J Syst Sci Complex (2012) 25: 802–820

A REVIEW AND PROSPECT OF READABLE MACHINE PROOFS FOR GEOMETRY THEOREMS*

Jianguo JIANG · Jingzhong ZHANG

DOI: 10.1007/s11424-012-2048-3
Received: 14 March 2012 / Revised: 20 March 2012

Abstract After half a century research, the mechanical theorem proving in geometries has become an active research topic in the automated reasoning field. This review involves three approaches on automated generating readable machine proofs for geometry theorems which include search methods, coordinate-free methods, and formal logic methods. Some critical issues about these approaches are also discussed. Furthermore, the authors propose three further research directions for the readable machine proofs for geometry theorems, including geometry inequalities, intelligent geometry softwares and machine learning.

Key words Automated geometry reasoning, coordinate-free method, formal logic method, geometric inequality, intelligent geometry software, machine learning, mechanical theorem proving, readable machine proof, search method.

1 Introduction

It has been a human dream to find a "silver bullet" or a cookbook for solving all geometry problems. Many prominent scientists have arduously explored the dream in history. About 300 BC, Euclid, the author of the classical book Elements of Geometry, said to Ptolemy I king: "There is no royal road to geometry". In the middle of the 17th century, Descartes tried to propose an unified approach for all geometry problems by introducing the coordinate system into geometry. In the early period of the 20th century, Hilbert presented an ambitious plan of establishing a complete formal system for each mathematic theory. He wanted to find a general decision method for all statements in the formal system. In his classic book Foundations of Geometry, he also outlined one of this kind of method just adaptable to a class of geometry statements in affine geometry[1−2]. However, the incompleteness theorem published by Gödel completely denied the feasibility of Hilbert's plan in 1931.

In 1951, Tarski proposed a noticeable decision method for elementary geometry and elementary algebra[3]. In spite of subsequent improvements by Seidenberg and others, Tarski's method

Jianguo JIANG
School of Mathematics, Liaoning Normal University, Dalian 116029, *China.* Email: jjgbox@sina.com.
Jingzhong ZHANG
Laboratory of Computer Reasoning and Trustworthy Compution, University of Electronic Science and Technology of China, Chengdu 610054, *China*; *Chengdu Institute of Computer Applications, Chinese Academy of Sciences, Chengdu* 610041, *China.* Email: zjz101@yahoo.com.cn.
*This research was supported by the Funds of the Chinese Academy of Sciences for Key Topics in Innovation Engineering under Grant No. KJCX2-YW-S02.
◇ *This paper was recommended for publication by Editor Xiao-Shan GAO.*

remained impractical for proving non-trivial geometry theorems[4]. In 1974, Collins made an important contribution to Tarski's method by proposing the cylindrical algebra decomposition algorithm (CAD)[5]. Based on the CAD, Arnon implemented a program which could prove several difficult geometry theorems[6].

A breakthrough in the automated geometry theorem proving is made by the famous Chinese mathematician Wu Wen-tsün in 1977[7]. Under the enlightenment of Chinese ancient idea of mathematical mechanization, he proposed a decision method for a class of geometry statements of equality type[8]. Later, it was clarified that the algebraic tools needed in Wu's method can be developed from Ritt's work[9]. The algebraic aspect of this approach is known as the Wu-Ritt's characteristic set method[10]. Wu's method can prove quite difficult geometry theorems efficiently. The prover based on Wu's method implemented by Chou has successfully proved 512 non-trivial geometry theorems[11−12], which made the Wu's method wildly known in the automated reasoning field.

The success of Wu's method has inspired many scholars' interests in the automated geometry theorem proving. Under the influence of Wu's success, many decision methods have been brought forward such as the Gröbner method (GB)[13−14], the resultant method[15−16], the elimination method[17], the proving-by-examples method[18−19], and the parallel numerical method[19−20].

The basic ideas of above methods are to firstly algebraize geometry statements by the coordinate system, and then decide the truth or falsity of statements using profound algebraic theories. Because of this, all of these methods are generally called algebraic approaches based on coordinate. These approaches can only decide the truth or falsity of geometry statements, but can not generate proofs which are from hypotheses to conclusion in the traditional style. The decision process of these approaches usually involves complex computation of some "large scale polynomials". Sometimes, several polynomials in the decision process have up to hundreds of items and even up to thousands of items. It is difficult for people to understand the geometric meaning of these polynomials, and it is also tedious and formidable for people to verify whether the computation of those large scale polynomials is correct or not.

How to generate readable machine proofs for geometry theorems automatically is a quite challenging and interesting research topic. Since the basic idea of automated reasoning is to replace the difficulty of quality with the complexity of quantity, some scientists think that it is very difficult to image automatically generating readable machine proofs for geometry theorems. In fact, early in 1958, Gelernter and his collaborators had began to research on how to generate traditional proofs for geometry theorems by computers[21−23]. Their program proved some simple geometry statements of middle school level. After nearly 30 years' development, only some very simple geometry statements had been readably proved using computer, while effective method was still not found.

In 1993, a breakthrough in automated generating readable machine proofs for geometry theorems is made by Chou, Gao, and Zhang[24−25]. For many famous geometry theorems, such as Butterfly theorem, Simson's theorem, Desargues' theorem and Feuerbach's theorem, the area method proposed by them can automatically and effectively generate very short and beautiful readable proofs[24−25].

The success of the area method had also unprecedentedly inspired research interests in the automated generating readable machine proofs for geometry theorems which has become an active field in the automated reasoning today. Many other methods that can generate readable proofs for geometry theorems have made great progress. Furthermore, more and more attention has been paid to the application value of the automated generation readable machine proofs for geometry theorems. The intelligent geometry software that can generate readable proofs for geometry theorems has been widely applied in the education field.

The rest of this paper is divided into two parts. Section 2 is a review of three kinds of approaches to automatically generating readable machine proofs for geometry theorems, including search methods, coordinate-free methods, and formal logic methods. In Section 3, we look forward into the future of the automated generating readable machine proofs for geometry theorems from three possible directions including geometry inequalities, intelligent geometry softwares, and machine learning.

2 Research Review

2.1 Search Methods

The basic idea of search methods is to simulate human solving process of geometry problems using artificial intelligent techniques. The synthetic approach and the analytic approach are two kinds of traditional techniques of solving geometry problems. The synthetic approach starts from the hypotheses of statement to deduce more conclusions step by step until the conclusion to be proved is deduced. The analytic approach starts from the conclusion of a statement to look for sufficient conditions step by step until one of the sufficient conditions could be derived from the hypotheses. When proving geometry statements, we usually look for the idea of the proof with the analytic approach, and express the proof itself in synthetic style.

Essentially, the search method is a rule-based expert system, whose basic structure is composed of three parts, including a rule database, a fact database, and a reasoning engine. All rules of inference used by the engine, which usually are axioms, theorems, lemmas, formulae, definitions and algebraic operation rules in geometry, are stored in the rule database. All geometric facts stored in the fact database are expressed by given geometry predicates such as equidistant, equiangular, parallel, perpendicular, and midpoint. The reasoning engine applies rules of inference into the facts database so as to deduce more new geometric facts again and again.

According to the direction of the deduction, there are three kinds of search method including forward chaining method[26−27], backward chaining method[21−23,28−29], and bidirectional chaining method[30−33]. The forward chaining method deduces from the hypotheses to the conclusion, the backward chaining method deduces from the conclusion to the hypotheses and the bidirectional chaining method deduces simultaneously from the hypotheses and the conclusion to the middle of the proof. Each kind of three search methods has different advantages and disadvantages. The forward chaining method is always feasible, but it does not have explicit reasoning goal. The backward chaining method has explicit reasoning goal, but it sometimes lacks feasibility. The bidirectional chaining method is feasible and has explicit reasoning goal, but it is difficult to implement it.

