big)^2 + \big(\frac{d\widehat{u}_0(x)}{dx}\big)^2 dx < +\infty$, the closed-loop system consisting of* (1), (15), *and* (33) *is exponentially stable in the sense of the following norm:*

$$\left(\|X(t)\|^2 + \|\widehat{X}(t)\|^2 + \int_0^D \big(u(x,t)^2 + u_x(x,t)^2 + \widehat{u}(x,t)^2 + \widehat{u}_x(x,t)^2\big)\,dx\right)^{\frac{1}{2}}.$$

Proof In order to prove the desired stability of system $(\widehat{X}, \widehat{u}, \widetilde{X}, \widetilde{u})$, we will first show that of the system $(\widehat{X}, \widehat{w}, \widetilde{X}, \widetilde{w})$. For this, we choose the following Lyapunov function:

$$\widehat{V}(t) = \widehat{X}(t)^{\mathrm{T}}\widehat{P}\widehat{X}(t) + \frac{1}{2}\int_0^D \widehat{w}(x,t)^2dx + \frac{1}{2}\int_0^D \widehat{w}_x(x,t)^2dx + \widehat{e}\widetilde{V}(t),$$

where $\widehat{e}$ is a to-be-specified positive constant, $\widehat{P} = \widehat{P}^{\mathrm{T}} > 0$ satisfies the following Lyapunov equation:

$$(A + BK)^{\mathrm{T}}\widehat{P} + \widehat{P}(A + BK) = -\widehat{Q}, \tag{36}$$

for some to-be-specified $\widehat{Q} = \widehat{Q}^{\mathrm{T}} > 0$.

By computing the time derivative of $\widehat{V}(t)$ along the solutions of (21), (35), and using integration by parts, we have

$$\begin{aligned}
\dot{\widehat{V}}(t) &= \dot{\widehat{X}}(t)^{\mathrm{T}}\widehat{P}\widehat{X}(t) + \widehat{X}(t)^{\mathrm{T}}\widehat{P}\dot{\widehat{X}}(t) + \int_0^D \widehat{w}(x,t)\widehat{w}_t(x,t)dx \\
&\quad + \int_0^D \widehat{w}_x(x,t)\widehat{w}_{xt}(x,t)dx + \widehat{e}\dot{\widetilde{V}}(t) \\
&= -\widehat{X}(t)^{\mathrm{T}}\widehat{Q}\widehat{X}(t) + 2\widehat{X}(t)^{\mathrm{T}}\widehat{P}B\widehat{w}(0,t) + 2\widehat{X}(t)^{\mathrm{T}}\widehat{P}(B+L)(\widetilde{w}(0,t) + K_1\widetilde{X}(t)) \\
&\quad - \int_0^D \widehat{w}_x(x,t)^2dx + \int_0^D \widehat{w}(x,t)M(x)dx(\widetilde{w}(0,t) + K_1\widetilde{X}(t)) \\
&\quad - \int_0^D \widehat{w}_{xx}(x,t)^2dx - \int_0^D \widehat{w}_{xx}(x,t)M(x)dx(\widetilde{w}(0,t) + K_1\widetilde{X}(t)) + \widehat{e}\dot{\widetilde{V}}(t).
\end{aligned}$$

Then, using Young's Inequality, we obtain

$$\begin{aligned}
\dot{\widehat{V}}(t) &\le -\lambda_{\min}(\widehat{Q})\|\widehat{X}(t)\|^2 + \frac{1}{16D}\widehat{w}(0,t)^2 + 16D\|\widehat{P}B\|^2\|\widehat{X}(t)\|^2 + \|\widehat{P}(B+L)\|^2\|\widehat{X}(t)\|^2 \\
&\quad + (\widetilde{w}(0,t) + K_1\widetilde{X}(t))^2 - \int_0^D \widehat{w}_x(x,t)^2dx + \frac{1}{16D^2}\int_0^D \widehat{w}(x,t)^2dx \\
&\quad + 4D^2\int_0^D M(x)^2dx(\widetilde{w}(0,t) + K_1\widetilde{X}(t))^2 - \int_0^D \widehat{w}_{xx}(x,t)^2dx \\
&\quad + \frac{1}{2}\int_0^D \widehat{w}_{xx}(x,t)^2dx + \frac{1}{2}\int_0^D M(x)^2dx(\widetilde{w}(0,t) + K_1\widetilde{X}(t))^2 + \widehat{e}\dot{\widetilde{V}}(t) \\
&= -\left(\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2\right)\|\widehat{X}(t)\|^2 + \frac{1}{16D}\widehat{w}(0,t)^2 \\
&\quad + \frac{1}{16D^2}\int_0^D \widehat{w}(x,t)^2dx + (\widetilde{w}(0,t) + K_1\widetilde{X}(t))^2\left(1 + \left(4D^2 + \frac{1}{2}\right)\int_0^D M(x)^2dx\right)
\end{aligned}$$

$$-\int_0^D \widehat{w}_x(x,t)^2dx - \frac{1}{2}\int_0^D \widehat{w}_{xx}(x,t)^2dx + \widehat{e}\dot{\widetilde{V}}(t). \tag{37}$$

Noting $\widehat{w}(D,t)=0$, by Poincaré's Inequality and Agmon's Inequality, there hold

$$\int_0^D \widehat{w}(x,t)^2dx \le 4D^2\int_0^D \widehat{w}_x(x,t)^2dx, \qquad \widehat{w}(0,t)^2 \le 4D\int_0^D \widehat{w}_x(x,t)^2dx. \tag{38}$$

