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HYPERGAMES AND BAYESIAN GAMES: A THEORETICAL COMPARISON OF THE MODELS OF GAMES WITH INCOMPLETE INFORMATION*

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Abstract The present study discusses the relationships between two independently developed models of games with incomplete information, hypergames (Bennett, 1977) and Bayesian games (Harsanyi, 1967). The authors first show that any hypergame can naturally be reformulated in terms of Bayesian games in an unified way. The transformation procedure is called Bayesian representation of hypergame. The authors then prove that some equilibrium concepts defined for hypergames are in a sense equivalent to those for Bayesian games. Furthermore, the authors discuss carefully based on the proposed analysis how each model should be used according to the analyzer's purpose.

Key words Bayesian game, Bayesian representations of hypergame, hypergame, incomplete information.

1 Introduction

Game theory provides mathematical models of interactive decision making. A game is called complete information if all the agents (decision makers) know all the components of the game. Otherwise, if some or all of them lack full information about it, the game is called incomplete information. Since people often do not have complete knowledge about games they play and thus many realistic interactive situations accompany incompleteness of information, theoretical frameworks that can analyze such situations have been required.

Hypergame theory deals with agents who may misperceive some components of the game[1] [1]. It is the basic idea of hypergames that each agent is assumed to have her own subjective view of it, which is formulated as a normal form game called her subjective game, and make decisions based on it. In this way it allows agents to hold different perceptions about the game. Although hypergames have been developed in several ways[2–5], in this paper we focus on the simplest model of it called simple hypergames. A simple hypergame is given as the set of agents and the collections of subjective games for each[6–7] (see Section 2.1). Henceforth we simply mean a simple hypergame when referring to a hypergame.

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1. For the earliest attempt to incorporate misperceptions in games, see [9].

On the other hand, Bayesian games have been proposed by Harsanyi[8], who argued that incompleteness of information about anything are captured without loss of generality by subjective probability distributions for each agent over the set of possible states (see Section 2.2 and Section 3). In his way of modeling, such possibilities are modeled as types of each agent, and a game with incomplete information is reformulated as a game of complete information called a Bayesian game by introducing a set of types as well as each agent's belief about them (in a form of probability distribution on the others' types).

Our aim is to compare the two models. In particular, we examine how hypergames can be differentiated from Bayesian games. Since they have been established and developed independently, the relationship has not been investigated rigorously enough.

Let us illustrate our motivation with an example. Consider a two-agent interactive decision of agents 1 and 2, where agent 1 believes the game they play is prisoners' dilemma (Table 1) while agent 2 believes chicken game (Table 2). The situation can be captured as a hypergame by defining each agent's subjective game as each table shows. Since they both see the same set of agents as well as actions but suppose different utility functions, we say that they perceive correctly agents and actions but misperceive utility functions.

Table 1 Prisoners' dilemma

1	2	
	l	r
t	3, 3	1, 4
b	4, 1	2, 2

Table 2 Chicken game

1	2	
	l	r
t	3, 3	2, 4
b	4, 2	1, 1

Now, we have two key questions that motivate our study. First, is it possible to formulate the situation as a Bayesian game as well? The answer is: The standard formulation of Bayesian game would allow us to do so. In the example, two types are introduced for each agent: One type associates the agent with prisoners' dilemma, and the other with chicken game. In Bayesian games, a type is characterized by subjective prior and utility function. In this case, for example, agent 1's type associated with prisoners' dilemma has a subjective prior (probability distribution) that assigns probability 1 to agent 2's type associated with prisoners' dilemma while probability 0 to the other type. It has the same utility function as in prisoners' dilemma. We can define the other types in a similar way and hence construct a Bayesian game (for details, see Example 3.2 in Section 3).

In the current study, we shall show that, in a similar way, any hypergames can naturally and uniquely reformulate in terms of Bayesian games and propose the general procedure which we call Bayesian representation of hypergames. In the transformation process, first each subjective game in a hypergame is extended in a unified way, and then, based on the extended subjective games for all the agents, a Bayesian game is constructed.

The second question is, then, can we analyze the situation with existing equilibrium concepts for Bayesian games? Would the Bayesian game analysis lead us to any different implication from analyzing it as a hypergame? To examine the problem, we investigate relations of equilibrium concepts for hypergames and Bayesian games. As a result, we shall argue that two equilibrium concepts for hypergames, hyper Nash equilibrium and best response equilibrium, lead us to the same implications as two for Bayesian games, Bayesian Nash equilibrium and Nash equilibrium of Bayesian games, respectively.

