J Syst Sci Complex (2013) 26: 886–901

STOCHASTIC MAXIMUM PRINCIPLE FOR MIXED REGULAR-SINGULAR CONTROL PROBLEMS OF FORWARD-BACKWARD SYSTEMS*

ZHANG Feng

DOI: 10.1007/s11424-013-0287-6
Received: 15 November 2010 / Revised: 3 May 2011

Abstract This paper considers a stochastic optimal control problem of a forward-backward system with regular-singular controls where the set of regular controls is not necessarily convex and the regular control enters the diffusion coefficient. This control problem is difficult to solve with the classical method of spike variation. The authors use the approach of relaxed controls to establish maximum principle for this stochastic optimal control problem. Sufficient optimality conditions are also investigated.

Keywords Forward-backward system, maximum principle, relaxed control, singular control.

1 Introduction

Stochastic singular control was first introduced by Bather and Chernoff[1] who considered a simplified model of spaceship control. Benes, et al.[2] were the first to solve rigorously an example of a finite-fuel singular control problem. Since then, stochastic singular control problem has attracted considerable research interest due to its wide applicability in a number of areas. See, for example, [3–7] and the references therein.

In most cases, stochastic optimal singular control problem was solved by dynamic programming. It was shown in particular that the value function is the solution of a variational inequality and the optimal state is a reflected diffusion at the free boundary.

Maximum principle for stochastic optimal control problems has been studied by many authors, including Peng[8, 9], Shi and Wu[10, 11], Wu[12], Xu[13], etc. It's worth pointing out that, Bahlali[14] reconsidered the stochastic optimal control problem studied in [8], and he solved the problem with the relaxed control method under certain conditions. In that paper, the author

ZHANG Feng
School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan 250014, China. Email: zhangfeng1104@yahoo.cn; zhangfeng1104@sdufe.edu.cn.
*This research was supported by the National Natural Science Foundation of China under Grant Nos. 11201268 and 61105077, and the Natural Science Foundation of Shandong Province under Grant Nos. ZR2011AQ018 and ZR2012AQ013.
◇ *This paper was recommended for publication by Editor ZHANG Jifeng.*

replaced the U-valued regular control process (u_t) by a $\mathbb{P}(U)$-valued process (q_t), where $\mathbb{P}(U)$ is the space of probability measures on U equipped with the topology of stable convergence. The set of these new processes, which are called relaxed controls, has a nice structure of convexity. Then the original optimal control problem is replaced by a new stochastic optimal relaxed control problem, and the maximum principle for this new problem can be obtained by the classic method of convex perturbation. With the help of the celebrated chattering lemma, the maximum principle for the original stochastic optimal control problem is easily obtained under certain conditions. Bahlali[14] established the maximum principle by using only the first-order expansion and the associated adjoint equation, which improved the result of Peng[8] to some extent.

Maximum principle for optimal control problems of forward-backward systems, in which the control variable enters the diffusion coefficient and the control domain is not convex, remained to be an open problem. The main difficulty of solving this open problem is how to use a suitable variational technique to treat the variable z. If the approach developed by Peng[8] is used, the second order expansion leads to a nonlinear problem, and hence it is difficult to deduce the second-order variational inequality. Recently, two works have made important progress in solving this problem. Wu[15] solved this problem by transferring it to a forward optimal control problem with state constraint. Yong[16] studied a stochastic optimal control problem of forward-backward system with mixed initial-terminal conditions. However, the proofs in the previous two works are lengthy and technical. It's worth pointing out that Bahlali[17] solved this problem with the relaxed control method under certain conditions.

The first result in stochastic maximum principle for singular optimal control problems was obtained in [18], in which linear dynamics, convex cost criterion, and convex state constraint were assumed. Bahlali and Chala[19] generalized the result of [18] to the nonlinear case with a convex state constraint, for which the maximum principle was obtained by a convex perturbation. Bahlali and Mezerdi[20] extended the previous two works to the nonlinear dynamics case, in which the regular control enters the diffusion coefficient and the domain of the regular controls is non-convex. The authors used the second order adjoint equation and the second order variational inequality to derive the maximum principle, which is a generalization of Peng[8] to singular control problems. It's worth pointing out that the control systems in these works are stochastic differential equations with singular controls, while stochastic singular control problems of forward-backward systems have not been studied.

In this paper, inspired by the references [14] and [17], we consider a stochastic optimal control problem of a forward-backward system in which the control variable consists of two components: the regular control and the singular control. It's assumed that the set of regular controls is not necessarily convex and the regular control enters the diffusion coefficient. We use the approach of relaxed controls to establish necessary as well as sufficient optimality conditions for this stochastic optimal control problem. Firstly, we replace the regular control by a relaxed control and consider a new stochastic relaxed-singular control problem. Since the set of relaxed controls has a nice structure of convexity, the maximum principle, and sufficient optimality conditions can be easily obtained by using the convex perturbation. Under certain conditions, necessary and sufficient optimality conditions for the original regular-singular control problem are obtained from the corresponding results for relaxed-singular controls by virtue of the celebrated chattering lemma. Our result can be regarded as a generalization of [14] and [17] to singular control problems.

This paper is organized as follows. In Section 2, we formulate the regular-singular control problem and the relaxed-singular control problem, and give the main assumptions. As a preliminary, we consider stochastic differential equations (SDEs) and backward stochastic differential equations (BSDEs) with singular controls. In Section 3, we establish necessary and sufficient optimality conditions for the optimal relaxed-singular control problem. In Section 4, we derive necessary and sufficient optimality conditions for the regular-singular control problem under certain conditions.

2 Preliminaries

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\mathbb{E}$ stand for the expectation with respect to the probability measure $\mathbb{P}$. Let $T>0$ be a fixed finite time and $\{\mathcal{F}_t,,0\le t\le T\}$ be the natural filtration of a d-dimensional standard Brownian motion $\{B_t,0\le t\le T\}$, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. We will use $\alpha\cdot\beta$ to denote the inner product of two vectors α and β which are of the same dimension. For $n\in\mathbb{N}$, we denote by $S^2(\mathbb{R}^n)$ the set of n-dimensional $\mathcal{F}_t$-adapted processes $\{\phi_t,0\le t\le T\}$ such that $\mathbb{E}\left[\sup_{0\le t\le T}|\phi_t|^2\right]<\infty$, and by $H^2(\mathbb{R}^n)$ the set of n-dimensional $\mathcal{F}_t$-adapted processes $\{\psi_t,0\le t\le T\}$ such that $\mathbb{E}\left[\int_0^T|\psi_t|^2dt\right]<\infty$. Let U_1 be a nonempty subset of $\mathbb{R}^k$ and $U_2=([0,\infty))^n$. Let $\mathcal{U}_1$ be the class of measurable adapted processes $v:[0,T]\times\Omega\to U_1$ such that $\sup_{0\le t\le T}\mathbb{E}|v_t|^2<\infty$. Denote by $\mathcal{U}_2$ the class of measurable adapted processes $\eta:[0,T]\times\Omega\to U_2$ such that $\eta(\cdot)$ is of bounded variation, nondecreasing, left-continuous with right limits, with $\eta_0=0$ and $\mathbb{E}|\eta_T|^2<\infty$. An admissible regular-singular control is a pair of processes $(v(\cdot),\eta(\cdot))\in\mathcal{U}\doteq\mathcal{U}_1\times\mathcal{U}_2$. In what follows, we denote by c a positive constant which can be varied in different lines.