The earliest work in the geometry theorem proving based on the search method can be traced back to Gelernter and his collaborators[21−23]. In 1958, they studied how to prove Euclid geometry theorems with the backward chaining method. If making aimless search, the backward chaining method would generate a huge proof tree during its searching process, which may probably cause not to find the goal to be proved within reasonable time and limited memory space. For this reason, they took numerical diagram as heuristic information to improve the search efficiency of the backward chaining method[22−23]. During backward searching, the generated sub-goals are verified numerically in the numerical diagram. Those invalid sub-goals, which are false in the numerical diagram, are deleted. Furthermore, they also discussed simply several important ideas in automated reasoning, such as constructing auxiliary points and using lemmas. Gelernter's Geometry Machine (GM) proved more than 50 geometry statements of middle school level in IBM704 machine[22−23]. The following Example 1 is one of the most difficult geometry theorem which could be proved by GM, given that the auxiliary point G (the intersection point of CE and AB) is artificially constructed in the diagram.

Example 1 If $ABCD$ is a trapezoid, $AB \| CD$, E and F are midpoints of BD and AC respectively, EF meets AD at M, then $AM = DM$ (Figure 1).

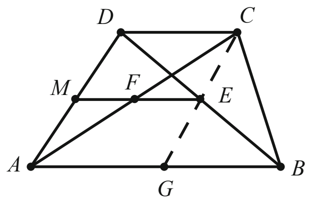

Figure 1 One of the most difficult geometry theorem proved by GM

The work of Gelernter and his collaborators has attracted wide attention in the artificial intelligence field[21−23]. Many scholars have made a large amount of research on the search method and most of these works could be regarded as the extension and improvement of Gelernter's work[28−35].

In 1975, Nevins began to prove geometry theorems with the bidirectional chaining method[31]. In the reasoning process, the forward chaining method was mainly used to search new geometric facts in the diagram and the backward chaining method is used only to prove the equilateral or equiangular. Besides, he adopted a series of structural knowledge representation techniques to improve the efficiency of the forward chaining method, such as canonical naming and storing equiangular or equidistant information by equivalence class. The prover implemented by Nevins had proved all geometry theorems in [22–23, 28] in PDP-10 computer. The running time each of these was less than 5, except one theorem.

In 1986, Coelho and Pereira implemented a prover GEOM based on the bidirectional chaining method[33]. What different from the prover implemented by Nevins is that the GEOM paid more attention to the application of backward chaining method in the reasoning process and only used the forward chaining method to search congruent triangles hidden in the diagram. Same as the Nevins' prover, the GEOM can prove some simple elementary plane geometry theorems. Coelho and Pereira also analyzed and compared reasoning efficiency of these two methods in detail. As they pointed out, the efficiency of search methods had something to do with the type of the geometry theorem to be proved. Some geometry theorems were adaptable to the backward chaining method, while others were adaptable to the forward chaining method. The general geometry theorem prover should combine the forward chaining method with the backward chaining method.

In 1995, Chou, Gao, and Zhang proposed a deductive database method by introducing deductive database techniques into search methods[26−27,36]. The deductive database method can find a fixpoint for a given geometric diagram, i.e., it can find all properties of the geometric diagrams that can be deduced using a fixed set of geometric rules. They effectively controlled the size of the facts database with the structural deductive database techniques. Their experiments showed that the structured deductive database technique can reduce the size of the database by one thousand times. Furthermore, they also improved the efficiency of the forward chaining method significantly by some other techniques, such as the data searching tactics, rules of adding auxiliary points, and full-angle reasoning rules. The prover Geometry Expert (GEX) implemented by them had successfully proved more than 150 difficult geometry theorems, of which including many famous geometry theorems such as Orthocenter theorem, Simson's theorem and Butterfly theorem[37]. It is wonderful that GEX can effectively prove

Miquel's five-circle theorem which is difficult for the algebraic approaches because excessively large memory is needed to prove this theorem[27].

Example 2 (Miquel's Five-Circle Theorem) Let P_0, P_1, P_2, P_3, P_4 be five points, P_0P_1 meets P_3P_2 and P_3P_4 at Q_1 and Q_4, respectively, P_1P_2 meets P_4P_0 and P_4P_3 at Q_0 and Q_2 respectively, and P_2P_3 meets P_0P_4 at Q_3. Let the other intersections of consecutive circumscribed of triangles $P_0P_4Q_4$, $P_1P_0Q_0$, $P_2P_1Q_1$, $P_3P_2Q_2$, $P_4P_3Q_3$ be M_0, M_1, M_2, M_3, M_4 respectively. Show that M_0, M_1, M_2, M_3 and M_4 are concyclic (Figure 2).

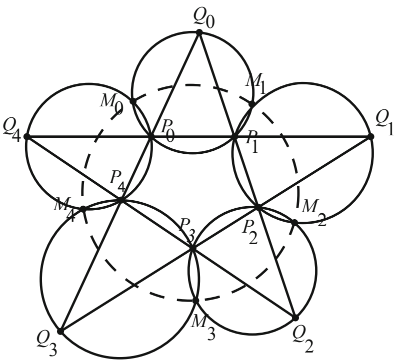

Figure 2 Miquel's five-circle theorem

The reasoning efficiency is still the key issue for search methods. A large amount of experiments show that heuristic[21−23,27−32,38−41] is one of the most effective techniques for improving the efficiency of search methods. Furthermore, eliminating redundant reasoning[27], eliminating redundant matches[29,39], and optimizing structures of the facts database can also significantly improve the reasoning efficiency of search methods.

One of the main reasons that search methods could not prove very difficult theorems is that it didn't have a construction algorithm for adding auxiliary lines or points automatically. Wong, Reiter and Robinson analyzed logic foundations of adding auxiliary lines or points, but none of them gave concrete implementation of adding auxiliary lines or points[42−44]. The way of Gelernter, Gilmore, Goldstein, Ullman, Nevins, Welham and others to implement the construction is just to connect two existing points in the diagram[21−23,28,30−31,41]. Elcock designed a method for constructing the intersection point of two existing line segments in the diagram[45]. Greeno found a method to construct auxiliary segments of congruent triangles with a common edge[46]. Coelho and Pereira added auxiliary lines for applying the quadrilateral axiom[33]. The prover GEX uses more than thirty rules of adding auxiliary points to enhance its power[27]. Though GEX proved about forty theorems by adding auxiliary points, it still can not find auxiliary lines or points for many non-trivial famous theorems, such as Pappus' hexagon theorem, Simson's theorem and Desargues' theorem. The prover GRAM of Matsuda and Vanlen proved 32 geometry theorems that require auxiliary lines or points[47]. Worthy of special mention is that GRAM found 4 different auxiliary lines for Example 1 mentioned above after running 3967 seconds[47].

It is very difficult to deduce algebraic expressions involving geometric quantities using search methods. The essential cause is that it will lead to combinatorial explosion of search space. Algebraic expressions involving geometric quantities can not be deduced basically using provers based on search methods[21−23,28,30−33]. The prover GEX can only express the geometric proportion with the given predicates but can not deduce a little more complex relational expressions of line segment[27]. The prover GRAM can not deduce algebraic expressions involving geometric quantities at all[47]. In [40], a heuristic algorithm which could deduce polynomial equality involving geometric quantities had been brought forward. The prover based on the heuristic algorithm could generate readable machine proofs for some geometry theorems whose conclusions are geometric equalities, such as Euler's theorem, Ptolemy's theorem and Stewart's theorem. This heuristic algorithm enjoyed high efficiency, but it is incomplete and still needs to be further improved[40].

2.2 Formal Logic Methods

One of the early applications of the automated reasoning is to prove mathematical theorems by computer. In 1956, thirty-eight theorems in the classic book Principia Mathematica of Whitehead and Russell were successfully proved by Logic Theory Machine of Newell and Simon. In 1958, the prover designed by Wang Hao spent only three minutes proving all theorems of propositional calculus in Principia Mathematica. In 1959, Gelernter's GM proved about fifty geometry statements of high school level. In 1965, the resolution principle proposed by Robinson has inspired a new wave of research on automated theorem reasoning. In 1996, the famous Robbins problem was proved by McCune using the theorem prover EQP.