Substituting this into (37) and by (39) and completing the square, we have

$$\begin{aligned}
\dot{\widehat{V}}(t) \le & -\Big(\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2\Big)\|\widehat{X}(t)\|^2 - \frac{1}{2}\int_0^D \widehat{w}_x(x,t)^2dx \\
& -\frac{1}{2}\int_0^D \widehat{w}_{xx}(x,t)^2dx + \overline{M}(\widetilde{w}(0,t)+K_1\widetilde{X}(t))^2 - \widehat{e}\varepsilon_2\widetilde{V}(t) \\
\le & -\Big(\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2\Big)\|\widehat{X}(t)\|^2 - \frac{1}{2}\int_0^D \widehat{w}_x(x,t)^2dx \\
& -\frac{1}{2}\int_0^D \widehat{w}_{xx}(x,t)^2dx + 2\overline{M}\widetilde{w}(0,t)^2 + 2\overline{M}\|K_1\|^2\|\widetilde{X}(t)\|^2 \\
& -\widehat{e}\varepsilon_2\Big(\lambda_{\min}(P)\|\widetilde{X}(t)\|^2 + \frac{\gamma}{2}\int_0^D \widetilde{w}(x,t)^2dx + \frac{1}{2}\int_0^D \widetilde{w}_x(x,t)^2dx\Big),
\end{aligned}$$

where $\overline{M} = 1 + D\left(4D^2+\frac{1}{2}\right)\left(M_3\|L\|(1+DM_1) + M_1\|B+L\|\right)^2$.

Noting that $\widehat{w}_x(0,t)=0$, by Poincare's Inequality, we have

$$\int_0^D \widehat{w}_x(x,t)^2dx \le 4D^2\int_0^D \widehat{w}_{xx}(x,t)^2dx.$$

Then, by (25) and (38), we have

$$\begin{aligned}
\dot{\widehat{V}}(t) \le & -\Big(\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2\Big)\|\widehat{X}(t)\|^2 - \frac{1}{8D^2}\int_0^D \widehat{w}(x,t)^2dx \\
& -\frac{1}{8D^2}\int_0^D \widehat{w}_x(x,t)^2dx + 2\overline{M}(1+2D)\|q(D)\|^2\|\widetilde{X}(t)\|^2 \\
& +2\overline{M}(1+4D^2)\int_0^D \widetilde{w}_x(x,t)^2dx + 2\overline{M}\|K_1\|^2\|\widetilde{X}(t)\|^2 \\
& -\widehat{e}\varepsilon_2\Big(\lambda_{\min}(P)\|\widetilde{X}(t)\|^2 + \frac{\gamma}{2}\int_0^D \widetilde{w}(x,t)^2dx + \frac{1}{2}\int_0^D \widetilde{w}_x(x,t)^2dx\Big) \\
= & -\Big(\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2\Big)\|\widehat{X}(t)\|^2 - \frac{1}{8D^2}\int_0^D \widehat{w}(x,t)^2dx \\
& -\frac{1}{8D^2}\int_0^D \widehat{w}_x(x,t)^2dx - \frac{\widehat{e}\varepsilon_2\gamma}{2}\int_0^D \widetilde{w}(x,t)^2dx \\
& -\Big(\frac{\widehat{e}\varepsilon_2}{2} - 2\overline{M}(1+4D^2)\Big)\int_0^D \widetilde{w}_x(x,t)^2dx \\
& -\Big(\widehat{e}\varepsilon_2\lambda_{\min}(P) - 2\overline{M}(1+2D)\|q(D)\|^2 - 2\overline{M}\|K_1\|^2\Big)\|\widetilde{X}(t)\|^2.
\end{aligned}$$

By choosing

$$\lambda_{\min}(\widehat{Q}) > 16D\|\widehat{P}B\|^2 + \|\widehat{P}(B+L)\|^2,$$

$$\widehat{e} > \frac{1}{\varepsilon_2} \max\left\{\frac{2\overline{M}(\|K_1\|^2 + (1+2D)M_3^2)}{\lambda_{\min}(P)}, 4\overline{M}(1+4D^2)\right\},$$

it is concluded that

$$\begin{aligned}
\dot{\widehat{V}}(t) \le & -\left(\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2\right)\frac{1}{\lambda_{\max}(\widehat{P})}\widehat{X}(t)^{\mathrm{T}}\widehat{P}\widehat{X}(t) \\
& -\frac{1}{8D^2}\int_0^D \widehat{w}(x,t)^2 dx - \frac{1}{8D^2}\int_0^D \widehat{w}_x(x,t)^2 dx \\
& -\left(\widehat{e}\varepsilon_2\lambda_{\min}(P) - 2\overline{M}(1+2D)\|q(D)\|^2 - 2\overline{M}\|K_1\|^2\right)\frac{1}{\lambda_{\max}(P)}\widetilde{X}(t)^{\mathrm{T}}P\widetilde{X}(t) \\
& -\frac{\widehat{e}\varepsilon_2\gamma}{2}\int_0^D \widetilde{w}(x,t)^2 dx - \left(\frac{\widehat{e}\varepsilon_2}{2} - 2\overline{M}(1+4D^2)\right)\int_0^D \widetilde{w}_x(x,t)^2 dx \\
\le & -\varepsilon_3\widehat{V}(t),
\end{aligned}$$

with

$$\begin{aligned}
\varepsilon_3 = \min\Bigg\{ & \frac{\lambda_{\min}(\widehat{Q}) - 16D\|\widehat{P}B\|^2 - \|\widehat{P}(B+L)\|^2}{\lambda_{\max}(\widehat{P})}, \frac{1}{4D^2}, \\
& \frac{\widehat{e}\varepsilon_2\lambda_{\min}(P) - 2\overline{M}((1+2D)\|q(D)\|^2 + \|K_1\|^2)}{\widehat{e}\lambda_{\max}(P)}, \varepsilon_2 - \frac{4\overline{M}(1+4D^2)}{\widehat{e}}\Bigg\},
\end{aligned}$$

which yields

$$\widehat{V}(t) \le \widehat{V}(0)\mathrm{e}^{-\varepsilon_3 t}.$$

This implies that $(\widehat{X}, \widehat{w}, \widetilde{X}, \widetilde{w})$ is exponentially stable in the sense of the following norm:

$$\left(\|\widetilde{X}(t)\|^2 + \|\widehat{X}(t)\|^2 + \int_0^D \left(\widetilde{w}(x,t)^2 + \widetilde{w}_x(x,t)^2 + \widehat{w}(x,t)^2 + \widehat{w}_x(x,t)^2\right) dx\right)^{\frac{1}{2}}.$$