Given $a\in\mathbb{R}^n$ and $\eta(\cdot)\in\mathcal{U}_2$, let us consider the following SDE:

$$x_t=a+\int_0^t b(s,x_s)ds+\int_0^t\sigma(s,x_s)dB_s+\int_0^t C_sd\eta_s,\quad 0\le t\le T,\tag{1}$$

where $b:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$ are measurable mappings, and $C:[0,T]\to\mathbb{R}^{n\times n}$ is a continuous function.

Proposition 2.1 *Assume that b, σ are uniformly Lipschitz in x, $b(\cdot,0)\in H^2(\mathbb{R}^n)$ and $\sigma(\cdot,0)\in H^2(\mathbb{R}^{n\times d})$. Then SDE (1) admits a unique solution $x(\cdot)\in S^2(\mathbb{R}^n)$.*

Proof Set $b_1(t,x)=b\big(t,x+\int_0^t C_sd\eta_s\big)$ and $\sigma_1(t,x)=\sigma\big(t,x+\int_0^t C_sd\eta_s\big)$. Then, b_1 and σ_1 are uniformly Lipschitz in x. Moreover, we can easily check that $b_1(\cdot,0)\in H^2(\mathbb{R}^n)$ and $\sigma_1(\cdot,0)\in H^2(\mathbb{R}^{n\times d})$. Consequently, the following SDE:

$$X_t=a+\int_0^t b_1(s,X_s)ds+\int_0^t\sigma_1(t,X_s)dB_s,\quad 0\le t\le T$$

has a unique solution $X(\cdot) \in S^2(\mathbb{R}^n)$. Let us define $x_t = X_t + \int_0^t C_s d\eta_s$. Then it's easy to check that $x(\cdot) \in S^2(\mathbb{R}^n)$ and it solves SDE (1). Thus, the existence of the solution is proved. Let $x^1(\cdot)$ and $x^2(\cdot)$ be two solutions of SDE (1). Then we have

$$x_t^1 - x_t^2 = \int_0^t \left[b(s, x_s^1) - b(s, x_s^2)\right] ds + \int_0^t \left[\sigma(s, x_s^1) - \sigma(s, x_s^2)\right] dB_s.$$

By the basic tools of stochastic calculus, it's easy to get $\mathbb{E}\left[\sup_{0\le t\le T} |x_t^1 - x_t^2|^2\right] = 0$. Thus, the uniqueness is also proved. ∎

Giving an $\mathcal{F}_T$-measurable random variable ζ, we consider the following BSDE:

$$y_t = \zeta + \int_t^T f(s, y_s, z_s)ds - \int_t^T z_s dB_s + \int_t^T D_s d\eta_s, \quad 0 \le t \le T, \tag{2}$$

where $f : [0,T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is a measurable mapping and $D : [0,T] \to \mathbb{R}^{m\times n}$ is a continuous function.

Proposition 2.2 *Assume that* $\mathbb{E}|\zeta|^2 < \infty$, *$f$ is uniformly Lipschitz in (y, z) and $f(\cdot, 0, 0) \in H^2(\mathbb{R}^m)$. Then BSDE (2) admits a unique solution $(y(\cdot), z(\cdot)) \in S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m\times d})$.*

Proof Set $A_t = \int_0^t D_s d\eta_s$ and $F(t, y, z) = f(t, y - A_t, z)$. Then it's easy to check that the following BSDE:

$$Y_t = \zeta + A_T + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \le t \le T$$

admits a unique solution $(Y(\cdot), Z(\cdot)) \in S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m\times d})$. Now, let us set $y_t = Y_t - A_t$ and $z_t = Z_t$. Then it follows that $(y(\cdot), z(\cdot)) \in S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m\times d})$ and it solves BSDE (2). Let $(y^1(\cdot), z^1(\cdot))$ and $(y^2(\cdot), z^2(\cdot))$ be two solutions of BSDE (2). By Itô's formula applied to $|y_s^1 - y_s^2|^2$, $t \le s \le T$, combined with Gronwall's Lemma, it's easy to prove the uniqueness. ∎

Now, let us formulate the stochastic optimal regular-singular control problem. The control system evolves by the following forward-backward stochastic differential equation (FBSDE):

$$\begin{cases} dx_t = b(t, x_t, v_t)dt + \sigma(t, x_t, v_t)dB_t + C_t d\eta_t, \\ dy_t = -f(t, x_t, y_t, z_t, v_t)dt + z_t dB_t - D_t d\eta_t, \\ x_0 = a, \quad y_T = \varphi(x_T), \end{cases} \tag{3}$$

where $b : [0,T]\times\mathbb{R}^n\times U_1 \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times U_1 \to \mathbb{R}^{n\times d}$, $f : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times U_1 \to \mathbb{R}^m$, $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, $C : [0,T] \to \mathbb{R}^{n\times n}$ and $D : [0,T] \to \mathbb{R}^{m\times n}$ are continuous functions. The pair $(v(\cdot), \eta(\cdot)) \in \mathcal{U}$ is called an admissible regular-singular control. The problem is to minimize the following cost functional over $\mathcal{U}$:

$$J(v(\cdot), \eta(\cdot)) = \mathbb{E}\left[g(x_T) + h(y_0) + \int_0^T l(t, x_t, y_t, v_t)dt + \int_0^T G_t \cdot d\eta_t\right], \tag{4}$$

where $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^m \to \mathbb{R}$, $l : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times U_1 \to \mathbb{R}$, and $G : [0,T] \to \mathbb{R}^n$ are continuous functions.

Let us assume

(H1) b, σ, f, and φ are bounded by $c(1+|x|+|y|+|z|+|v|)$. They are continuously differentiable in (x, y, z), and the partial derivatives are continuous and uniformly bounded.

(H2) g, h, and l are continuously differentiable in (x, y), with derivatives bounded by $c(1+|x|+|y|+|v|)$. Moreover, l has linear growth in (x, y, z, v).

By Propositions 2.1 and 2.2, FBSDE (3) admits a unique solution $(x(\cdot), y(\cdot), z(\cdot)) \in S^2(\mathbb{R}^n)\times S^2(\mathbb{R}^m)\times H^2(\mathbb{R}^{m\times d})$, and the functional J is well defined.

In this stochastic regular-singular control problem, the control system is a forward-backward system, the regular control appears in the diffusion coefficient and the set of regular controls is not necessarily convex. As mentioned before, this problem is difficult to solve with the classical method of spike variation. We will consider it with the relaxed control method.

Definition 2.3 A relaxed control $(q_t)_t$ is a $\mathbb{P}(U_1)$-valued, $(\mathcal{F}_t)$-progressively measurable process such that $\chi_{(0,t]}\cdot q_t$ is $\mathcal{F}_t$-measurable for any t.

For more details on relaxed controls, one can refer to [17], [21], and [22]. We denote by $\mathcal{R}_1$ the set of relaxed controls and by $\mathcal{R} \doteq \mathcal{R}_1 \times \mathcal{U}_2$ the set of admissible relaxed-singular controls.