Currently, automated theorem proving has been the best developed subfield of automated reasoning. Many theorem provers based on the logic method were successful developed, such as first-order logic theorem provers Otter, E, SPASS and Vampire, as well as higher-order logic theorem provers Coq, Isabelle, HOL and ACL2. There are many popular deductive methods used by theorem provers based on logic, including Hilbert-style deductive systems, natural deduction, resolution, tableaux methods, sequence calculus, and constructive type calculus.

As early as in 1976, McCharen and Wos have done an experiment of proving theorems in Tarski's geometry with their prover ITP[48]. They have only proved several trivial geometry theorems. ITP is an interactive theorem prover based on the resolution principle and the proof generated by it is a proof of resolution-style. The proof of resolution style is machine-oriented and has certain readability since some steps are also coincident with the traditional proof. In 1989, Quaife continued the work of McCharen with Otter[49]. The Otter is an automated theorem prover based on resolution and a series of tactics used by it can improve the resolution efficiency such as Hyper-resolution, UR-resolution, paramodulation, support set and weight of clause. In order to further improve the proving efficiency of Otter, Quaife had also adopted the weight of clause as heuristic to reduce the search space, especially to reduce the quantity of the useless clauses generated by the paramodulation. He set the maximum weight value of the reserved clause as 25 and the clauses exceeding this maximum weight value would be abandoned. The other weight heuristic tactics was just to reserve the ground clause and abandon the clauses containing the variables. Quaife had proved 62 Tarski's geometry theorems, most of which could not be proved by previous provers based on resolution. One of the most difficult Tarski's geometry theorem which was proved by Otter is the bisecting diagonals theorem[49]. This theorem had been successfully proved after running Otter 555.59 seconds and total 23764 clauses had been generated in the proving process, but only 1615 clauses were reserved after using the weight heuristic tactics. Furthermore, Quaife had also written a post processing program which could convert the output of Otter into more readable proofs[49].

Example 3 (Bisecting Diagonals Theorem) The diagonals of a non-degenerate rectangle bisect each other (Figure 3).

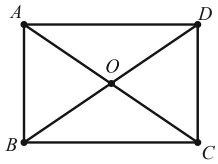

Figure 3 Bisecting diagonals theorem

In 2003, Meikle and Fleuriot developed Hilbert's geometry with the theorem prover Isabelle/ Isar[50]. Isabelle is a semi-automated theorem prover, which not only supports interactive theorem proving but also supports semi-automated theorem proving to some extent. Isar is a structural formal proof language, which can convert the outputs of Isabelle into readable proofs which can be understood more easily by people. Meikle and Fleuriot have just formalized the first three groups of axioms in the Hilbert's axiom system and proved some direct inferences of these axioms with Isabelle/Isar such as SAS theorem. In 2008, Scott continued the work of Meikle and Fleuriot and further studied how to improve the proof quality generated by Isabelle[51]. Compared with the traditional proofs given by people, the proofs generated by Isabelle are extraordinary verbose. Even if proving those trivial geometry theorems, the proofs generated by it would probably contain very long steps. Though tactics can shorten the length of reasoning to some extent, it was difficult to process the non-trivial high-level proofs with tactics. In order to eliminate a great number of trivial reasoning steps and further shorten the proof length, Scott used some abstract geometric predicates such as collinear and planar. He not only viewed the collinearity or coplanarity as the relationship of points but also as a set of points. In this way, the trivial reasoning relevant to collinearity or coplanarity could be eliminated through operations of sets, such as subset, intersection, union, set difference and set complement. For example, the subsets of collinearity or coplanarity are still collinear or planar.

Example 4 (SAS Theorem) If AB and $A'B'$ are congruent, $\angle BAC$ and $\angle B'A'C'$ are congruent, and AC and $A'C'$ are congruent in $\triangle ABC$ and $\triangle A'B'C'$, then $\triangle ABC$ is congurent to $\triangle A'B'C'$ (Figure 4).

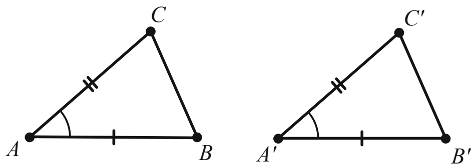

Figure 4 SAS theorem

Coq is a proving assistant based on the calculus of inductive construction. Its high-level language can be used to define axioms, parameters, types, functions, predicates and so on. Coq is not an automated theorem prover, but it contains many tactics and various decision processes which can be used for automated theorem proving. The tactic is a program consist of basic logic steps needed for theorem proving. Kahn has formalized the constructive geometry of Plato and Coq. Dehlinger has also formalized incidence axioms and order axioms in Hilbert axiom system[52]. In 2004, Narboux implemented a proving tactic for geometry theorem based on the area method[53]. Since Narboux's tactic uses only two geometric invariants, signed area and directed line segment ratio, and does not use the Pythagorean difference, it could only prove some constructive affine geometry theorems other than those geometry theorems involving metric property such as perpendicular or equiangular. For the convenience of implementation, Narboux divided his tactic into 6 sub-tactics executing different proving tasks:

Initialization tactic, simplification tactic, unification tactic, elimination tactic, free point elimination tactic and conclusion tactic. This tactic implemented by Narboux was slower than the prover Euclid implemented by Chou, Gao, and Zhang[25,54], but more than 20 non-trivial constructive affine geometry theorems could also be proved within several minutes such as Ceva's theorem, Menelaus' theorem, Pascal's theorem and Desargues' theorem[53]. In 2007, Narboux had integrated the dynamic geometric software GeoProof, the automated theorem prover and the interactive proving system Coq to one software used for teaching and learning geometry theorem[55]. Users can explore, measure and discover the hypothesis with GeoProof, verify the facts with Gröbner method or Wu's method and mechanically check user's proof with Coq.

The greatest disadvantage of formal logic methods is their low reasoning efficiency. Though the general theorem provers, such as Otter, Isabelle and Coq, have been able to prove some relatively simple geometry theorems in high-speed computer, they do not have the capability of proving any class of difficult geometry theorems. The essential cause for low efficiency of formal logic methods is the combination explosion of its search space. The use of heuristic is one effective technique to improve the reasoning efficiency of formal logic methods. For example, the proof search can be guided to advance towards the correct direction with the geometric diagrams as the semantic model. The search space is reduced through deleting those subgoals which are checked false in the geometric diagram [23,33,43]. In [56], the possible lemmas were proposed through the geometric diagrams to plan the proof. How to realize the semantic resolution and support set tactics with the geometric diagrams is discussed in the paper so as to further enhance the proving efficiency of general theorem provers based on the resolution principle. Furthermore, the formal logic methods and other high-efficient proving methods can be combined into a more effective hybrid deduction method. The classical first-order logic method and the algebraic methods can be combined to develop a hybrid deduction system, which can generate the easily understood readable proofs of different levels[57−58].

The readability of proofs produced by formal logic methods is still needed for further improving. Usually, those proofs generated with formal logic methods contain a great number of redundant reasoning steps, which makes them appear extraordinary verbose and be not good for understanding and reading. The elimination of the redundant reasoning steps can further enhance the readability of the proof. The symmetry and transitivity of geometric predicates is the main reason for generating redundant reasoning[21,27,31,38,51]. The equivalence class reasoning can significantly eliminate the redundant reasoning steps generated by geometric equivalent predicates such as equiangular, equidistant, congruent and similar[38]. The redundant reasoning steps generated by the geometric predicates such as collinear and planar, are eliminated through operations among sets[51]. The redundant reasoning steps was eliminated with the deductive database technique[27]. It can effectively eliminate redundant reasoning generated by those symmetry geometric predicates through introducing an order relation between geometric objects (points, line segments, straight lines, angles, triangles, etc) to unify the geometric predicate information of various equivalent forms into one standard format[21,31,49,56]. Furthermore, those traditional geometric proving techniques, such as proving by contradiction, the identity method and the analogy method, can be used to further improve readability of formal logic methods. For example, the analogy reasoning is used to generate more easily understood readable machine proofs for geometry theorems[56,59]. Lemmas can be introduced through checking analogy between compositions of the proof to further make the proof shorter and more readable.