By the similar way in deriving (32), we obtain that system $(\widehat{X}, \widehat{u}, \widetilde{X}, \widetilde{u})$ is exponentially stable in the same sense. Therefore, by noting the fact $u(x,t) = \widehat{u}(x,t) + \widetilde{u}(x,t)$ and $X(t) = \widehat{X}(t) + \widetilde{X}(t)$, we conclude that system $(X, u, \widehat{X}, \widehat{u})$ is exponentially stable in the sense of the norm defined by (36). This completes the proof. ∎

4 Simulation Results

In this section, an example is given to verify the effectiveness of theoretical results for the following simple system:

$$\begin{cases}
\dot{X}(t) = X(t) + u(0,t), \\
y(t) = X(t), \\
u_t(x,t) = u_{xx}(x,t) + xX(t), \\
u_x(0,t) = 0 \\
u(1,t) = U(t),
\end{cases} \tag{39}$$

where $X(t) \in \boldsymbol{R}$, the initial conditions are $X_0 = 0.5$ and $u_0(x) = x^2$.

From (4) and (33), we see that, to design controllers for (39), the controller parameters, i.e. $\gamma(\cdot)$ and $k(\cdot)$ should be determined. However, the sum of the infinite series defined in (7) is difficult to calculate even for simple nonconstant function $\lambda(x)$. On the other hand, appropriate truncation of the series is sufficient for the practical implementation. Therefore, we replace $\gamma(x)$ by its approximation $\overline{\gamma}(x) = \sum_{i=0}^{4} \gamma_i(x)$ in the controllers. Choosing $K = -2$ and by (8), we have

$$\begin{cases} \gamma_0(x) = -2 - \dfrac{x^3}{3!}, \\ \gamma_1(x) = -x^2 - \dfrac{3x^5}{5!} - \dfrac{x^8}{8!}, \\ \gamma_2(x) = -\dfrac{2x^4}{4!} - \dfrac{5x^7}{7!} - \dfrac{4x^{10}}{10!} - \dfrac{x^{13}}{13!}, \\ \gamma_3(x) = -\dfrac{2x^6}{6!} - \dfrac{7x^9}{9!} - \dfrac{9x^{12}}{12!} - \dfrac{5x^{15}}{15!} - \dfrac{x^{18}}{18!}, \\ \gamma_4(x) = -\dfrac{2x^8}{8!} - \dfrac{9x^{11}}{11!} - \dfrac{16x^{14}}{14!} - \dfrac{14x^{17}}{17!} - \dfrac{6x^{20}}{20!} - \dfrac{x^{23}}{23!}. \end{cases}$$

Then by (4) and (7), we obtain the state-feedback controller:

$$U(t) = \int_0^1 \int_0^{1-y} \overline{\gamma}(\xi) d\xi u(y,t) dy - 3.2789 X(t). \tag{40}$$

Moreover, by (18) and choosing $K_1 = 1$, we conclude $q(x) = x + \mathrm{e}^{-x}$. Then, by choosing $L = 3$, $p(x) = 3(x + \mathrm{e}^{-x})$ follows directly. Hence, by (15), we obtain the following observer for System (39) when only $u(0,t)$ is available for measurement:

$$\begin{cases} \dot{\widehat{X}}(t) = \widehat{X}(t) + u(0,t) + 3(u(0,t) - \widehat{u}(0,t)), \\ \widehat{y}(t) = \widehat{X}(t), \\ \widehat{u}_t(x,t) = \widehat{u}_{xx}(x,t) + x\widehat{y}(t) + 3(x + \mathrm{e}^{-x})(u(0,t) - \widehat{u}(0,t)), \\ \widehat{u}_x(0,t) = 0, \\ \widehat{u}(1,t) = U(t), \end{cases} \tag{41}$$

with initial estimates $\widehat{X}(0) = 0.2$ and $\widehat{u}(x,0) = \mathrm{e}^x$. Then, by (33), we obtain the following output-feedback controller:

$$U(t) = \int_0^1 \int_0^{1-y} \overline{\gamma}(\xi) d\xi \widehat{u}(y,t) dy - 3.2789 \widehat{X}(t). \tag{42}$$

By using the explicit forward Euler method (see, e.g., Page 406 of [19]) with 20-step discretization in space, four simulation figures are obtained for the closed-loop system signals. Specifically, Figures 1 and 2 show that states $u(x,t)$ and $X(t)$ of (39) with state-feedback controller (40) in the loop converge to zero, and meanwhile, Figures 3 and 4 show that both the states $u(x,t)$ and $X(t)$ of closed-loop system (39), (41), and (42) converge to zero.