Now, let us turn to the study of the stochastic optimal relaxed-singular control problem which corresponds to the regular-singular optimal control problem. The control system is

$$\begin{cases} dx_t = \int_{U_1} b(t, x_t, a) q_t(da)dt + \int_{U_1} \sigma(t, x_t, a) q_t(da) dB_t + C_t d\eta_t, \\ dy_t = -\int_{U_1} f(t, x_t, y_t, z_t, a) q_t(da)dt + z_t dB_t - D_t d\eta_t, \\ x_0 = a, \quad y_T = \varphi(x_T), \end{cases} \tag{5}$$

and the expected cost to be minimized over the class $\mathcal{R}$ is defined by

$$\mathcal{J}(q(\cdot), \eta(\cdot)) = \mathbb{E}\left[g(x_T) + h(y_0) + \int_0^T \int_{U_1} l(t, x_t, y_t, a) q_t(da)dt + \int_0^T G_t \cdot d\eta_t \right]. \tag{6}$$

Let us impose the following condition:

(H3)

$$\mathbb{E}\int_0^T \left[\int_{U_1} |a| q_t(da) \right]^2 dt < \infty, \quad \forall q \in \mathcal{R}_1.$$

Now, let us show that FBSDE (5) admits a unique solution and $\mathcal{J}$ is well defined on $\mathcal{R}$. In fact, for $\psi = b, \sigma, f, l$, set

$$\overline{\psi}(t, x, y, z, q) = \int_{U_1} \psi(t, x, y, z, a) q(da), \quad (t, x, y, z, q) \in [0, T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{P}(U_1).$$

Then FBSDE (5) can be rewritten as

$$\begin{cases} dx_t = \overline{b}(t, x_t, q_t)dt + \overline{b}(t, x_t, q_t) dB_t + C_t d\eta_t, \\ dy_t = -\overline{f}(t, x_t, y_t, z_t, q_t)dt + z_t dB_t - D_t d\eta_t, \\ x_0 = a, \quad y_T = \varphi(x_T), \end{cases}$$

and the cost functional becomes

$$\mathcal{J}(q(\cdot),\eta(\cdot)) = \mathbb{E}\bigg[g(x_T) + h(y_0) + \int_0^T \bar{l}(t,x_t,y_t,q_t)dt + \int_0^T G_t \cdot d\eta_t\bigg].$$

From the assumptions (H1) and (H3) it follows that $\overline{b}$, $\overline{\sigma}$, $\overline{f}$ are Lipschitz in (x,y,z) and satisfy

$$\mathbb{E}\int_0^T \left[|\overline{b}(t,0,q_t)|^2 + |\overline{\sigma}(t,0,q_t)|^2 + |\overline{f}(t,0,0,0,q_t)|^2\right] dt < \infty.$$

Consequently, FBSDE (5) admits a unique solution $(x(\cdot),y(\cdot),z(\cdot)) \in S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m\times d})$ and $\mathcal{J}$ is well defined on $\mathcal{R}$.

Remark 2.4 The relaxed control finds its interest in two essential points. The first is that an optimal relaxed control exists under general conditions on the coefficients, while the existence of regular controls is difficult to insure. See, e.g., [17] and [22]. The second is that the relaxed-singular control problem is a generalization of the regular-singular control problem. In fact, if we take $q_t(da) = \delta_{v_t}(da)$, where δ_v is a Dirac measure concentrated at a single point v, then for $\psi = b,\sigma,f,l$, we have

$$\int_{U_1} \psi(t,x,y,z,a)\delta_{v_t}(da) = \psi(t,x,y,z,v_t).$$

Thus, FBSDE (3) and the functional $J(v(\cdot),\eta(\cdot))$ are special cases of (5) and $\mathcal{J}(q(\cdot),\eta(\cdot))$ respectively.

3 Necessary and Sufficient Conditions for the Stochastic Optimal Relaxed-Singular Control Problem

3.1 Maximum Principle for the Stochastic Optimal Relaxed-Singular Control Problem

Since $\mathcal{R}$ is convex, the convex perturbation method is used to derive the maximum principle. Let $(\mu(\cdot),\xi(\cdot)) \in \mathcal{R}$ be an optimal relaxed-singular control and $(\widehat{x}(\cdot),\widehat{y}(\cdot),\widehat{z}(\cdot))$ be the corresponding solution of FBSDE (5). A perturbed control can be defined by

$$\left(\mu_t^\theta,\xi_t^\theta\right) = (\mu_t + \theta(q_t - \mu_t), \xi_t + \theta(\eta_t - \xi_t)), \quad \forall\theta\in[0,1],\ (q(\cdot),\eta(\cdot))\in\mathcal{R}.$$

Let us denote by $\left(x^\theta(\cdot),y^\theta(\cdot),z^\theta(\cdot)\right)$ the solution of FBSDE (5) associated with $\left(\mu^\theta(\cdot),\xi^\theta(\cdot)\right)$.

Lemma 3.1 *Assume* (H1) *and* (H3). *Then we have*

$$\lim_{\theta\to 0}\left[\sup_{0\le t\le T}\mathbb{E}|x_t^\theta - \widehat{x}_t|^2\right] = 0, \tag{7}$$

$$\lim_{\theta\to 0}\left[\sup_{0\le t\le T}\mathbb{E}|y_t^\theta - \widehat{y}_t|^2\right] = 0, \tag{8}$$

$$\lim_{\theta\to 0}\mathbb{E}\int_0^T |z_t^\theta - \widehat{z}_t|^2 dt = 0. \tag{9}$$

Proof From Section 2, we know that both $\big(x^\theta(\cdot), y^\theta(\cdot), z^\theta(\cdot)\big)$ and $(\widehat{x}(\cdot), \widehat{y}(\cdot), \widehat{z}(\cdot))$ belong to $S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m\times d})$. By (H1) and (H3), it's easy to deduce that

$$\mathbb{E}|x_t^\theta|^2 \le c(1+\theta^2)\left[1+\mathbb{E}\int_0^t |x_s^\theta|^2 ds\right] \le 2c + 2c\mathbb{E}\int_0^t |x_s^\theta|^2 ds,$$

where $c > 0$ is independent of θ. Thus, by Gronwall's Lemma,

$$\sup_\theta \sup_{0\le t\le T} \mathbb{E}|x_t^\theta|^2 \le c. \tag{10}$$

Now, squaring both sides of

$$y_t^\theta + \int_t^T z_s^\theta dB_s = \varphi\big(x_T^\theta\big) + \int_t^T \int_{U_1} f\big(s, x_s^\theta, y_s^\theta, z_s^\theta, a\big)\mu_s^\theta(da)ds + \int_t^T D_s d\xi_s^\theta,$$

and then using (H1) and (10) we derive

$$\begin{aligned}\mathbb{E}|y_t^\theta|^2 + \mathbb{E}\int_t^T |z_s^\theta|^2 ds &\le c(1+\theta^2)\left[1+\mathbb{E}\int_t^T |y_s^\theta|^2 ds + (T-t)\mathbb{E}\int_t^T |z_s^\theta|^2 ds\right]\\ &\le 2c\left[1+\mathbb{E}\int_t^T |y_s^\theta|^2 ds + (T-t)\mathbb{E}\int_t^T |z_s^\theta|^2 ds\right],\end{aligned}$$

where $c > 0$ is independent of θ and t. By choosing $\delta = \frac{1}{4c}$, we have

$$\mathbb{E}|y_t^\theta|^2 + \frac{1}{2}\mathbb{E}\int_t^T |z_s^\theta|^2 ds \le 2c + 2c\mathbb{E}\int_t^T |y_s^\theta|^2 ds, \quad T-\delta \le t \le T.$$

Then Gronwall's Lemma yields

$$\sup_{T-\delta\le t\le T} \mathbb{E}|y_t^\theta|^2 + \mathbb{E}\int_{T-\delta}^T |z_t^\theta|^2 dt \le c.$$

With the same procedure, we obtain

$$\sup_{T-2\delta\le t\le T-\delta} \mathbb{E}|y_t^\theta|^2 + \mathbb{E}\int_{T-2\delta}^{T-\delta} |z_t^\theta|^2 dt \le c.$$