2.3 Coordinate-Free Methods

Coordinate-free methods do not use low-level coordinates but use high-level geometric invariants to translate geometry statements into algebraic forms. Roughly speaking, this kind of method is adaptable to the constructive geometry statements of equality type. The so-called constructive geometry statement refers to the geometry statement that the hypotheses can be described by a sequence of geometry construction steps and the conclusion can be represented with the rational equality of high-level geometric invariants. According to the difference of high-level geometric invariants being used, coordinate-free methods can be developed into various proving methods of different styles, such as the area method[25,54,60–62], the full-angle method[36,61] and the complex number method[61] applying to Euclid plane geometry, the volume method[61] applying to Euclid solid geometry, the vector method[61] applying to Euclidean geometry and the argument method[61] applying to non-Euclidean geometry.

The area method is an age-old solving approach for geometric problems. The famous Pythagorean theorem in plane geometry was proved with the area method at the earliest. Usually, the area method is only viewed as a special technique of solving geometric problems; however, Zhang had gradually recognized the universality of the area method in the long-term teaching practice and developed it into a systematic method to solve geometric problems[54,61–65]. In 1992, cooperating with Chou and Gao, Zhang further improved the area method and developed it into a mechanical algorithm which could be well run in computer[25,54,60–61].

The area method uses three geometric invariants, including the signed area, the directed line segment ratio and the Pythagorean difference. Since the main geometric invariant is the signed area, this method is known as the area method. The area method is complete for constructive geometry statements. If the constructive geometry statement to be proved is true, it can be certainly proved with the area method. If the geometry statement can not be proved with the area method, this statement must be false. The proof generated automatically by the area method is usually short and elegant, its readability can be comparable with the traditional proofs and its efficiency is also almost the same with those powerful algebraic methods based on coordinate, such as the Wu's method and the GB method[60]. Furthermore, the area method can be used to search multiple proofs and the shortest proof[66]. The geometry theorem prover Euclid which was implemented based on the area method by Chou, Gao, and Zhang had successfully proved more than four hundred non-trivial geometry theorems[60], of which including many famous geometry theorems, such as Butterfly theorem, Pappus' theorem, Simson's theorem, Desargues' theorem, Pascal's theorem and Feuerbach's theorem. This work made the readable machine proofs in geometry which didn't have any progress for several tens of years achieve an important breakthrough[25,54,60].

Example 5 (Pappus' Hexagon Theorem) If A, B and C are three points on one line, X, Y and Z are three points on another line, and AY meets XB at P, AZ meets XC at Q, BZ meets YC at R, then P, Q and R are collinear (Figure 5).

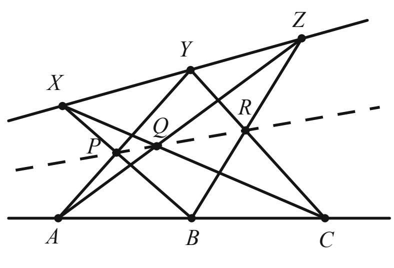

Figure 5 Pappus' hexagon theorem

Not all proofs generated by the area method are short and elegant. Especially, the area method doesn't work well for those geometry theorems only involving traditional angle essentially, such as the Miquel's five-circle theorem and the Miquel's theorem[60,67]. One of the most essential factors is that the traditional angle is a concept involving ordering relation in geometry. To overcome this shortcoming of the area method, Chou, Gao, and Zhang proposed a method based on full-angle and extended the idea of eliminating variables or points to the idea of eliminating lines[60,67−69]. The concept of full-angle was explicitly used by Wu to express the predicate of angle congruence as an algebraic equation[8]. The prover based on the full-angle method can produce short and elegant proofs for more than one hundred geometry theorems[67]. Furthermore, the proofs produced with full-angles were more like traditional proofs. The full-angle method is not complete for constructive geometry theorems, but it could be developed as a complete method by being integrated into the area method. So the full-angle method could be taken as a complement to the area method[67].

Since the signed area is equivalent to the outer product of vectors, Pythagorean difference is equivalent to the inner product of vectors and the length of the directed line segment is equivalent to the modulus of vectors, it is natural to think that taking the vector as the geometric invariants can also be developed to a new eliminating-point method. In 1993, Chou, Gao, and Zhang proposed a proving method based on the vector[60,70]. The advantage of the vector method is that it could be easily generalized to high dimensional geometry, while its disadvantage is that the readability of proofs is poorer than that of the area method. The prover based on the vector method has readably proved more than 410 non-trivial geometry theorems[70].

The plane vector can be represented by complex number, so the vector method could be translated into the complex number method through a little modification[60]. Since the complex number can carry out operations such as plus, minus, multiplication and division, which makes the complex number method have more flexibility than the vector method and the area method. For some special geometry theorems, the proofs given by the complex number method are shorter than those given by the vector method and the area method. For example, if Morley's trisector theorem is proved by the area method[60], its result would be very bad. The area method could produce a proof after running 1086.8 seconds and there were up to 3125 polynomials in the proof! If this theorem is proved by the complex number method, a short and elegant readable proof would be produced.

Example 6 (Morley's Trisector Theorem) In $\triangle ABC$, let P, Q and R be three points of intersection of the adjacent angle trisector. Prove that $\triangle PQR$ is an equilateral triangle (Figure 6).

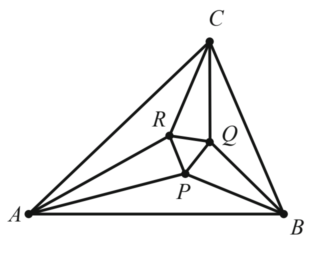

Figure 6 Morley's trisector theorem

In 1995, Chou, Gao, and Zhang had further developed the area method to solid geometry. They proposed a coordinate-free method based on volume with which many solid geometry theorems could be proved readably[71]. Besides using those geometric invariants of the area method, this method also uses signed volume as geometric invariants. More than 80 solid geometry theorems have been proved with this method, of which including many non-trivial solid geometry theorems, such as the centroid theorem, Ceva's theorem, Menelaus' theorem, Desargues' theorem, and Monge's theorem.

In 1997, Yang, Gao, Chou, and Zhang had further generalized the area method to non-Euclidean geometry and proposed a coordinate-free method based on argument[72]. The argument method mainly uses two geometric invariants: one is the argument(roughly speaking, it is sine value of the area) and the other is cosine value of the distance. The argument method is complete and effective for constructive non-Euclidean geometry statements. The prover based on the argument method generated readable proofs for more than 90 non-Euclidean geometry theorems. Worthy of special mention is that they have discovered several tens of new non-Euclidean geometry theorems with this method automatically.

Almost above coordinate-free methods deal with geometry theorems using some geometric quantities. However, the mass point method proposed by Zou and Zhang directly deals with geometric points rather than geometric quantities[69]. The mass point method is also a coordinate-free and diagram-independent method. It could be implemented as provers capable of proving many difficult geometry theorems, including Pappus' Theorem, Pascal's Theorem, Feuerbach's Theorem and even Morley's Trisector Theorem. The proofs generated by the mass point method are indeed human readable and easily understood by a mathematician, and moreover, each expression in a proof has clear and intuitive geometrical meaning, although some of them may involve seemingly huge expressions. Since it is feasible to apply arithmetic operations directly to geometric points, the algorithms and implementations for the mass point method are much easier and more concise than that of the area method.