5 Concluding Remarks

In this paper, the stabilization of a class of coupled PDE-ODE systems with spatially varying coefficient has been investigated. By infinite-dimensional backstepping method, both state-feedback and output-feedback controllers have been successfully constructed, which ensure the desirable stability of the closed-loop systems. It is worthy pointing out that, the control design is more difficult to solve since the presence of the spatially varying coefficient makes controllers parameters can not be derived by the method in the related literature. Moreover, restriction on the ODE sub-system parameter in the related literature is completely removed in the paper. Since spatially varying coefficients arise frequently in PDEs, extension of the methods and ideas in the paper to more complicated coupled systems, such as coupled PDE-PDE systems with spatially varying coefficients, will be meaningful and deserve investigation.

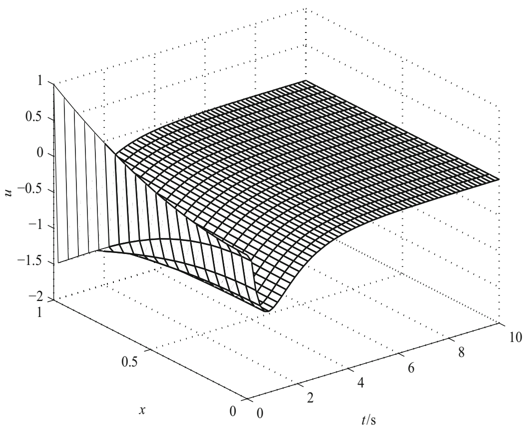

Figure 1 Trajectory of $u(x,t)$ with state-feedback controller (40)

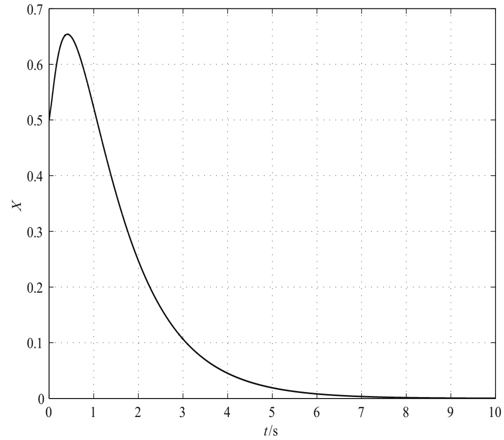

Figure 2 Trajectory of $X(t)$ with state-feedback controller (40)

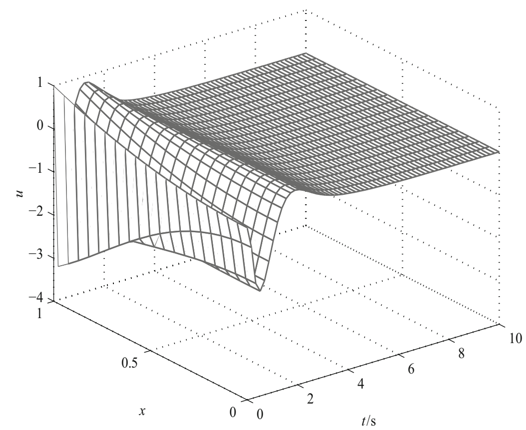

Figure 3 Trajectory of $u(x,t)$ with output-feedback controller (42)

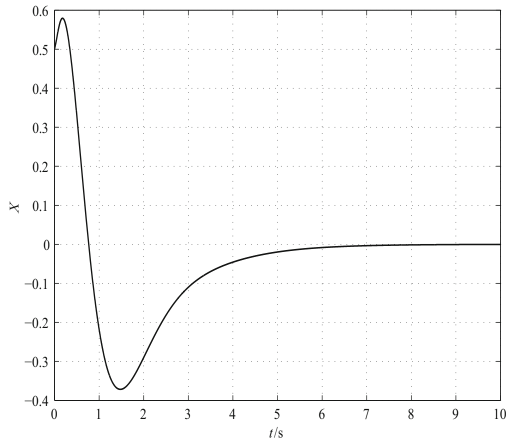

Figure 4 Trajectory of $X(t)$ with output-feedback controller (42)

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Appendix

A Useful Inequalities

Lemma A.1 *For any matrix function* $A(x) = (a_{ij}(x)) : [0, D] \to \boldsymbol{R}^{m\times n}$ *which is continuous* *and integrable on* $[0,\, D]$, *the following inequality holds:*

$$\Big\| \int_0^D A(x)dx \Big\| \le \sqrt{m}\, n \int_0^D \|A(x)\| dx.$$

Proof Let $\|A(x)\|_F = \sqrt{\mathrm{Tr}(A(x)^{\mathrm{T}} A(x))}$. Then noting that $\lambda_i(A(x)^{\mathrm{T}} A(x)) \ge 0$, $i = 1, 2, \cdots, n$, we have

$$\|A(x)\| = \sqrt{\lambda_{\max}(A(x)^{\mathrm{T}} A(x))} \le \sqrt{\sum_{i=1}^{n} \lambda_i(A(x)^{\mathrm{T}} A(x))} = \sqrt{\mathrm{Tr}(A(x)^{\mathrm{T}} A(x))} = \|A(x)\|_F. \quad \text{(A.1)}$$

Moreover,

$$\|A(x)\|_F = \sqrt{\sum_{i=1}^{n} \lambda_i(A(x)^{\mathrm{T}} A(x))} \le \sqrt{n\, \lambda_{\max}(A(x)^{\mathrm{T}} A(x))} = \sqrt{n} \|A(x)\|. \quad \text{(A.2)}$$