After a finite number of iterations, it follows

$$\sup_\theta \sup_{0\le t\le T} \mathbb{E}|y_t^\theta|^2 + \sup_\theta \mathbb{E}\int_0^T |z_t^\theta|^2 dt \le c. \tag{11}$$

By the definition of μ^θ, we get

$$\begin{aligned}
x_t^\theta - \widehat{x}_t = &\int_0^t \int_{U_1} \left[b(s, x_s^\theta, a) - b(s, \widehat{x}_s, a)\right] \mu_s(da)ds \\
&+ \int_0^t \int_{U_1} \left[\sigma(s, x_s^\theta, a) - \sigma(s, \widehat{x}_s, a)\right] \mu_s(da)dB_s \\
&+ \theta \int_0^t \int_{U_1} b(s, x_s^\theta, a) q_s(da)ds - \theta \int_0^t \int_{U_1} b(s, x_s^\theta, a)\mu_s(da)ds \\
&+ \theta \int_0^t \int_{U_1} \sigma(s, x_s^\theta, a) q_s(da)dB_s - \theta \int_0^t \int_{U_1} \sigma(s, x_s^\theta, a)\mu_s(da)dB_s \\
&+ \theta \int_0^t C_s d\eta_s - \theta \int_0^t C_s d\xi_s.
\end{aligned}$$

By standard arguments, using (H1), (H3), and (10), we deduce

$$\mathbb{E}|x_t^\theta - \widehat{x}_t|^2 \leq c\theta^2 + c(1+\theta^2)\mathbb{E}\int_0^t |x_s^\theta - \widehat{x}_s|^2 ds \leq c\theta^2 + 2c\mathbb{E}\int_0^t |x_s^\theta - \widehat{x}_s|^2 ds.$$

From Gronwall's Lemma, it follows that $\sup_{0\leq t\leq T} \mathbb{E}|x_t^\theta - \widehat{x}_t|^2 \leq c\theta^2$. Then the result (7) follows immediately.

From the definition of μ^θ, it follows

$$\begin{aligned}
&(y_t^\theta - \widehat{y}_t) + \int_t^T (z_s^\theta - \widehat{z}_s)dB_s \\
&= \left[\varphi(x_T^\theta) - \varphi(\widehat{x}_T)\right] + \theta \int_t^T D_s d(\eta_s - \xi_s) \\
&\quad + \theta \int_t^T \int_{U_1} f\left(s, x_s^\theta, y_s^\theta, z_s^\theta, a\right) q_s(da)ds - \theta \int_t^T \int_{U_1} f(s, x_s^\theta, y_s^\theta, z_s^\theta, a)\mu_s(da)ds \\
&\quad + \int_t^T \int_{U_1} \left[f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) - f(s, \widehat{x}_s, y_s^\theta, z_s^\theta, a)\right] \mu_s(da)ds \\
&\quad + \int_t^T \int_{U_1} \left[f(s, \widehat{x}_s, y_s^\theta, z_s^\theta, a) - f(s, \widehat{x}_s, \widehat{y}_s, z_s^\theta, a)\right] \mu_s(da)ds \\
&\quad + \int_t^T \int_{U_1} \left[f(s, \widehat{x}_s, \widehat{y}_s, z_s^\theta, a) - f(s, \widehat{x}_s, \widehat{y}_s, \widehat{z}_s, a)\right] \mu_s(da)ds.
\end{aligned} \tag{12}$$

By squaring both sides of (12), using (H1) and (H3), we obtain that there exists $c > 0$ which is independent of θ such that

$$\mathbb{E}|y_t^\theta - \widehat{y}_t|^2 + \mathbb{E}\int_t^T |z_s^\theta - \widehat{z}_s|^2 \mathrm{d}s \leq c\mathbb{E}\int_t^T |y_s^\theta - \widehat{y}_s|^2 \mathrm{d}s + c(T-t)\mathbb{E}\int_t^T |z_s^\theta - \widehat{z}_s|^2 \mathrm{d}s + \alpha^\theta,$$

where α^θ is given by

$$\alpha^\theta = c\mathbb{E}|x_T^\theta - \widehat{x}_T|^2 + c\mathbb{E}\int_0^T |x_t^\theta - \widehat{x}_t|^2 \mathrm{d}t + c\theta^2.$$

It follows from (7) that $\lim_{\theta\to 0}\alpha^\theta = 0$. Finally, applying the iteration procedure, we obtain the results (8) and (9). ∎

Let $x^1(\cdot)$ be the solution of

$$\begin{cases} dx_t^1 = \int_{U_1} b_x(t,\widehat{x}_t,a)\mu_t(da)\cdot x_t^1 dt + \int_{U_1}\sigma_x(t,\widehat{x}_t,a)\mu_t(da)x_t^1\cdot dB_t \\ \qquad + \left[\int_{U_1} b(t,\widehat{x}_t,a)q_t(da) - \int_{U_1} b(t,\widehat{x}_t,a)\mu_t(da)\right]dt \\ \qquad + \left[\int_{U_1}\sigma(t,\widehat{x}_t,a)q_t(da) - \int_{U_1}\sigma(t,\widehat{x}_t,a)\mu_t(da)\right]dB_t + C_t d(\eta_t-\xi_t), \\ x_0^1 = 0, \end{cases} \tag{13}$$

and $(y^1(\cdot),z^1(\cdot))$ be the solution of

$$\begin{cases} -dy_t^1 = \left[\int_{U_1} f(t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,a)q_t(da) - \int_{U_1} f(t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,a)\mu_t(da)\right]dt \\ \qquad + \int_{U_1}\Big[f_x(t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,a)\cdot x_t^1 + f_y(t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,a)\cdot y_t^1 \\ \qquad + f_z(t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,a)\cdot z_t^1\Big]\mu_t(da)dt - z_t^1 dB_t + D_t d(\eta_t-\xi_t), \\ y_T^1 = \varphi_x(\widehat{x}_T)x_T^1. \end{cases} \tag{14}$$

Set

$$X_t^\theta = \theta^{-1}(x_t^\theta - \widehat{x}_t) - x_t^1,\quad Y_t^\theta = \theta^{-1}(y_t^\theta - \widehat{y}_t) - y_t^1,\quad Z_t^\theta = \theta^{-1}(z_t^\theta - \widehat{z}_t) - z_t^1.$$

Lemma 3.2 *Under* (H1)–(H3), *we have*

$$\lim_{\theta\to 0}\left[\sup_{0\le t\le T}\mathbb{E}|X_t^\theta|^2\right] = 0,\quad \lim_{\theta\to 0}\left[\sup_{0\le t\le T}\mathbb{E}|Y_t^\theta|^2\right] = 0,\quad \lim_{\theta\to 0}\mathbb{E}\int_0^T |Z_t^\theta|^2 dt = 0.$$

Proof It's easy to check that X_t^θ, Y_t^θ, and Z_t^θ do not depend on the singular part. Hence, the result follows immediately from Lemma 9 in [17]. ∎

From the optimality of $(\mu(\cdot),\xi(\cdot))$, we derive the following variational inequality.