After all, it seems that Bayesian games are general enough in the sense that any hypergames can be captured in terms of Bayesian games. But our conclusion is not so simple. We also discuss carefully based on our analyses how each model should be used according to the investigator's purpose.

Following the introduction, Section 2 introduces the frameworks of hypergames and Bayesian games. Section 3 provides the procedure of Bayesian representation, the general way of transformation of hypergames into Bayesian games. Then we prove our main results in Section 4 that refer to the relevance between equilibrium concepts of hypergames and Bayesian games. In Section 5, we discuss several topics on implications of our analysis. Finally, the conclusion is added.

2 Models of Games with Incomplete Information: Hypergames and Bayesian Games

Here we introduce two models of games with incomplete information which we shall study in the present paper, hypergames and Bayesian games. The frameworks as well as several equilibrium concepts are defined.

Since the basis of the both models is normal form game, let us begin with it. A normal form game is a formal model of interactive decision making in which each agent makes a decision simultaneously and independently, and once and for all. It consists of three components: a set of agents, sets of actions available to each agent, and utility functions for each agent that associate real values (utilities) with outcomes.

Definition 2.1 (Normal form games) $G = (I, A, u)$ is a normal form game, where
1) I is the finite set of agents.
2) $A = \times_{i \in I} A_i$, where A_i is the finite set of agent i's actions[2]. $a \in A$ is called an outcome.
3) $u = (u_i)_{i \in I}$, where $u_i : A \to \Re$ is agent i's utility function[3].

Nash equilibrium has been the central equilibrium concept for analyses of games. The notion is also important to understand equilibrium concepts for hypergames and Bayesian games introduced later.

Definition 2.2 (Nash equilibrium) Let $G = (I, A, u)$ be a normal form game. $a^* = (a_i^*, a_{-i}^*) \in A$ is a Nash equilibrium of G iff $\forall i \in I, \forall a_i \in A_i, u_i(a^*) \geq u_i(a_i, a_{-i}^*)$. Let us denote the set of Nash equilibria of a normal form game G by $N(G)$.

We say that $a_i' \in A_i$ is a best response of agent i to $a_{-i} \in A_{-i}$, some given choices of the others, iff $u_i(a_i', a_{-i}) \geq u_i(a_i, a_{-i})$ for any $a_i \in A_i$, that is, a_i' maximizes i's utility function when the others take a_{-i}. In a Nash equilibrium, each agent chooses a best response to the choices of the others. Therefore, nobody has an incentive to change the action as long as the others do not change their choices.

We call an agent's action that constitutes some Nash equilibrium her Nash action, that is, $a_i^* \in A_i$ is called agent i's Nash action in a normal form game G iff there exists $a_{-i} \in A_{-i}$ such that $(a_i^*, a_{-i}) \in N(G)$. Let us denote the set of agent i's Nash actions in G by $N_i(G)$.

Next, we introduce two independently established and developed models of games with incomplete information, hypergames (in Section 2.1) and Bayesian games (in Section 2.2).

2.1 Hypergames

In hypergames, each agent is assumed to have her own subjective view of the game she faces,

2. We do not deal with mixed extensions of games.
3. We note that, in our study, we do not need to assume utility functions to be cardinal, or of von Neumann-Morgenstern. One can suppose ordinal utility functions as the hypergame literature typically does.

which is given as a normal form game called her subjective game. A hypergame is defined as a collection of subjective games for each agent involved in the situation.

Definition 2.3 (Hypergames) $H = (I, (G^i)_{i\in I})$ is a hypergame, where I is the finite set of agents and $G^i = (I^i, A^i, u^i)$ is a normal form game called agent i's subjective game, where

1) I^i is the finite set of agents perceived by agent i. We assume $I^i \subseteq I$.

2) $A^i = \times_{j\in I^i} A^i_j$, where A^i_j is the finite set of agent j's actions perceived by agent i.

3) $u^i = (u^i_j)_{j\in I^i}$, where $u^i_j : A^i \to \Re$ is agent j's utility function perceived by agent i.