Therefore, by (A.1), we have

$$\Big\| \int_0^D A(x)dx \Big\|^2 \le \Big\| \int_0^D A(x)dx \Big\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \Big| \int_0^D a_{ij}(x)dx \Big|^2,$$

by which, and noting that $\big| \int_0^D a_{ij}(x)dx \big| \le \int_0^D |a_{ij}(x)|dx$, after some direct calculations, we obtain

$$\begin{aligned}
\Big\| \int_0^D A(x)dx \Big\|^2 &\le \sum_{i=1}^{m} \sum_{j=1}^{n} \Big(\int_0^D |a_{ij}(x)|dx \Big)^2 \\
&\le \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^D |a_{ij}(x)|dx \Big)^2 = \Big(\int_0^D \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}(x)|dx \Big)^2 \\
&\le mn \Big(\int_0^D \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}(x)|^2 dx} \Big)^2 = mn \Big(\int_0^D \|A(x)\|_F dx \Big)^2.
\end{aligned}$$

Substituting (A.2) into the above inequality yields

$$\Big\| \int_0^D A(x)dx \Big\|^2 \le mn^2 \Big(\int_0^D \|A(x)\| dx \Big)^2,$$

which directly implies the desirable inequality. ∎

Lemma A.2[20] (*Poincaré's Inequality*) *For any* $w \in C^1[0,\, D]$§, *there hold*

$$\begin{cases} \displaystyle\int_0^D w(x)^2 dx \le 2Dw(0)^2 + 4D^2 \int_0^D w_x(x)^2 dx, \\ \displaystyle\int_0^D w(x)^2 dx \le 2Dw(D)^2 + 4D^2 \int_0^D w_x(x)^2 dx. \end{cases}$$

§ $C^1[0,\, D]$ denotes the set of all continuously differentiable functions defined on $[0,\, D]$.

Lemma A.3[20] *(Agmon's Inequality) For any* $w \in C^1[0, D]$*, there hold*

$$\begin{cases} w(x)^2 \leq w(0)^2 + 2\sqrt{\int_0^D w(x)^2 dx \int_0^D w_x(x)^2 dx}, \\ w(x)^2 \leq w(D)^2 + 2\sqrt{\int_0^D w(x)^2 dx \int_0^D w_x(x)^2 dx}. \end{cases}$$

B Proof of Proposition 1

Letting $x = 0$ in (2), we obtain $u(0, t) = w(0, t) + \gamma(0)X(t)$. Substituting this into the first equation of (1), we have

$$\dot{X}(t) = (A + B\gamma(0))X(t) + Bw(0, t).$$

Hence, to obtain the first equation of (3), there must hold $\gamma(0) = K$.

To obtain the other three equations of (3), we first compute $w_x(x, t)$, $w_{xx}(x, t)$, and $w_t(x, t)$ from (2), that is,

$$w_x(x,t) = u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy - \gamma(x)'X(t), \tag{B.1}$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) - \frac{d}{dx}k(x,x)u(x,t) - k(x,x)u_x(x,t) \\ &\quad -k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy - \gamma(x)''X(t) \end{aligned} \tag{B.2}$$

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)u_t(y,t)dy - \gamma(x)\dot{X}(t) \\ &= u_t(x,t) - \int_0^x k(x,y)u_{yy}(y,t)dy - \int_0^x k(x,y)\lambda(y)CX(t)dy - \gamma(x)\dot{X}(t) \\ &= u_{xx}(x,t) + \lambda(x)CX(t) - k(x,x)u_x(x,t) + k_y(x,x)u(x,t) \\ &\quad -k_y(x,0)u(0,t) - \int_0^x k_{yy}(x,y)u(y,t)dy - \int_0^x k(x,y)\lambda(y)CX(t)dy \\ &\quad -\gamma(x)(AX(t) + Bu(0,t)). \end{aligned} \tag{B.3}$$

Then, letting $x = 0$ in (B.1) and noting $u_x(0, t) = 0$, we have

$$w_x(0,t) = k(0,0)u(0,t) + \gamma(0)'X(t) = 0.$$

Hence, the sufficient condition to guarantee the trueness of the third equation of (3) is

$$k(0,0) = 0, \qquad \gamma(0)' = 0. \tag{B.4}$$

Moreover, subtracting the two sides of (B.2) from the two sides of (B.3) separately, there holds

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) &= \left(\gamma(x)'' - \gamma(x)A - \int_0^x k(x,y)\lambda(y)Cdy + \lambda(x)C\right)X(t) \\ &\quad - (k_y(x,0) + \gamma(x)B)\,u(0,t) + 2\frac{d}{dx}k(x,x)u(x,t) \\ &\quad + \int_0^x (k_{xx}(x,y) - k_{yy}(x,y))\,u(y,t)dy, \end{aligned}$$

by which and noting (B.4) and $\gamma(0)=K$, the sufficient condition for the trueness of the second equation of (3) is that $\gamma(x)$ and $k(x,y)$ must satisfy the following equations (called kernel equations):

$$\begin{cases} \gamma(x)'' - \gamma(x)A - \displaystyle\int_0^x k(x,y)\lambda(y)Cdy + \lambda(x)C = 0, \\ \gamma(0)' = 0, \\ \gamma(0) = K, \end{cases} \tag{B.5}$$

and

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = 0, \\ k(x,x) = 0, \\ k_y(x,0) = -\gamma(x)B. \end{cases} \tag{B.6}$$

It is easily to verify that (6) is the solution of Equation (B.6). Then, substituting this into the first equation of (B.5) directly concludes (5).