Lemma 3.3 *Let $(\mu(\cdot),\xi(\cdot))$ be an optimal control of the stochastic relaxed-singular control problem and $(\widehat{x}(\cdot),\widehat{y}(\cdot),\widehat{z}(\cdot))$ be the corresponding trajectory. Then $\forall(q(\cdot),\eta(\cdot))\in\mathcal{R}$,*

$$\begin{aligned} 0 \le&\, \mathbb{E}\left[g_x(\widehat{x}_T)\cdot x_T^1\right] + \mathbb{E}\left[h_y(\widehat{y}_0)\cdot y_0^1\right] + \mathbb{E}\int_0^T G_t\cdot d(\eta_t-\xi_t) \\ &+ \mathbb{E}\int_0^T\left[\int_{U_1} l(t,\widehat{x}_t,\widehat{y}_t,a)q_t(da) - \int_{U_1} l(t,\widehat{x}_t,\widehat{y}_t,a)\mu_t(da)\right]dt \\ &+ \mathbb{E}\int_0^T\left[\int_{U_1} l_x(t,\widehat{x}_t,\widehat{y}_t,a)\mu_t(da)\cdot x_t^1 + \int_{U_1} l_y(t,\widehat{x}_t,\widehat{y}_t,a)\mu_t(da)\cdot y_t^1\right]dt. \end{aligned}$$

Proof Since $\theta^{-1}\left[\mathcal{J}(\mu^\theta(\cdot),\xi^\theta(\cdot))-\mathcal{J}(\mu(\cdot),\xi(\cdot))\right]\geq 0$, from simple calculation, it follows

$$\begin{aligned}0\leq&\mathbb{E}\left[x_T^1\cdot\int_0^1 g_x(\widehat{x}_T+\lambda\theta(X_T^\theta+x_T^1))d\lambda\right]+\mathbb{E}\left[y_0^1\cdot\int_0^1 h_y(\widehat{y}_0+\lambda\theta(Y_0^\theta+y_0^1))d\lambda\right]\\&+\mathbb{E}\int_0^T\left[\int_{U_1} l(t,\widehat{x}_t,\widehat{y}_t,a)q_t(da)-\int_{U_1} l(t,\widehat{x}_t,\widehat{y}_t,a)\mu_t(da)\right]dt+\rho_t^\theta\\&+\theta^{-1}\mathbb{E}\int_0^T\int_{U_1}\left[l(t,x_t^\theta,y_t^\theta,a)-l(t,\widehat{x}_t,\widehat{y}_t,a)\right]\mu_t(da)dt+\mathbb{E}\int_0^T G_t\cdot d(\eta_t-\xi_t),\end{aligned}\tag{15}$$

where ρ_t^θ is given by

$$\begin{aligned}\rho_t^\theta=&\mathbb{E}\left[X_T^\theta\cdot\int_0^1 g_x(\widehat{x}_T+\lambda\theta(X_T^\theta+x_T^1))d\lambda\right]+\mathbb{E}\left[Y_0^\theta\cdot\int_0^1 h_y(\widehat{y}_0+\lambda\theta(Y_0^\theta+y_0^1))d\lambda\right]\\&+\mathbb{E}\int_0^T\int_{U_1}\left[l(t,x_t^\theta,y_t^\theta,a)-l(t,\widehat{x}_t,\widehat{y}_t,a)\right](q_t(da)-\mu_t(da))dt.\end{aligned}$$

Set $\varLambda_t^\theta(a)=\left(t,\widehat{x}_t+\lambda(x_t^\theta-\widehat{x}_t),\widehat{y}_t+\lambda(y_t^\theta-\widehat{y}_t),a\right)$. Then we have

$$\begin{aligned}&\mathbb{E}\int_0^T\int_{U_1}\left[l(t,x_t^\theta,y_t^\theta,a)-l(t,\widehat{x}_t,\widehat{y}_t,a)\right](q_t(da)-\mu_t(da))dt\\=&\mathbb{E}\int_0^T\int_0^1\int_{U_1}\left[l_x(\varLambda_t^\theta(a))\cdot(x_t^\theta-\widehat{x}_t)+l_y(\varLambda_t^\theta(a))\cdot(y_t^\theta-\widehat{y}_t)\right](q_t(da)-\mu_t(da))d\lambda dt.\end{aligned}$$

Hence, by Lemmas 3.1 and 3.2, we can use Hölder's inequality to get $\rho_t^\theta\to 0$ as $\theta\to 0$. We also have

$$\begin{aligned}&\theta^{-1}\mathbb{E}\int_0^T\int_{U_1}\left[l(t,x_t^\theta,y_t^\theta,a)-l(t,\widehat{x}_t,\widehat{y}_t,a)\right]\mu_t(da)dt\\=&\mathbb{E}\int_0^T\int_0^1\int_{U_1}\left[l_x(\varLambda_t^\theta(a))\cdot(X_t^\theta+x_t^1)+l_y(\varLambda_t^\theta(a))\cdot(Y_t^\theta+x_t^1)\right]\mu_t(da)d\lambda dt.\end{aligned}$$

Then the proof can be concluded from Lemmas 3.1, 3.2, and the dominated convergence theorem by letting θ go to 0 in (15). ∎

Let us define $\mathcal{H}:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{P}(U_1)\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}$ by

$$\begin{aligned}\mathcal{H}(t,x,y,z,q,k,p,P)=&p\cdot\int_{U_1} b(t,x,a)q_t(da)+P\cdot\int_{U_1}\sigma(t,x,a)q_t(da)\\&+k\cdot\int_{U_1} f(t,x,y,z,a)q_t(da)+\int_{U_1} l(t,x,y,a)q_t(da).\end{aligned}$$

We denote by $(x^{q,\eta}(\cdot),y^{q,\eta}(\cdot),z^{q,\eta}(\cdot))$ the trajectory corresponding to $(q(\cdot),\eta(\cdot))\in\mathcal{R}$. Let us introduce the following FBSDE (called adjoint equation):

$$\begin{cases} dk_t^{\mu,\xi} = \mathcal{H}_y\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big)dt \\ \qquad\qquad + \mathcal{H}_z\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big)dB_t, \\ dp_t^{\mu,\xi} = -\mathcal{H}_x\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big)dt + P_t^{\mu,\xi} dB_t, \\ k_0^{\mu,\xi} = h_y\big(y_0^{\mu,\xi}\big), \quad p_T^{\mu,\xi} = g_x\big(x_T^{\mu,\xi}\big) + \varphi_x\big(x_T^{\mu,\xi}\big)k_T^{\mu,\xi}. \end{cases} \tag{16}$$

It's easy to check that the adjoint equation admits a unique solution $\big(k^{\mu,\xi}(\cdot), p^{\mu,\xi}(\cdot), P^{\mu,\xi}(\cdot)\big)$.

Now, we are ready to establish the maximum principle for the stochastic relaxed-singular control problem.

Theorem 3.4 *Let $(\mu(\cdot), \xi(\cdot))$ be an optimal control of the relaxed-singular optimal control problem, $\big(x^{\mu,\xi}(\cdot), y^{\mu,\xi}(\cdot), z^{\mu,\xi}(\cdot)\big)$ be the corresponding trajectory and $\big(k^{\mu,\xi}(\cdot), p^{\mu,\xi}(\cdot), P^{\mu,\xi}(\cdot)\big)$ be the solution of* (16) *associated with $(\mu(\cdot), \xi(\cdot))$. Then for any $(q(\cdot), \eta(\cdot)) \in \mathcal{R}$, we have*

$$\begin{aligned} \mathbb{E}\int_0^T \Big[&\mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, q_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) \\ &-\mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big)\Big]\, dt \geq 0, \end{aligned} \tag{17}$$

$$\mathbb{E}\int_0^T \big(C_t^T p_t^{\mu,\xi} + D_t^T k_t^{\mu,\xi} + G_t\big)\cdot d(\eta_t - \xi_t) \geq 0. \tag{18}$$