The coordinate-free methods can also be further generalized to more general geometric algebras, such as bracket algebra and Clifford algebra[73]. A method based on bracket algebra for proving projective geometry theorems was given in [74–75], with which many difficult geometry theorems in projective geometry have been proved. Though proofs generated with bracket algebra are also very short, its readability reduces compared with the area method. Some researchers proposed a method based on Clifford algebra to generate readable proofs for geometry theorems[76−78]. It can not only prove extremely difficult theorems but also solve famous open problems with Clifford algebra. For example, one open problem put forward by Erdös has been more successfully solved by the Clifford algebra method[79]. Neither the Miquel's five-circle theorem nor the Miquel theorem in the plane geometry can be proved with those powerful algebraic method based on coordinate, but both theorems can be proved by the Clifford algebra method with very short and very beautiful readable proofs. Furthermore, the Clifford algebra method can not only prove theorems in Euclidean geometry[58,80−81], but also theorems in other geometries, such as projective geometry[74], affine geometry[82], and differential geometry[83−84]. Some geometry theorems relevant to conic curves have also been readably proved by the Clifford algebra method[81].

These existing coordinate-free methods can also be further improved. On one hand, these methods could be expanded by introducing more geometric invariants or more geometry diagram constructions; on the other hand, the idea of eliminating point could be developed to ideas of eliminating other geometric objects, such as eliminating line, eliminating circle, eliminating plane or eliminating solid, in this way might the readability of proofs be further improved. Furthermore, coordinate-free methods could only prove the constructive geometry theorems of equality type, while could not prove the constructive geometric inequalities. In our opinion, it would be a research direction with hope of gaining new breakthrough to discuss coordinate-free methods being able to prove geometric inequalities. Finally, it would probably be a very challenging task to combine coordinate-free methods, search methods and pure algebraic methods organically so as to design a higher effective and stronger method that could produce readable machine proofs for geometry theorems, but every step towards this direction would be very valuable.

3 Further Research Directions

3.1 Geometric Inequalities

The mechanical proving geometric inequalities has been regarded as one of the most difficult problems in the automated reasoning field. Though geometric inequalities could be proved with the quantifier elimination method of Tarski theoretically[3], there are not any non-trivial geometric inequality being proved by it in computers at all since its complexity is too high. The efficiency of the quantifier elimination method had been improved a little by Seidenberg, Collins, Arnon and Dolzmannin later[4−6,85]. Especially, the cylindrical algebra decomposition algorithm (CAD) proposed by G. Collins had been able to prove some simple geometric inequalities[5]. Wu and Wang had studied how to prove geometric inequalities with the characteristic set method (CS)[86−88]. Though the CS method is not common as the CAD, the CS method has higher efficiency than CAD for some types of inequality. The dimension-decreasing algorithm (DDA) proposed by Yang can effectively prove many classes of non-trivial geometric inequalities[89−91]. The prover BOTTEMA implemented by Yang took more than two seconds to verify more than one hundred basic inequalities in the Bottema's book [92] in Pentium IV/2200 computer, of which including some classical geometric inequalities, such as Euler's inequality, Finsler-Hadwiger's inequality and Gerretseen's inequality[89−91]. This work of Yang had made a significant progress for mechanical proving geometric inequalities. Recently, Yang and his collaborators have proposed a successive difference substitution method(SDS) which could decide nonnegativity of polynomials with nonnegative real variables[93−96]. SDS uses nothing but linear transformation to split variables into smaller positive quantities so that degrees and numbers of variables of resulting polynomials in the whole computation process could never increase, which is the reason why SDS could solve a great many of difficult problems with large polynomials. Yang and his collaborators plan to prove more difficult geometric inequalities using their SDS.

The quantifier elimination method of Tarski[3], the CAD method of Collins[5,85], the CS method of Wu[86−87], the DDA method and the SDS method of Yang[89−96] are algebraic decision method for geometric inequalities, which firstly transform geometric inequalities into equivalent algebraic forms and then decide whether they are true or false. For human, such decision process does not have any readability at all. The readable proofs for triangular inequalities are generated by the cell-decomposition algorithm based on geometric invariants[97]. This kind of readable machine proofs generated by the method still have great difference with traditional proofs. Furthermore, how to readably prove some simple triangular inequalities is preliminarily discussed in [98]. Up to now, the success of readable machine proving for geometric inequalities has not been reported.

The automated generation of readable proofs for geometric inequality is still a great challenge today. In our opinion, it is a very hopeful research direction to expand search methods, coordinate-free methods and formal logic methods to prove geometric inequalities automatically.

3.2 Intelligent Geometry Softwares

The readable theorem proving in geometries has high application value on education. It can be used to develop intelligent geometry softwares[99−100] which is a kind of dynamic geometric softwares with function of automated reasoning. The intelligent geometry softwares not only have powerful dynamic drawing function but also can prove the geometry theorems readably.

In recent years, the research on this kind of softwares has become a focus. Some intelligent geometry softwares are developed successively by some countries in the world, such as Geometry Expert (China)[37,101−103], MMP/Geometer (China)[104], Super Sketchpad (China)[105], Geometry Turtor (USA)[29], Advanced Geometry Turtor (USA)[47], $HOARD_{ATINF}$ (France)[56], Geometry Explorer(UK)[106] and WinGCLC (Serbia)[107].

However, so far as we know, none of these geometry theorem provers is sustainable, which means that users can not add new knowledge to the provers and can not modify the added knowledge in provers when using these provers. What is more, users can not further develop the provers so as to realize integrating use of of different reasoning methods. So, when the geometry theorem provers developed by others are unable to meet one's requirement, he has to develop a new prover according to his own idea, which demands a great deal of repeated programming work. Recently, Zheng and Zhang proposed and realized a sustainable geometry automated reasoning platform[108−109]. The platform consists of six parts, including knowledge base, knowledge editor, information base, reasoning engine, information query and dynamic geometry drawing system. On the platform, the users can add geometry objets, predicates and rules, and integrated use of different methods.

The intelligent geometry softwares can help students learning geometry proofs better. It is not an easy thing for students to prove geometry statements. Students can interact friendly with computers to explore the ideas of proof using the intelligent geometry softwares. They can observe, guess and discover those geometric properties hidden in geometric diagrams, which enables the students not only to understand the geometry statements more visually and more deeply but also to improve their interests in learning geometry. Furthermore, the students can also explore ideas of proof step by step with intelligent geometry softwares. This revolution of learning geometry would certainly improve the learning efficiency of students significantly.

The educational application has posed higher requirements for the geometry theorem provers. Not only are the provers required to generate readable proofs but also are the generated proofs required to enable the students to read, understand and grasp. It had better be able to generate readable proofs of traditional styles[37,47], namely, the proofs is the same as those in the usual geometric textbooks of middle school. Provers are not only required to be able to generate the geometry proofs automatically but also help the students to analyze and check whether the proofs given by them are correct or complete[55−56]. The prover of the intelligent geometry softwares had better be interactive. In this way, students can prove the extremely difficult geometry statements through the mutual cooperation with the computer when using the interactive geometry theorem provers. Students can design and plan the main steps of proofs and the computer can complete the trifling reasoning of each step automatically. Therefore, there are broad application prospects to develop interactive readable prover for geometry theorems. If an intelligent geometry software could be enabled to support the traditional geometric problem solving methods such as adding auxiliary line, proving by contradiction, identity method, analogy method and induction method, it would satisfy the requirements for the educational application better. It is believed that the research on this aspect would achieve breakthrough in the near future.

3.3 Machine Learning

None of the existing geometry theorem provers has any self-learning capability today. The prover's power of solving problems can not be enhanced through a large amount of training. Whether the geometry theorem prover with self-learning capability could be designed remains an important subject well worth of research.

In fact, the prover stores geometry theorems proved by itself into its database for future use, which could be understood as a kind of relatively rough rote learning. If the prover could be enabled to select the theorems needed to be permanently stored according to some standard itself and refuse these theorems that would probably be useless in future, the style of this rote learning could apparently be further improved . There are two difficulties for implementing this kind of rote learning: one is how to determine the theorems needed to be permanently stored and the other is how to decide whether the theorem to be proved has been in the theorem base or not. If the prover acquires this rote learning capability, its problem-solving capability would become stronger and stronger with more and more geometry theorems being stored. An intelligent geometry theorems database would be probably developed if research keeps going on from this direction.