C Proof of Proposition 2

By the first equation of (5), we have

$$\gamma(x)'' = \gamma(x)A + \int_0^x \int_0^{x-y} \gamma(\xi)Bd\xi\lambda(y)Cdy - \lambda(x)C.$$

Integrating both sides of the above equation on $[0,\, x]$ twice and noting $\gamma(0)=K$, $\gamma(0)'=0$, after some simple managements, we conclude

$$\gamma(x) = \gamma_0(x) + \int_0^x \int_0^\eta \gamma(\xi)Ad\xi d\eta + \frac{1}{2}\int_0^x \int_0^{x-y} (x-y-\xi)^2\gamma(\xi)Bd\xi\lambda(y)Cdy. \tag{C.1}$$

Thus, to prove the proposition, it suffices to show that (7) is the unique solution of the above equation, and the absolute and uniform convergence of the series defined by (7) must be ensured. For this, we will estimate $\gamma_i(x)$ by induction. First, for $\gamma_0(x)$, using Lemma A.1 and noting $0\le x\le D$, we have

$$\begin{aligned} \|\gamma_0(x)\| &\le \|K\| + \left\|\int_0^x \int_0^\eta \lambda(\xi)Cd\xi d\eta\right\| \\ &\le \|K\| + n^2\int_0^x \int_0^\eta |\lambda(\xi)|\cdot\|C\|d\xi d\eta \\ &\le \|K\| + n^2D^2\|C\|\max_{x\in[0,\,D]}|\lambda(x)| = M_4. \end{aligned} \tag{C.2}$$

Then, suppose that for all $x\in[0,\, D]$, there holds

$$\|\gamma_i(x)\| \le M_4M_5^i\frac{x^{2i}}{(2i)!}, \tag{C.3}$$

where $M_5 = n^2\left(\|A\| + \frac{1}{2}D^2\|B\|\cdot\|C\|\max_{x\in[0,\,D]}|\lambda(x)|\right)$, by which and (8), we have,

$$\begin{aligned} &\|\gamma_{i+1}(x)\| \\ \le& \left\|\int_0^x \int_0^\eta \gamma_i(\xi)Ad\xi d\eta\right\| + \frac{1}{2}\left\|\int_0^x \int_0^{x-y} (x-y-\xi)^2\gamma_i(\xi)Bd\xi\lambda(y)Cdy\right\| \end{aligned}$$

$$
\begin{aligned}
&\le n^2 \int_0^x \int_0^\eta \|\gamma_i(\xi)A\| d\xi d\eta + \frac{1}{2} n^2 D^2 \int_0^x \int_0^{x-y} |\gamma_i(\xi)B| d\xi |\lambda(y)| \cdot \|C\| dy \\
&\le n^2 \|A\| M_4 M_5^i \int_0^x \int_0^\eta \frac{\xi^{2i}}{2i!} d\xi d\eta + \frac{1}{2} n^2 D^2 \|B\| \cdot \|C\| \max_{x\in[0,\,D]} |l(x)| M_4 M_5^i \int_0^x \int_0^{x-y} \frac{\xi^{2i}}{2i!} d\xi dy \\
&= n^2 \|A\| M_4 M_5^i \frac{x^{2(i+1)}}{2(i+1)!} + \frac{1}{2} n^2 D^2 \|B\| \cdot \|C\| \max_{x\in[0,\,D]} |l(x)| M_4 M_5^i \frac{x^{2(i+1)}}{2(i+1)!} \\
&= n^2 \left(\|A\| + \frac{1}{2} D^2 \|B\| \cdot \|C\| \max_{x\in[0,\,D]} |l(x)| \right) M_4 M_5^i \frac{x^{2(i+1)}}{2(i+1)!} \\
&= M_4 M_5^{i+1} \frac{x^{2(i+1)}}{2(i+1)!}.
\end{aligned} \tag{C.4}
$$

Therefore, (C.3) is proven. Then, noting $0 \le x \le D$, we have

$$
\sup_{x\in[0,\,D]} \sum_{i=0}^{+\infty} \|\gamma_i(x)\| \ \le \ \sum_{i=0}^{+\infty} M_4 \frac{(D\sqrt{M_5})^{2i}}{(2i)!}. \tag{C.5}
$$

It is not hard to verify that the series on the right-hand side of (C.5) converges. Hence, by the well known Weierstrass M-test, the series defined by (7) converges absolutely and uniformly on $[0,\,D]$. By substituting (7) into (C.1) and noting (8), it is not hard to verify that (7) is the solution of (C.1). Then, the existence of the solution to Equation (5) is concluded.

To show the uniqueness, we assume that $\overline{\gamma}(x)$ and $\widetilde{\gamma}(x)$ are two different solutions of (5) with error $\Delta\gamma(x) = \overline{\gamma}(x) - \widetilde{\gamma}(x)$. Substituting these two solutions into (C.1) and after some direct calculation, we have

$$
\Delta\gamma(x) = \int_0^x \int_0^\eta \Delta\gamma(\xi) A d\xi d\eta + \frac{1}{2} \int_0^x \int_0^{x-y} (x-y-\xi)^2 \Delta\gamma(\xi) B d\xi \lambda(y) C dy. \tag{C.6}
$$

From (C.3) and (C.5), there holds $\sup_{x\in[0,\,D]} \|\gamma(x)\| \le M_4 \exp(D\sqrt{M_5})$, and then $\sup_{x\in[0,\,D]} \|\Delta\gamma(x)\| \le 2M_4 \exp(D\sqrt{M_5})$. Next, we will estimate $\Delta\gamma(x)$ by induction. Suppose that for all $x \in [0,\,D]$, there holds

$$
\|\Delta\gamma(x)\| \le 2M_4 \exp(D\sqrt{M_5}) M_5^i \frac{x^{2i}}{(2i)!}. \tag{C.7}
$$