Proof Applying Itô's formula to $p_t^{\mu,\xi}\cdot x_t^1 + k_t^{\mu,\xi}\cdot y_t^1$, combining with Lemma 3.3, we get

$$\begin{aligned} &\mathbb{E}\int_0^T \Big[\mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, q_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) - \mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big)\Big]\, dt \\ &+ \mathbb{E}\int_0^T \big(C_t^T p_t^{\mu,\xi} + D_t^T k_t^{\mu,\xi} + G_t\big)\cdot d(\eta_t - \xi_t) \geq 0. \end{aligned}$$

We can conclude (17) and (18) by choosing $\eta(\cdot) = \xi(\cdot)$ and $q(\cdot) = \mu(\cdot)$, respectively. ∎

For simplicity, let us set $M_t = C_t^T p_t^{\mu,\xi} + D_t^T k_t^{\mu,\xi} + G_t$, $M_t = (M_t^1, M_t^2, \cdots, M_t^n)^{\mathrm{T}}$. Then similar to Theorem 4.2 in [18] we can get

Theorem 3.5 *Assume the conditions in Theorem* 3.4 *still hold. Then for any $q \in \mathbb{P}(U_1)$,*

$$\begin{aligned} &\mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, q, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) \\ &\geq \mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big), \quad \text{a.e. a.s.}, \end{aligned} \tag{19}$$

$$\mathbb{P}\left\{\sum_{i=1}^n \chi_{[M_t^i \geq 0]} d\xi_t^i = 0\right\} = 1, \tag{20}$$

$$\mathbb{P}\big\{\forall t \in [0,T], \forall i; M_t^i \geq 0\big\} = 1. \tag{21}$$

3.2 Sufficient Optimality Conditions for the Stochastic Relaxed-Singular Control Problem

Let us still denote by $(x^{q,\eta}(\cdot), y^{q,\eta}(\cdot), z^{q,\eta}(\cdot))$ the trajectory associated with $(q(\cdot), \eta(\cdot)) \in \mathcal{R}$.

Theorem 3.6 *Assume that g, h, and $\mathcal{H}(t,\cdot,\cdot,\cdot,q,k,p,P)$ are convex, and moreover, $y_T^{q,\eta}$ takes the following particular form: $y_T^{q,\eta} = Rx_T^{q,\eta}+\zeta$, where $R \in \mathbb{R}^{m\times n}$ and $\zeta \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^m)$. Then $(\mu(\cdot),\xi(\cdot)) \in \mathcal{R}$ is an optimal relaxed-singular control if it satisfies* (19), (20), *and* (21).

Proof Let us denote $\widehat{\mathcal{J}} = \mathcal{J}(q(\cdot),\eta(\cdot)) - \mathcal{J}(\mu(\cdot),\xi(\cdot))$ for $(q(\cdot),\eta(\cdot)) \in \mathcal{R}$. Then from the convexity of g and h, it follows

$$\begin{aligned}\widehat{\mathcal{J}} \geq & \mathbb{E}\big[g_x\big(x_T^{\mu,\xi}\big)\cdot\big(x_T^{q,\eta}-x_T^{\mu,\xi}\big)\big] + \mathbb{E}\big[h_y\big(y_0^{\mu,\xi}\big)\cdot\big(y_0^{q,\eta}-y_0^{\mu,\xi}\big)\big] + \mathbb{E}\int_0^T G_t\cdot d(\eta_t-\xi_t) \\ & + \mathbb{E}\int_0^T\bigg[\int_{U_1} l\big(t,x_t^{q,\eta},y_t^{q,\eta},a\big)q_t(da) - \int_{U_1} l\big(t,x_t^{\mu,\xi},y_t^{\mu,\xi},a\big)\mu_t(da)\bigg]\,dt.\end{aligned}$$

Let us set $\mathbb{H}^{q,\eta}(t) = \mathcal{H}\big(t,x_t^{q,\eta},y_t^{q,\eta},z_t^{q,\eta},q_t,k_t^{\mu,\xi},p_t^{\mu,\xi},P_t^{\mu,\xi}\big)$. Then by Itô's formula applied to $p_t^{\mu,\xi}\cdot\big(x_t^{q,\eta}-x_t^{\mu,\xi}\big)+k_t^{u,\xi}\cdot\big(y_t^{q,\eta}-y_t^{\mu,\xi}\big)$, we get

$$\widehat{\mathcal{J}} \geq \mathbb{E}\bigg[\int_0^T M_t\cdot d(\eta_t-\xi_t) + \int_0^T \Theta_t\,dt\bigg],$$

where

$$\Theta_t = \mathbb{H}^{q,\eta}(t) - \mathbb{H}^{\mu,\xi}(t) - \mathbb{H}_x^{\mu,\xi}(t)\cdot\big(x_t^{q,\eta}-x_t^{\mu,\xi}\big) - \mathbb{H}_y^{\mu,\xi}(t)\cdot\big(y_t^{q,\eta}-y_t^{\mu,\xi}\big) - \mathbb{H}_z^{\mu,\xi}(t)\cdot\big(z_t^{q,\eta}-z_t^{\mu,\xi}\big).$$

From (20) and (21), it's easy to get

$$\mathbb{E}\int_0^T M_t\cdot d(\eta_t-\xi_t) \geq 0.$$

By the minimum condition (19) and Lemma 2.3 (iii) of Chapter 3 in [23], we have $0 \in \partial_q\mathbb{H}^{\mu,\xi}(t)$. Then by Lemma 2.4 of Chapter 3 in [23], we can further conclude that

$$\big(\mathbb{H}_x^{\mu,\xi}(t),\mathbb{H}_y^{\mu,\xi}(t),\mathbb{H}_z^{\mu,\xi}(t),0\big) \in \partial_{x,y,z,q}\mathbb{H}^{\mu,\xi}(t).$$

Consequently, by the assumption that $\mathcal{H}$ is convex in (x,y,z) and is linear in q we can conclude from Lemma 2.3 (v) of Chapter 3 in [23] that $\Theta_t \geq 0$. Hence, it follows that $\widehat{\mathcal{J}} \geq 0$ and thus the proof is complete. ∎

4 Necessary and Sufficient Conditions for the Stochastic Regular-Singular Control Problem

In this section, we aim to derive the necessary and sufficient conditions for the stochastic regular-singular control problem from the corresponding results for the stochastic relaxed-singular control problem obtained in Section 3.

Let us define $\delta(\mathcal{U}_1)\times\mathcal{U}_2 = \big\{(q(\cdot),\eta(\cdot)) \in \mathcal{R} : q(\cdot) = \delta_{v(\cdot)}, v(\cdot) \in \mathcal{U}_1\big\} \subset \mathcal{R}$, and denote by $\delta(U_1)\times U_2$ the action set of all relaxed-singular controls in $\delta(\mathcal{U}_1)\times\mathcal{U}_2$.

The following result can be easily obtained by Remark 2.4.

Lemma 4.1 *Under* (H1)–(H3)*, we have*

$$\min_{(v(\cdot),\eta(\cdot))\in\mathcal{U}} J(v(\cdot),\eta(\cdot)) = \min_{(q(\cdot),\eta(\cdot))\in\delta(\mathcal{U}_1)\times\mathcal{U}_2} \mathcal{J}(q(\cdot),\eta(\cdot)).$$

Thus, $(u(\cdot),\xi(\cdot))$ *minimizes* J *over* $\mathcal{U}$ *if and only if* $(\delta_{u(\cdot)},\xi(\cdot))$ *minimizes* $\mathcal{J}$ *over* $\delta(\mathcal{U}_1)\times\mathcal{U}_2$*.*

We need the following result, which is called chattering lemma. It can also be seen in [17] and [21].