A hypergame assumes that each agent believes that it is common knowledge among all the agents (who she thinks participate in the game) that the game they play is her own subjective game[4], that is, agent i believes not only that the situation is G^i but also that everyone perceives it as well. Therefore, an agent never knows another agent's subjective game and hence the whole structure of the hypergame, which are described only from an analyzer's point of view. We say that agent i misperceives the set of agents iff $I^i \neq I$, some agent j's action set iff $A^i_j \neq A^j_j$, and some agent j's utility function iff $u^i_j \neq u^j_j$, with $j \neq i$. Since in a hypergame each agent i chooses an action from A^i_i, $\times_{i\in I} A^i_i$ is interpreted as the set of all the realizable outcomes from an objective viewpoint. Note that utilities of an agent may not be defined on some of its elements.

We use two equilibrium concepts for hypergames in the subsequent analysis.

The first one is called hyper Nash equilibrium[7].

Definition 2.4 (Hyper Nash equilibrium) Let $H = (I, (G^i)_{i\in I})$ be a hypergame. $a^* = (a^*_i, a^*_{-i}) \in \times_{i\in I} A^i_i$ is a hyper Nash equilibrium of H iff $\forall i \in I, a^*_i \in N_i(G^i)$. Let us denote the set of hyper Nash equilibria of H by $HN(H)$.

In a hyper Nash equilibrium, every agent chooses a Nash action. It can be interpreted as follows: if we assume every agent adopts Nash action as the decision criteria[5], that is, always chooses some Nash action, an outcome that obtains is necessarily a hyper Nash equilibrium (as long as every agent has at least one Nash action). Therefore, the set of hyper Nash equilibria provides us with all the candidates of outcomes likely to happen under the assumption. In fact, by definition, $HN(H) = \times_{i\in I} N(G^i)$.

Next, the second equilibrium concept we use is best response equilibrium.

Definition 2.5 (Best response equilibrium) Let $H = (I, (G^i)_{i\in I})$ be a hypergame. $a^* = (a^*_i, a^*_{-i}) \in \times_{i\in I} A^i_i$ is a best response equilibrium of H iff $\forall i \in I, \forall a_i \in A^i_i, u^i_i(a^*) \geq u^i_i(a_i, a^*_{-i})$. Let us denote the set of best response equilibria of H by $BE(H)$.

A best response equilibrium is such an outcome in which each agent chooses a best response to the choices of the others (in each subjective game). Although the definition apparently looks like Nash equilibrium for normal form games, the implication is largely different. Best response equilibrium does not assure that, from a particular agent's point of view, the other's choices are also their best responses. Therefore, even if an agent takes a best response, she might consider that she should change her choice as some other agent who she thinks does not take a best response might change the choice. The notion of best response equilibrium refers to nothing more than the fact that every agent chooses a best response[6].

We give an example of hypergames and those equilibria.

4. Something is called common knowledge if everyone knows it, everyone knows everyone knows it, everyone knows everyone knows everyone knows it, and so on[10]. Individual belief of common knowledge is discussed in terms of epistemic logic in [11].

5. Game theoretically, this assumption is not perfectly convincing: the precise implication of common knowledge of the game structure and rationality is that an agent chooses a rationalizable action, a somewhat weaker notion of Nash action[12].

6. Best response equilibrium is mathematically equivalent to the concept of group stability based on rationality in first-level hypergames provided in [5].

Example 2.1 Consider a two-agent hypergame of 1 and 2, namely $H = (I, G^i)_{i \in I}$ with $I = \{1, 2\}$. Table 3 illustrates 1's subjective game, G^1, where $A_1^1 = \{a, b, c, d\}, A_2^1 = \{p, q, r\}$ and each entry expresses utilities for the both which reflect the utility functions perceived by 1, i.e., u_1^1 and u_2^1. Similarly, Table 4 represents 2's subjective game, G^2. In the hypergame, the both agents perceive the set of agents correctly but misperceive each other's action set and utility function.

Since each subjective game is a normal form game, we can derive its Nash equilibrium. In this case, $N(G^1) = \{(a, p), (b, q)\}$ while $N(G^2) = \{(b, q)\}$. Hence, $HN(H) = \{(a, q), (b, q)\}$. On the other hand, $BE(H) = \{(b, q), (c, r)\}$.

Table 3 1's subjective game, G^1

1	2		
	p	q	r
a	3, 3	0, 0	0, 0
b	0, 0	2, 2	0, 0
c	0, 0	0, 0	1, −1
d	1, 1	0, 0	0, 0

Table 4 2's subjective game, G^2

1	2			
	p	q	r	s
a	3, 3	0, 0	−5, 0	0, 5
b	0, 0	2, 2	0, 0	1, 0
c	5, 0	0, 0	−1, 1	0, 0

2.2 Bayesian Games

Bayesian games are defined as follows.