Developing the prover's analogy learning capability is a relatively high level learning style. The analogy is a traditional problem-solving technique. When solving a geometric problem, people often recall the similar problems that they have solved before. If the geometric problem to be proved is similar to some geometric problem proved by someone before, he/she would try to solve the present geometric problem according to former successful problem-solving experiences. If the geometric problem has not ever been solved in the past, one would try to solve this new problem with more general problem-solving method. If the problem is successfully solved, the problem and its solutions could be remembered for solving other similar problems. The core problem of analogy learning is how to define the similarity of problems and how to apply the problem-solving methods of some problems to similar problems. The provers with analogy learning capability could effectively prove geometry theorems by the analogy method. How to implement the geometry theorem prover with analogy reasoning function is preliminarily discussed in [56].

It is another challenge to enable geometry theorems provers to have induction learning capability. The induction is one of the most important learning capability for human to acquire new knowledge and new techniques. People often summarize tactics or problem-solving patterns which are commonly effective against some kinds of problems by a large amount of problem-solving practice. For example, those geometry statements whose conclusion only involving segment proportion are often successfully proved using those theorems of similar triangles. Though some tactics probably are not completely effective, they can help people find problem-solving ideas quickly. We had better also let geometry theorem prover induce and analyze some high-level heuristic problem-solving tactics or problem-solving pattern as the geometry experts of human automatically so that it can select the suitable problem-solving tactics according to the characteristics of the problem to be solved.

The existing provers can effectively prove many difficult geometry theorems and even can effectively generate very short and beautiful readable proofs, but they can not improve their own problem-solving capability through a large of practices since they do not have any self-learning capability. No matter how powerful its functions are, the geometry theorem prover short of self-learning capability can not be regarded as intelligent or compared with human geometricians. Can we design a geometry theorem prover with sufficiently powerful self-learning capability to enable it reach or exceed the human geometricians? Just as the checker program of A. Samuel, it defeated Samuel himself after three years and has been able to defeat a state champion of USA after another three years[110–111].

References

[1] D. Hilbert, *Foundations of Geometry*, Open Court Publishing Company, La Salla, Illinois, 1971.
[2] W. T. Wu, *Mathematics Mechanization*, Science Press, Beijing, 2003.

[3] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, The RAND Corporation Press, Santa Monica, 1948.

[4] A. Seidenberg, A new decision method of elementary algebra, *Annals of Mathematics*, 1954, **60**: 365–371.

[5] G. Collins, Quantifier elimination for real closed fields by cylindrical algebraic decomposition, *Lecture Notes Computer Sciences* 33, Springer-Verlag, Berlin, Heidelberg, 1975.

[6] D. Arnon, Geometric reasoning with logic and algebra, *Artificial Intelligence*, 1988, **37**: 37–60.

[7] W. T. Wu, On the decision problem and the mechanization of theorem proving in elementary geometry, *Journal of Systems Science and Mathematical Science*, 1978, **21**(16): 157–179 (in Chinese).

[8] W. T. Wu, *Basic Principles of Mechanical Theorem Proving in Geometries*, Science Press, Beijing, 1984 (in Chinese).

[9] J. Ritt, *Differential Algebra*, American Mathematical Society, New York, 1950.

[10] S. C. Chou, An introduction to Wu's method for mechanical theorem proving in geometry, *Journal of Automated Reasoning*, 1988, **4**: 237–267.

[11] S. C. Chou, Proving Elementary Geometry Theorems Using Wu's Algorithm, *Automated Theorem Proving: After* 25 *Years* (ed. by W. Bledsoe and D. Loveland), AMS Contemporary Mathematics Series, 1984, **29**: 243–286.

[12] S. C. Chou, *Mechanical Geometry Theorem Proving*, D. Reidel Publishing Company, Dordrecht, Netherlands, 1988.

[13] B. Kutzler and S. Stifter, On the application of Buchberger's algorithm to automated geometry theorem proving, *Journal of Symbolic Computation*, 1986, **2**: 389–397.

[14] B. Buchberger, G. Collins, and B. Kutzler, Algebraic methods for geometric reasoning, *Annual Review of Computer Sciences*, 1995, **3**(19): 85–119.

[15] D. Kapur, T. Saxena, and L. Yang, Algebraic and geometric reasoning with dixon resultants, *Proceedings of International Symposium on Symbolic and Algebraic Computation* (Oxford, 1994), ACM Press, New York, 1994 99–107.

[16] J. Z. Zhang, L. Yang, and X. R. Hou, The sub-resultant method for automated theorem proving, *Journal of Systems Science and Mathematical Sciences*, 1995, **15**(1): 10–15. (in Chinese).

[17] D. M. Wang, Elimination procedures for mechanical theorem proving in geometry, *Annals of Mathematics and Artificial Intelligence*, 1995, **13**: 1–24.

[18] J. W. Hong, Can geometry be proved by an example? *Scientia Sinica*, 1986, **29**: 824–834.

[19] J. Z. Zhang and L. Yang, Principles of parallel numerical method and single-instance method of mechanical theorem proving, *Mathematics in Practice and Theory*, 1989, **1**: 34–43.

[20] J. Z. Zhang, L. Yang, and M. Deng, The parallel numerical method of mechanical theorem proving, *Theoretical Computer Science*, 1990, **74**: 253–271.

[21] H. Gelernter and N. Rochester, Intelligent behavior in problem-solving machines, *IBM Journal*, 1958: 336–345.

[22] H. Gelernter, Realization of a Geometry Theorem Machine, *Computers and Thought* (ed. by E. Feigenbaum and J. Feldman), McGraw-Hill, New York, 1963.

[23] H. Gelernter, J. Hansen, and D. Loveland, Empirical Explorations of the Geometry Theorem Proving Machine, *Computers and Thought* (ed. by E. Feigenbaum and J. Feldman), McGraw-Hill, New York, 1963.

[24] S. C. Chou, X. S. Gao, and J. Z. Zhang, *Automated Production of Traditional Proofs for Theorems in Euclidean Geometry, IV, A Collection of* 400 *Geometry Theorems*, TR-92-7, Department of Computer Science, WSU, 1992.

[25] S. C. Chou, X. S. Gao, and J. Z. Zhang, Automated production of traditional proofs for constructive geometry theorems, *Proceedings of Eighth IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 1993.

[26] J. Z. Zhang, X. S. Gao, and S. C. Chou, The geometric information search system by forward reasoning, *Chinese Journal of Computers*, 1996, **19**(10): 721–727 (in Chinese).

[27] S. C. Chou, X. S. Gao, and J. Z. Zhang, A deductive database approach to automated geometry theorem proving and discovering, *Journal of Automated Reasoning*, 2000, **25**(3): 219–246.

[28] H. Goldstein, *Elementary Geometry Theorem Proving*, AIM-280, MIT, Cambridge, Massachusetts, 1973.

[29] J. Anderson, C. Boyle, A. Corbett, and M. Lewis, The geometry tutor, *Proceedings of IJCAI'85*, Los Angles, 1985.

[30] P. Gilmore, An examination of the geometry theorem proving machine, *Artificial Intelligence*, 1970, **1**(2): 171–187.

[31] A. Nevins, Plane geometry theorem proving using forward chaining, *Artificial Intelligence*, 1975, **6**(1): 1–23.

[32] S. Ullman, *Model-Driven Geometry Theorem Prover*, AIM-321, MIT, Cambridge, Massachusetts, 1975.

[33] H. Coelho and L. Pereira, Automated reasoning in geometry theorem proving with prolog, *Journal of Automated Reasoning*, 1986, **2**(4): 329–390.

[34] T. Evans, A Heuristic Program to Solve Geometry Analogy Problems, *Semantic Information Processing* (ed. by M. Minsky), MIT Press, Cambridge, 1969.

[35] J. Anderson, Tuning of Search of the Problem Space for Geometry Proofs, *Proceeding of IJCAI'81*, Vancouver, 1981.