Lemma 4.2 *Let* $q(\cdot)$ *be a predictable process with values in* $\mathbb{P}(U_1)$*. Then there exists a sequence of predictable processes* $(u^n(\cdot))\subset U_1$ *such that the sequence of measures* $dt\delta_{u_t^n}(da)$ *converges weakly to* $dtq_t(da)$*,* $\mathbb{P}$*-a.s.*

In what follows, we need an additional assumption:

(H4) The set U_1 is compact. The functions b,σ,f,l are bounded.

By Lemma 4.2, it's easy to get the following result, which connects the stochastic regular-singular control problem with the stochastic relaxed-singular control problem. The proof of this lemma is very similar to that of Lemma 15 in [17], so we omit it.

Lemma 4.3 *There exists a sequence* $u^n(\cdot)\subset\mathcal{U}_1$ *such that* $(x^{n,\eta}(\cdot),y^{n,\eta}(\cdot),z^{n,\eta}(\cdot))$ *converges to* $(x^{q,\eta}(\cdot),y^{q,\eta}(\cdot),z^{q,\eta}(\cdot))$ *in* $S^2(\mathbb{R}^n)\times S^2(\mathbb{R}^m)\times H^2(\mathbb{R}^{m\times d})$ *and* $J(u^n(\cdot),\eta(\cdot))$ *converges to* $\mathcal{J}(q(\cdot),\eta(\cdot))$*, where* $(x^{n,\eta}(\cdot),y^{n,\eta}(\cdot),z^{n,\eta}(\cdot))$ *is the trajectory corresponding to* $(u^n(\cdot),\eta(\cdot))$*.*

The following lemma will play a key role in deriving the maximum principle in this section.

Lemma 4.4 *The regular-singular control* $(u(\cdot),\xi(\cdot))$ *minimizes* J *over* $\mathcal{U}$ *if and only if the relaxed-singular control* $(\delta_{u(\cdot)},\xi(\cdot))$ *minimizes* $\mathcal{J}$ *over* $\mathcal{R}$*.*

Proof Let us set $\mu(\cdot)=\delta_{u(\cdot)}$. Then it's easy to check that $J(u(\cdot),\xi(\cdot))=\mathcal{J}(\mu(\cdot),\xi(\cdot))$. Firstly, we assume that $(u(\cdot),\xi(\cdot))$ minimizes J over $\mathcal{U}$. For any $(q(\cdot),\eta(\cdot))\in\mathcal{R}$, by Lemma 4.3 there exists a sequence of processes $(u^n(\cdot))\subset\mathcal{U}_1$ such that $J(u^n(\cdot),\eta(\cdot))\to\mathcal{J}(q(\cdot),\eta(\cdot))$ as $n\to\infty$. Since $J(u(\cdot),\xi(\cdot))\le J(u^n(\cdot),\eta(\cdot))$ for any n, by letting n go to infinity we get $J(u(\cdot),\xi(\cdot))\le\mathcal{J}(q(\cdot),\eta(\cdot))$, $\forall(q(\cdot),\eta(\cdot))\in\mathcal{R}$. So $\mathcal{J}(\mu(\cdot),\xi(\cdot))\le\mathcal{J}(q(\cdot),\eta(\cdot))$, $\forall(q(\cdot),\eta(\cdot))\in\mathcal{R}$, from which we conclude that $(\mu(\cdot),\xi(\cdot))$ minimizes $\mathcal{J}$ over $\mathcal{R}$. Secondly, let us assume that $(\mu(\cdot),\xi(\cdot))$ minimizes $\mathcal{J}$ over $\mathcal{R}$. Then for any $(v(\cdot),\eta(\cdot))\in\mathcal{U}$, we have $\mathcal{J}(\mu(\cdot),\xi(\cdot))\le\mathcal{J}(\delta_{v(\cdot)},\eta(\cdot))=J(v(\cdot),\eta(\cdot))$, $\forall(v(\cdot),\eta(\cdot))\in\mathcal{U}$. So we can obtain $J(u(\cdot),\xi(\cdot))\le J(v(\cdot),\eta(\cdot))$, $\forall(v(\cdot),\eta(\cdot))\in\mathcal{U}$, from which it follows that $(u(\cdot),\xi(\cdot))$ minimizes J over $\mathcal{U}$. ∎

Now, let us define $H:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times U_1\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}$ by

$$H(t,x,y,z,v,k,p,P)=p\cdot b(t,x,v)+P\cdot\sigma(t,x,v)+k\cdot f(t,x,y,z,v)+l(t,x,y,v).$$

Let $(u(\cdot),\xi(\cdot))\in\mathcal{U}$ be an optimal control of the regular-singular optimal control problem and $\big(x^{u,\xi}(\cdot),y^{u,\xi}(\cdot),z^{u,\xi}(\cdot)\big)$ be the corresponding trajectory. Then it's easy to check that the following adjoint equation

$$\begin{cases} dk_t^{u,\xi} = H_y\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, u_t, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big)dt \\ \qquad\qquad +H_z\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, u_t, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big)dB_t, \\ dp_t^{u,\xi} = -H_x\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, u_t, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big)dt + P_t^{u,\xi}dB_t, \\ k_0^{u,\xi} = h_y\big(y_0^{u,\xi}\big), \quad p_T^{u,\xi} = g_x\big(x_T^{u,\xi}\big) + \varphi_x\big(x_T^{u,\xi}\big)k_T^{u,\xi} \end{cases} \tag{22}$$

admits a unique solution $(k^{u,\xi}(\cdot), p^{u,\xi}(\cdot), P^{u,\xi}(\cdot))$. We are ready to state the maximum principle for the optimal regular-singular control problem. Let us denote $N_t = C_t^T p_t^{u,\xi} + D_t^T k_t^{u,\xi} + G_t$ and $N_t = (N_t^1, N_t^2, \cdots, N_t^n)^{\mathrm{T}}$.

Theorem 4.5 *Let $(u(\cdot), \xi(\cdot)) \in \mathcal{U}$ be an optimal control of the regular-singular optimal control problem and $\big(x^{u,\xi}(\cdot), y^{u,\xi}(\cdot), z^{u,\xi}(\cdot)\big)$ be the corresponding trajectory. Then for all $v \in U_1$,*

$$\begin{aligned} & H\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, v, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big) \\ \geq\; & H\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, u_t, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big), \quad \text{a.e. a.s.,} \end{aligned} \tag{23}$$

$$\mathbb{P}\left\{\sum_{i=1}^n \chi_{[N_t^i \geq 0]} d\xi_t^i = 0\right\} = 1, \tag{24}$$

$$\mathbb{P}\big\{\forall t \in [0,T], \forall i; N_t^i \geq 0\big\} = 1. \tag{25}$$

Proof Let us set $\mu(\cdot) = \delta_{u(\cdot)}$. Since $(u(\cdot), \xi(\cdot)) \in \mathcal{U}$, we have $(\mu(\cdot), \xi(\cdot)) \in \delta(\mathcal{U}_1) \times \mathcal{U}_2$. If $(u(\cdot), \xi(\cdot))$ minimizes J over $\mathcal{U}$, then by Lemma 4.4, $(\mu(\cdot), \xi(\cdot))$ minimizes $\mathcal{J}$ over $\mathcal{R}$. Hence, by Theorem 3.5, the adjoint equation (16) admits a unique solution $(k^{\mu,\xi}(\cdot), p^{\mu,\xi}(\cdot), P^{\mu,\xi}(\cdot))$ such that (19), (20), and (21) hold. Since $\delta(\mathcal{U}_1) \subset \mathcal{R}_1$, by (19) we conclude that for any $\rho \in \delta(U_1)$,

$$\mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \rho, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) \geq \mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big), \quad \text{a.e. a.s.} \tag{26}$$