Substituting this into (C.6) and along the similar process to obtain the estimate (C.4), for all $x \in [0,\,D]$, we have

$$
\begin{aligned}
\|\Delta\gamma(x)\| &\le \left\| \int_0^x \int_0^\eta \Delta\gamma(\xi) A d\xi d\eta \right\| + \frac{1}{2} \left\| \int_0^x \int_0^{x-y} (x-y-\xi)^2 \Delta\gamma(\xi) B d\xi \lambda(y) C dy \right\| \\
&\le n^2 \int_0^x \int_0^\eta \|\Delta\gamma(\xi)A\| d\xi d\eta + \frac{1}{2} n^2 D^2 \int_0^x \int_0^{x-y} |\Delta\gamma(\xi)B| d\xi |\lambda(y)| \cdot \|C\| dy \\
&\le 2n^2 \|A\| M_4 \exp(D\sqrt{M_5}) M_5^i \int_0^x \int_0^\eta \frac{\xi^{2i}}{2i!} d\xi d\eta \\
&\quad + n^2 D^2 \|B\| \cdot \|C\| \max_{x\in[0,\,D]} |\lambda(x)| M_4 \exp(D\sqrt{M_5}) M_5^i \int_0^x \int_0^{x-y} \frac{\xi^{2i}}{2i!} d\xi d\eta \\
&= n^2 \left(\|A\| + \frac{1}{2} D^2 \|B\| \cdot \|C\| \max_{x\in[0,\,D]} |\lambda(x)| \right) 2M_4 \exp(D\sqrt{M_5}) M_5^i \frac{x^{2(i+1)}}{2(i+1)!} \\
&= 2M_4 \exp(D\sqrt{M_5}) M_5^{i+1} \frac{x^{2(i+1)}}{2(i+1)!},
\end{aligned} \tag{C.8}
$$

Therefore, (C.7) is proven. Noting that $0 \le x \le D$, by (C.8), we have

$$\sup_{x\in[0,\,D]} \|\Delta\gamma(x)\| \le 2M_4 \exp(D\sqrt{M_5}) M_5^{i+1} \frac{D^{2(i+1)}}{2(i+1)!}, \tag{C.9}$$

which implies $\Delta\gamma(x) \equiv 0$ since $\lim_{i\to+\infty} 2M_4 \exp(D\sqrt{M_5}) M_5^{i+1} \frac{D^{2(i+1)}}{2(i+1)!} = 0$. Then (7) is the unique solution of (C.1).

Next, we turn to showing the estimates in (9). Noting that $0 \le x \le D$ and $\sup_{x\in[0,\,D]} \|\gamma(x)\| \le M_4 \exp(D\sqrt{M_5})$, we have

$$\sup_{x\in[0,\,D],y\in[0,\,D]} |k(x,y)| \le \sup_{x\in[0,\,D],y\in[0,\,D]} \int_0^{x-y} |\gamma(\xi)B| d\xi \le DM_4 \|B\| \exp(D\sqrt{M_5}) \tag{C.10}$$

and

$$\sup_{x\in[0,\,D],y\in[0,\,D]} |k_x(x,y)| = \sup_{x\in[0,\,D],y\in[0,\,D]} |\gamma(x-y)B| \le M_4 \|B\| \exp(D\sqrt{M_5}).$$

Moreover, integrating the first equation of (B.5) over $[0,\, x]$, we obtain

$$\gamma(x)' = \int_0^x (\gamma(x)A - \lambda(x)C)\, dx + \int_0^x \int_0^\xi k(\xi,y)\lambda(y)C dy d\xi,$$

which together with (C.10) yields

$$\begin{aligned}\sup_{x\in[0,\,D],y\in[0,\,D]} \|\gamma(x)'\| \le nD \max_{x\in[0,\,D]} |l(x)| \cdot \|C\| + nDM_4 \mathrm{e}^{D\sqrt{M_5}} \\ \cdot \left(\|A\| + nD^2 \max_{x\in[0,\,D]} |l(x)| \cdot \|B\| \cdot \|C\| \right) = M_6.\end{aligned}$$

Then, choosing $M_1 = \max\{M_4 \exp(D\sqrt{M_5}), DM_4\|B\| \exp(D\sqrt{M_5}), M_4\|B\| \exp(D\sqrt{M_5}), M_6\}$, we directly conclude (11).

D Proof of Proposition 3

We will first show the first, third, and fourth equations of (21), and then prove the second one.

Letting $x = 0$ in (17) yields $\widetilde{u}(0,t) = \widetilde{w}(0,t) + q(0)\widetilde{X}(t)$, substituting this into the first equation of (16) and noting $q(0) = K_1$ directly yield the first equation of (21).

Computing $\widetilde{w}_x(x,t)$ from (17) and letting $x = 0$, we have

$$\widetilde{w}_x(0,t) = \widetilde{u}_x(0,t) - q(0)'\widetilde{X}(t).$$

Noting $\widetilde{u}_x(0,t) = q(0)' = 0$, $\widetilde{w}_x(0,t) = 0$ follows from the above equation.

Moreover, letting $x = D$ in (17) and noting $\widetilde{u}(D,t) = 0$, the fourth equation of (21) can be directly concluded.

To show the second equation of (21), we first compute $\widetilde{w}_t(x,t)$ and $\widetilde{w}_{xx}(x,t)$, respectively,

$$\begin{aligned}\widetilde{w}_t(x,t) &= \widetilde{u}_t(x,t) - q(x)\dot{\widetilde{X}}(t) \\ &= \widetilde{u}_{xx}(x,t) + \lambda(x)C\widetilde{X}(t) - p(x)\widetilde{u}(0,t) - q(x)A\widetilde{X}(t) + q(x)L\widetilde{u}(0,t), \\ \widetilde{w}_{xx}(x,t) &= \widetilde{u}_{xx}(x,t) - q(x)''\widetilde{X}(t).\end{aligned}$$

Thus, we have

$$\widetilde{w}_t(x,t)-\widetilde{w}_{xx}(x,t)=(q(x)''-q(x)A+\lambda(x)C)\,\widetilde{X}(t)-(p(x)-q(x)L)\widetilde{u}(0,t).$$

Then, by (19) and noting $p(x)=q(x)L$, $\widetilde{w}_t(x,t)=\widetilde{w}_{xx}(x,t)$ can be obtained, which is the second equation of (21).