[36] Y. Zheng, S. C. Chou, and X. S. Gao, Visually dynamic presentation of proofs in plane geometry - part 2. Automated generation of visually dynamic presentation with the Full-Angle method and the deductive database method, *Journal of Automated Reasoning*, 2010, **45**(3): 243–266

[37] X. S. Gao, J. Z. Zhang, and S. C. Chou, *Geometry Expert*, Nine Chapters Publishing, TaiPei, Taiwan, 1998 (in Chinese).

[38] J. G. Jiang and J. Z. Zhang, The automated geometry reasoning network based on equivalent class reasoning, *Patttern Recognition and Artificial Intelligence*, 2006, **19**(5): 617–622 (in Chinese).

[39] J. G. Jiang and J. Z. Zhang, The automated geometry reasoning system based on RETE algorithm, *Journal of Sichuan University: Engineering Science Edition*, 2006, **38**(3): 135–139 (in Chinese).

[40] J. G. Jiang, J. Z. Zhang, and X. J. Wang, Readable proving for a class of geometric theorems of polynomial equality type, *Chinese Journal of Computers*, 2008, **31**(2): 207–213 (in Chinese).

[41] R. Welham R, Geometry Problem Solving, D.A.I. Research Report No.14, 1976.

[42] R. Wong, *Construction Heuristics for Geometry and a Vector Algebra Representation of Geometry*, Technical Memorandum 28, Project MAC, MIT, Cambridge, Massachusetts, 1973.

[43] R. Reiter, A semantically guided deductive system for automatic theorem proving, *IEEE Transactions on Computers*, 1977, **C-25**(4): 328–334.

[44] A. Robinson, Proving a theorem (as done by Man, Logician, or Machine), *Automation of Reasoning* (ed. by J. Siekmann and G. Wrightson), Springer-Verlag, Berlin, Heidelberg, 1983.

[45] W. Elcock, Representation of knowledge in geometry machine, *Machine Intelligence* (ed. by W. Elcock and D. Michie), John Wiley and Sons, 1977, **8**: 11–29.

[46] G. Greeno, M. Magone, and S. Chaiklin, Theory of constructions and set in problem solving, *Memory and Cognition*, 1979, **7**(6): 445–461.

[47] N. Matsuda and K. VanLehn, GRAMY: A geometry theorem prover capable of construction, *Journal of Automated Reasoning*, 2004. **32**(1): 3–33.

[48] J. McCharen, R. Overbeek, and L. Wos, Problems and experiments for and with automated theorem-proving programs, *IEEE Transactions on Computers*, 1976, **C-25**: 773–782.

[49] A. Quaife, Automated Development of Tarski's geometry, *Journal of Automated Reasoning*, 1989, **5**: 97–118.

[50] L. Meikle and J. Fleuriot, Formalizing Hilbert's grundlagen in Isabelle/Isar, *Theorem Proving in Higher Order Logics*, 2003: 319–334.

[51] P. Scott, Mechanizing Hilbert's Foundations of Geometry in Isabelle, Master Thesis, School of Informatics, University of Edinburgh, 2008.

[52] C. Dehlinger, J. Dufourd, and P. Schreck, Higher-order intuitionistic formalization and proofs in Hilbert's elementary geometry, *Automated Deduction in Geometry* (2000), *LNAI* 2061 (ed. by J. Richter-Gebert and D. M. Wang), Springer-Verlag, Berlin, Heidelberg, 2001.

[53] J. Narboux, A decision procedure for geometry in Coq, *Proceedings of TPHOLs'* 2004, *LNCS* 3223 (ed. by S. Konrad, B. Annett, and G. Genesh), 2004.

[54] J. Z. Zhang, S. C. Chou, and X. S. Gao, Automated production of traditional proofs for theorems in Euclidean geometry, I: The Hilbert intersection point theorems, *Annals of Mathematics and Artificial Intelligence*, 1995, **13**: 109–137.

[55] J. Narboux, A graphical user interface for formal proofs in geometry, *Journal of Automated Reasoning*, 2007, **39**: 161–180.

[56] R. Caferra, N. Peltier, and F. Puitg, Emphasizing human techniques in automated geometry theorem proving: A practical realization, *Automated Deduction in Geometry* (2000)*, LNAI* 2061 (ed. by J. Richter-Gebert and D. M. Wang), Springer-Verlag, Berlin, Heidelberg, 2001.

[57] S. Fèvre, Integration of reasoning and algebraic calculus in geometry, *Automated Deduction in Geometry* (1996)*, Lecture Notes in Artificial Intelligence* 1360, Springer-Verlag, Berlin, Heidelberg, 1998.

[58] S. Fèvre and D. M. Wang, Combining algebraic computing and term-rewriting for geometry theorem proving. *Proceedings of AISC*'98, Pittsburgh, USA, 1998.

[59] G. Défourneaux, C. Bourely, and N. Peltier, Semantic generalizations for proving and disproving conjectures by analogy, *Journal of Automated Reasoning*, 1998, **20**(1–2): 27–45.

[60] S. C. Chou, X. S. Gao, and J. Z. Zhang, *Machine Proofs in Geometry: Automated Production of Readable Proofs for Geometry Theorems*, World Scientific, Singapore, 1994.

[61] J. Z. Zhang, Points Elimination Methods for Geometric Problem Solving, *Mathematics Mechanization and Applications* (ed. by X. S. Gao and D. M. Wang), Academic Press, London, 2000.

[62] P. Janičić, J. Narboux, and P. Quaresma, The area method: A recapitulation, *Journal of Automated Reasoning*, 2012, **48**(4): 489–532.

[63] J. Z. Zhang, *How to Solve Geometry Problems Using Areas*, Shanghai Educational Publishing Inc., Shanghai, 1982 (in Chinese).

[64] J. Z. Zhang and P. Cao, *From Education of Mathematics to Mathematics for Education*, Sichun Educational Publishing Inc., Chengdu, 1988 (in Chinese).

[65] J. Z. Zhang, *A New Approach to Plane Geometry*, Sichuan Educational Publishing Inc., Chengdu, 1992 (in Chinese).

[66] S. C. Chou, X. S. Gao, and J. Z. Zhang, Automated generation of readable proofs with geometric invariants, I: Multiple and shortest proofs generation, *Journal of Automated Reasoning*, 1996, **17**(3): 325–347.

[67] S. C. Chou, X. S. Gao, and J. Z. Zhang, Automated generation of readable proofs with geometric invariants, II: Theorem proving with full-angles, *Journal of Automated Reasoning*, 1996, **17**(3): 349–370.

[68] S. C. Chou, X. S. Gao, and J. Z. Zhang, *A Collection of* 110 *Geometry Theorems and Their Machine Proofs Based on Full-Angles*, TR-94-4, Department of Computer Science, WSU, 1994.

[69] Y. Zou and J. Z. Zhang, Automated generation of readable proofs for constructive geometry statements with the mass point method, *Automated Deduction in Geometry* (2010)*, Lecture Notes in Artificial Intelligence* 6877, Springer-Verlag, Berlin Heidelberg, 2011.

[70] S. C. Chou, X. S. Gao, and J. Z. Zhang, Mechanical geometry theorem proving by vector calculation, *Proceedings of International Symposium on Symbolic and Algebraic Computation* (Kiev, Ukraine, July 6–8, 1993), ACM Press, New York, 1993.

[71] S. C. Chou, X. S. Gao, and J. Z. Zhang, Automated production of traditional proofs in solid geometry, *Journal of Automated Reasoning*, 1995, **14**: 257–291.

[72] L. Yang, X. S. Gao, S. C. Chou, and J. Z. Zhang, Automated production of readable proofs for theorems in non-euclidean geometries, *Automated Deduction in Geometry, Lecture Notes Computer Sciences* 1360, Springer-Verlag, Berlin, Heidelberg, 1997.

[73] H. B. Li, Clifford algebra approaches to mechanical geometry theorem proving, *Mathematics Mechanization* (ed. by X. S. Gao and D. M. Wang), Academic Press, London, 2000.