On the other hand, for any $\rho \in \delta(U_1)$, there exists $v \in U_1$ such that $\rho = \delta_v$. Then, by the fact that $\mu(\cdot) = \delta_{u(\cdot)}$ and $\rho = \delta_v$, it's easy to get

$$\begin{cases} \big(x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}\big) = \big(x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}\big), \\ \big(k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) = \big(k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big), \\ \mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \mu_t, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) = H\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, u_t, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big), \\ \mathcal{H}\big(t, x_t^{\mu,\xi}, y_t^{\mu,\xi}, z_t^{\mu,\xi}, \rho, k_t^{\mu,\xi}, p_t^{\mu,\xi}, P_t^{\mu,\xi}\big) = H\big(t, x_t^{u,\xi}, y_t^{u,\xi}, z_t^{u,\xi}, v, k_t^{u,\xi}, p_t^{u,\xi}, P_t^{u,\xi}\big). \end{cases} \tag{27}$$

Hence, the result (23) follows from (26) and (27). The results (24) and (25) are immediate consequences of (20) and (21). ∎

Finally, we establish the sufficient optimality conditions for the optimal regular-singular control problem.

Theorem 4.6 *Assume that g, h, and $H(t,\cdot,\cdot,\cdot,v,k,p,P)$ are convex, and moreover, $y_T^{v,\eta}$ takes the following particular form: $y_T^{v,\eta}=Rx_T^{v,\eta}+\zeta$, where $R\in\mathbb{R}^{m\times n}$ and $\zeta\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^m)$. Then $(u(\cdot),\xi(\cdot))$ is an optimal control of the regular-singular optimal control problem if it satisfies* (23)*,* (24)*, and* (25)*.*

Proof For $(u(\cdot),\xi(\cdot))$, $(v(\cdot),\eta(\cdot))\in\mathcal{U}$, there exist $\mu(\cdot),\rho(\cdot)\in\delta(\mathcal{U}_1)$ such that $\mu(\cdot)=\delta_{u(\cdot)}$, $\rho(\cdot)=\delta_{v(\cdot)}$. This implies that (27) holds. Then it follows from (23), (24), and (25) that

$$\begin{aligned}&\mathcal{H}\big(t,x_t^{\mu,\xi},y_t^{\mu,\xi},z_t^{\mu,\xi},\rho_t,k_t^{\mu,\xi},p_t^{\mu,\xi},P_t^{\mu,\xi}\big)\\ \geq\ &\mathcal{H}\big(t,x_t^{\mu,\xi},y_t^{\mu,\xi},z_t^{\mu,\xi},\mu_t,k_t^{\mu,\xi},p_t^{\mu,\xi},P_t^{\mu,\xi}\big),\quad \text{a.e. a.s.},\\ &\mathbb{P}\left\{\sum_{i=1}^n\chi_{[M_t^i\geq 0]}d\xi_t^i=0\right\}=1,\\ &\mathbb{P}\big\{\forall t\in[0,T],\forall i;M_t^i\geq 0\big\}=1.\end{aligned}$$

Then similar to the proof of Theorem 3.6, it's easy to deduce that $(\mu(\cdot),\xi(\cdot))$ minimizes $\mathcal{J}$ over $\delta(\mathcal{U}_1)\times\mathcal{U}_2$. Finally, by Lemma 4.1, we conclude that $(u(\cdot),\xi(\cdot))$ minimizes J over $\mathcal{U}$. ∎

References

[1] Bather J and Chernoff H, Sequential decision in the control of a spaceship (finite fuel), *Journal of Applied Probability*, 1967, **4**(3): 584–604.

[2] Benes V E, Shepp L A, and Witsenhausen H S, Some solvable stochastic control problems, *Stoch. Stoch. Rep.*, 1980, **4**: 39–83.

[3] Alvarez L H, Singular stochastic control, linear diffusions, and optimal stopping: A class of solvable problems, *SIAM J. Control Optim.*, 2001, **39**(6): 1697–1710.

[4] Haussmann U G and Suo W L, Singular optimal stochastic controls I: Existence, *SIAM J. Control Optim.*, 1995, **33**(3): 916–936.

[5] Haussmann U G and Suo W L, Singular optimal stochastic controls II: Dynamic programming, *SIAM J. Control Optim.*, 1995, **33**: 937–959.

[6] Karatzas I and Shreve S E, Connections between optimal stopping and singular stochastic control I: Monotone follower problems, *SIAM J. Control Optim.*, 1984, **22**(6): 856–877.

[7] Karatzas I and Shreve S E, Connections between optimal stopping and singular stochastic control II: Reflected follower problems, *SIAM J. Control Optim.*, 1985, **23**(3): 433–451.

[8] Peng S G, A general stochastic maximum principle for optimal control problems, *SIAM J. Control Optim.*, 1990, **28**(4): 966–979.

[9] Peng S G, Backward stochastic differential equations and applications to optimal control, *Appl. Math. Optim.*, 1993, **27**(2): 125–144.

[10] Shi J T and Wu Z, The maximum principle for fully coupled forward-backward stochastic control systems, *Acta Automatica Sinica*, 2006, **32**(2): 161–169.

[11] Shi J T and Wu Z, Maximum principle for forward-backward stochastic control system with

random jumps and applications to finance, *Journal of Systems Science and Complexity,* 2010, **23**(2): 219–231.

[12] Wu Z, Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems, *Systems Sci. Math. Sci.*, 1998, **11**: 249–259.

[13] Xu W S, Stochastic maximum principle for optimal control problem of forward and backward system, *J. Austral. Math. Soc. Ser. B*, 1995, **37**(2): 172–185.

[14] Bahlali S, Necessary and sufficient optimality conditions for relaxed and strict control problems, *SIAM J. Control Optim.*, 2008, **47**(4): 2078–2095.

[15] Wu Z, A general stochastic maximum principle for optimal control problems of forward-backward stochastic control systems, *Automatica*, 2013, **49**, DOI: 10.1016/j.automatica.2013.02.005.

[16] Yong J M, Optimality variational principle for controlled forward-backward stochastic differential equations with mixed initial-terminal conditions, *SIAM J. Control Optim.*, 2010, **48**(6): 4119–4156.

[17] Bahlali S, A general stochastic maximum principle for optimal control problems of forward-backward systems, arXiv: 0801.4326v6 [math.PR].

[18] Cadenillas A and Haussmann U G, The stochastic maximum principle for a singular control problem, *Stochastics: An International Journal of Probability and Stochastic Processes*, 1994, **49**(3–4): 211–237.

[19] Bahlali S and Chala A, The stochastic maximum principle in optimal control of singular diffusions with nonlinear coefficients, *Random Oper. and Stoch. Equ.*, 2005, **13**(1): 1–10.

[20] Bahlali S and Mezerdi B, A general stochastic maximum principle for singular control problems, *Electron J. Probab.*, 2005, **10**: 988–1004.

[21] El Karoui N, Huu Nguyen N, and Jeanblanc Piqué M, Compactification methods in the control of degenerate diffusions: Existence of an optimal control, *Stochastics*, 1987, **20**(3): 169–219.

[22] Lou H W, Analysis of the optimal relaxed control to an optimal control problem, *Appl. Math. Optim.*, 2009, **59**(1): 75–97.

[23] Yong J M and Zhou X Y, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.