E Proof of Proposition 4

We first show the first, third, and fourth equations of (35), and then prove the second one.

Letting $x=0$ in (34), we have $\widehat{u}(0,t)=\widehat{w}(0,t)+K\widehat{X}(t)$. Then, there holds

$$u(0,t)=\widetilde{u}(0,t)+\widehat{u}(0,t)=\widetilde{u}(0,t)+\widehat{w}(0,t)+K\widehat{X}(t).$$

Substituting this into the first equation of (15) yields

$$\dot{\widehat{X}}(t)=(A+BK)\widehat{X}(t)+B\widehat{w}(0,t)+(B+L)\widetilde{u}(0,t).$$

Letting $x=0$ in (17), we have $\widetilde{u}(0,t)=\widetilde{w}(0,t)+K_1\widetilde{X}(t)$. Substituting this into the above equation directly yields the first equation of (35).

Computing $\widehat{w}_x(x,t)$ from (34) and letting $x=0$ concludes

$$\widehat{w}_x(0,t)=\widehat{u}_x(0,t)-k(0,0)\widehat{u}(0,t)-\gamma(0)'\widehat{X}(t),$$

By (B.5), (B.6), and noting $\widehat{u}_x(0,t)=0$, $\widehat{w}_x(0,t)=0$ can be directly obtained.

Letting $x=D$ in (34) and by (33) directly yield the fourth equation of (35).

To derive the second equation of (35), we first compute $\widehat{w}_t(x,t)$ along the solutions of System (15),

$$\begin{aligned}\widehat{w}_t(x,t)&=\widehat{u}_t(x,t)-\int_0^x k(x,y)\widehat{u}_t(y,t)dy-\gamma(x)\dot{\widehat{X}}(t)\\&=\widehat{u}_{xx}(x,t)+\lambda(x)C\widehat{X}(t)+p(x)\widetilde{u}(0,t)-\gamma(x)\big(A\widehat{X}(t)+Bu(0,t)+L\widetilde{u}(0,t)\big)\\&\quad-\int_0^x k(x,y)\big(\widehat{u}_{yy}(y,t)+\lambda(y)C\widehat{X}(t)+p(y)\widetilde{u}(0,t)\big)dy.\end{aligned}$$

Using integration by parts twice yields

$$\begin{aligned}\widehat{w}_t(x,t)=&\ \widehat{u}_{xx}(x,t)+\lambda(x)C\widehat{X}(t)+p(x)\widetilde{u}(0,t)-\gamma(x)\big(A\widehat{X}(t)+Bu(0,t)+L\widetilde{u}(0,t)\big)\\&-k(x,x)\widehat{u}_x(x,t)+k_y(x,x)\widehat{u}(x,t)-k_y(x,0)\widehat{u}(0,t)-\int_0^x k_{yy}(x,y)\widehat{u}(y,t)dy\\&-\int_0^x k(x,y)\lambda(y)Cdy\widehat{X}(t)-\int_0^x k(x,y)p(y)dy\widetilde{u}(0,t).\end{aligned}$$

Noting $k_y(x,0)=-\gamma(x)B$ and $\widehat{u}(0,t)=u(0,t)-\widetilde{u}(0,t)$, after some simple managements, there holds

$$\begin{aligned}\widehat{w}_t(x,t)=&\ \widehat{u}_{xx}(x,t)+\left(\lambda(x)C-\gamma(x)A-\int_0^x k(x,y)\lambda(y)Cdy\right)\widehat{X}(t)\\&+\left(p(x)-\int_0^x k(x,y)p(y)dy-\gamma(x)(B+L)\right)\widetilde{u}(0,t)\\&-k(x,x)\widehat{u}_x(x,t)+k_y(x,x)\widehat{u}(x,t)-\int_0^x k_{yy}(x,y)\widehat{u}(y,t)dy.\end{aligned}$$

Moreover, by computing $\widehat{w}_{xx}(x,t)$ from (34), there holds

$$\begin{aligned}\widehat{w}_{xx}(x,t) = \widehat{u}_{xx}(x,t) - \frac{d}{dx}k(x,x)\widehat{u}(x,t) - k(x,x)\widehat{u}_x(x,t) \\ -k_x(x,x)\widehat{u}(x,t) - \int_0^x k_{xx}(x,y)\widehat{u}(y,t)dy - \gamma(x)''\widehat{X}(t).\end{aligned}$$

Using the above equations, we obtain

$$\begin{aligned}\widehat{w}_t(x,t) - \widehat{w}_{xx}(x,t) = {} & 2\frac{d}{dx}k(x,x)\widehat{u}(x,t) - \int_0^x \left(k_{yy}(x,y) - k_{xx}(x,y)\right)\widehat{u}(y,t)dy \\ & +\left(p(x) - \int_0^x k(x,y)p(y)dy - \gamma(x)(B+L)\right)\widetilde{u}(0,t) \\ & +\left(\gamma(x)'' + \lambda(x)C - \gamma(x)A - \int_0^x k(x,y)\lambda(y)Cdy\right)\widehat{X}(t).\end{aligned}$$

By (B.5), (B.6), and noting $\widetilde{u}(0,t) = \widetilde{w}(0,t) + K_1\widetilde{X}(t)$, we yield

$$\widehat{w}_t(x,t) - \widehat{w}_{xx}(x,t) = \left(p(x) - \int_0^x k(x,y)p(y)dy - \gamma(x)(B+L)\right)(\widetilde{w}(0,t) + K_1\widetilde{X}(t)),$$

which is the second equation of (35).