[74] B. Sturmfels, Computational algebraic geometry of projective configurations, *Journal of Symbolic Computation*, 1993, **11**: 595–618.

[75] J. Righter-Gebert, Mechanical theorem proving in projective geometry, *Annals of Mathematics and Artificial Intelligence*, 1995, **13**: 139–172.

[76] H. B. Li, New Explorations in Automated Theorem Proving in Geometries, Ph.D. Thesis, Peking University, China, 1994.

[77] H. B. Li, Some applications of clifford algebra to geometries, *Automated Deduction in Geometry* (1989), *Lecture Notes in Artificial Intelligence* 1669, Springer-Verlag, Berlin, Heidelberg, 1998.

[78] D. M. Wang, Clifford algebraic calculus for geometric reasoning with application to computer vision, *Automated Deduction in Geometry* (1996), *Lecture Notes in Artificial Intelligence* 1360, Springer-Verlag, Berlin, Heidelberg, 1998.

[79] H. B. Li and H. Shi, On Erdös' ten-point problem, *Acta Mathematica Sinica, New Series*, 1997, **13**(2): 221–230.

[80] S. Fèvre and D. M. Wang, Proving geometric theorems using clifford algebra and rewrite rules, *Lecture Notes in Artificial Intelligence* 1421 (ed. by D. M. Wang), Springer-Verlag, Berlin, Heidelberg, 1997.

[81] H. Q. Yang, S. G. Zhang, and G. C. Feng, A Clifford algebraic method for geometric reasoning, *Automated Deduction in Geometry* (Beijing, 1998), *Lecture Notes in Artificial Intelligence* 1669 (ed. by X. S. Gao, D. M. Wang, and L. Yang), Springer-Verlag, Berlin, Heidelberg, 1999.

[82] H. B. Li, Vectorial equation-solving for mechanical geometry theorem proving, *Journal of Automated Reasoning*, 2000, **25**: 83–121.

[83] H. B. Li and M. Cheng, Clifford algebraic reduction method for automated theorem proving in differential geometries, *Journal of Automated Reasoning*, 1998, **21**: 1–21.

[84] H. B. Li, Mechanical theorem proving in differential geometry, *Mathematics Mechanization* (ed. by X. S. Gao and D. M. Wang), Academic Press, London, 2000.

[85] A. Dolzmann, T. Sturm, and V. Weispfenning, A new approach for automatic theorem proving in real geometry, *Journal of Automated Reasoning*, 1998, **21**(3): 357–380.

[86] W. T. Wu, On problems involving inequalities, *MM-Res, Preprints, MMRC*, 1992, **7**: 103–138.

[87] W. T. Wu, On a finiteness theorem about problems involving inequalities, *Journal of Systems Science and Mathematical* Sciences, 1994, **7**(2): 193–200.

[88] D. M. Wang, Polynomial Equations Solving and Mechanical Geometric Theorem Proving, Ph.D. Thesis, Institute of Systems Science, Chinese Academy of Sciences, Beijing, 1993.

[89] L. Yang, Recent advances in automated theorem proving on inequalities, *Journal of Computer Science and Technology*, 1999, **14**(5): 434–446.

[90] L. Yang, X. R. Hou, and B. C. Xia, A complete algorithm for automated discovering of a class of inequality-type theorems, *Science in China* (*Series F*), 2001, **44**(1): 33–49.

[91] L. Yang and J. Z. Zhang, A practical program of automated proving for a class of geometric inequalities, *Automated Deduction in Geometry, Lecture Notes in Artificial Intelligence* 2061 (ed. by J. Richter-Gebert and D. M. Wang), Springer-Verlag, Berlin, Heidelberg, 2001.

[92] O. Bottema, et al, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, Netherlands, 1969.

[93] L. Yang, Solving harder problems with lesser mathematics, *Proceedings of the* 10*th Asian Technology Conference in Mathematics*, ATCM Inc., Blacksburg, 2005.

[94] L. Yang, Difference substitution and automated inequality proving, *Journal of Guangzhou University: Natural Science Edition*, 2006, **5**(2): 1–7 (in Chinese).

[95] L. Yang and Y. Yao, Difference substitution matrices and decision on nonnegativity of polynomials, *Journal of Systems Science and Mathematical Sciences*, 2009, **29**(9): 1169–1177 (in Chinese).

[96] Y. Yao, Infinite product convergence of column stochastic mean matrix and machine decision for positive semi-definite forms, *Science China Mathematics* (*Series A*), 2010, **53**(3): 251–264 (in Chinese).

[97] S. H. Xia, Automated Production of Readable Proof for a Class of Inequality-type Theorems, Ph.D, Thesis, Chinese Academy of Sciences, Beijing, 2002.

[98] S. Chen and J. Z. Zhang, Automated production of elementary and readable proof of inequality, *Journal of Sichuan University: Engineering Science Education*, 2003, **35**(4): 86–93 (in Chinese).

[99] S. C. Chou, X. S. Gao, and J. Z. Zhang, Automated geometry theorem proving and geometry education, *Proceedings of Asian Technology Conference in Mathematics*, Association of Mathematics Educators, Singapore, 1995.

[100] C. Z. Li and J. Z. Zhang, Readable machine solving in geometry and ICAI software MSG, *Automated Deduction in Geometry* (1998), *Lecture Notes in Artificial Intelligence* 1669, Springer-Verlag, Berlin, Heidelberg, 1998.

[101] S. C. Chou, X. S. Gao, and J. Z. Zhang, An introduction to geometry expert, *Proceedings of CADE–13, Lecture Notes in Artifficial Intelligence* (ed. by M. McRobbie and J. Slaney), Springer-Verlag, Berlin, Heidelberg, 1996.

[102] Y. Zheng, S. C. Chou, and X. S. Gao, An introduction to java geometry expert-(extended abstract), *Automated Deduction in Geometry — 7th International Workshop* (2008), *LNCS* 6301 (ed. by T. Sturm and C. Zengler), Springer, 2011.

[103] Y. Zheng, S. C. Chou, and X. S. Gao, Visually dynamic presentation of proofs in plane geometry - part 1. Basic features and the manual input method, *Journal of Automated Reasoning*, 2010, **45**(3): 213–241.

[104] X. S. Gao, Q. Lin, MMP/Geometer-A software package for automated geometry reasoning, *Automated Deduction in Geometry* (Hagenberg Castle, Austria, 2002) (ed. by F. Winkler), Springer-Verlag, Berlin, Heidelberg, 2004: 44–66.

[105] C. Z. Li and J. Z. Zhang, *Super Sketchpad*, Beijing Normal University Press, Beijing, 2004 (in Chinese).

[106] S. Wilson and J. Fleuriot, Geometry explorer: combining dynamic geometry, automated geometry theorem proving and diagrammatic proofs, *Proceedings of the 12th Workshop on Automated Reasoning* (ARW), Edinburgh, UK, 2005.

[107] P. Janicic and P. Quaresma, System description: GCLCprover + GeoThms, *International Joint Conference on Automated Reasoning* (IJCAR-2006), *Lecture Notes in Artificial Intelligence* 4130 (ed. by U. Furbach and N. Shankar), Springer-Verlag, Berlin, Heidelberg, 2006.

[108] H. Zheng and J. Z. Zhang, Reasoning of algorithm of geometry automatic reasoning platform with sustainable development by user, *Journal of Computer Applications*, 2011, **31**(8): 2101–2107 (in Chinese).

[109] H. Zheng, The sustainable geometry automated reasoning platform, *Journal of System Science and Mathematical Sciences*, 2011, **31**(12): 1622–1632 (in Chinese).

[110] A. Samuel, Some studies in machine learning using the game of checkers, *IBM Journal of Research and Development*, 1959, **3**(3): 210–229.

[111] A. Samuel, Some studies in machine learning using the game of checkers II — Recent progress, *IBM Journal of Research and Development*, 1967, **11**(6): 601–617.