Definition 2.6 (Bayesian games) $G^b = (I, A, T, p, u)$ is a Bayesian game, where

1) I is the finite set of agents.

2) $A = \times_{i \in I} A_i$, where A_i is the finite set of agent i's actions.

3) $T = \times_{i \in I} T_i$, where T_i is the finite set of agent i's types.

4) $p = (p_i)_{i \in I}$, where p_i is agent i's subjective prior, which is a joint probability distribution on T_{-i} for each $t_i \in T_i$.

5) $u = (u_i)_{i \in I}$, where $u_i : A \times T \to \Re$ is agent i's utility function.

A type of an agent is characterized by subjective prior and utility function. A subjective prior describes the type's perception about the game: Each type is assumed to have a probability distribution on the types of the other agents. Unlike normal form games, an agent's utility is determined not only by actions but also by types of the agents.

Note that a Bayesian game itself is a game with complete information in the sense that, although one might not know exactly the actual type of another agent, the type set of the agent as well as each type's subjective prior is now common knowledge: everyone knows all the components of the (Bayesian) game. Thus, we can analyze Bayesian games with the notion of Bayesian Nash equilibrium, a natural generalization of Nash equilibrium.

To define it, we need to introduce "action plans" for each type of each agent called strategies. A strategy of agent i, s_i, is a mapping from her types to her actions, namely, $s_i : T_i \to A_i$. Let us denote the set of agent i's strategies by S_i and let $S = \times_{i \in I} S_i$. We may write $s_{-i}(t_{-i})$ as meaning $(s_j(t_j))_{j \in I \setminus \{i\}}$ with $s_j \in S_j$ and $t_j \in T_j$. Then Bayesian Nash equilibrium is defined as follows.

Definition 2.7 (Bayesian Nash equilibrium) Let $G^b = (I, A, T, p, u)$ be a Bayesian game. $s^* = (s_i^*, s_{-i}^*) \in S$ is a Bayesian Nash equilibrium of G^b iff $\forall i \in I, \forall t_i \in T_i, \forall s_i \in S_i$,

$$\sum_{t_{-i}\in T_{-i}} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})),\ (t_i, t_{-i}))p_i(t_{-i}|t_i) \geq \sum_{t_{-i}\in T_{-i}} u_i((s_i(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i}))p_i(t_{-i}|t_i).$$

Let us denote the set of Bayesian Nash equilibria of a Bayesian game G^b by $BN(G^b)$.

In a Bayesian Nash equilibrium, each type of each agent maximizes its expected utility given her belief, i.e., subjective prior.

For a Bayesian game, we can consider a joint probability distribution p^o on the type set T, which describes probabilities for which a particular combination of types for each agent is chosen actually. We call it the objective prior of the Bayesian game. In particular, we say subjective priors are consistent in a Bayesian game iff each agent's subjective prior is given as the conditional probability distributions computed from the objective prior by Bayes formula[7].

We can also formulate Bayesian games by using objective priors instead of subjective priors as $G^b = (I, A, T, p^o, u)$, and define Nash equilibrium of it as follows:

Definition 2.8 (Nash equilibrium of Bayesian games) Let $G^b = (I, A, T, p^o, u)$ be a Bayesian game (with objective prior). $s^* = (s_i^*, s_{-i}^*) \in S$ is a Nash equilibrium of G^b iff $\forall i \in I, \forall s_i \in S_i$,

$$\sum_{t\in T} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), t)p^o(t) \geq \sum_{t\in T} u_i((s_i(t_i), s_{-i}^*(t_{-i})), t)p^o(t).$$

Let us denote the set of Nash equilibria of G^b by $N(G^b)$.

In a Nash equilibrium of a Bayesian game, each type of each agent maximizes its expected utility given an objective prior. In any Bayesian games with consistent priors, the set of Bayesian Nash equilibria coincides with that of Nash equilibria[8].

We refer to the both formulation, one with subjective priors and the other with objective priors, as Bayesian games and write them as G^b unless it may cause any confusion (when we write $BN(G^b)$ and $N(G^b)$, we suppose $(p_i)_{i\in I}$ and p^o, respectively). Furthermore, we may say that p^o is the objective prior of $G^b = (I, A, T, p, u)$. Since most literature assumes consistency of priors, the distinction between the two ways of formulating Bayesian games as well as between the two equilibrium concepts do not matter in practice[8]. On the other hand, we deal with Bayesian games without consistency in our study.

3 Bayesian Representation of Hypergames

In this section, we propose a general way to transform hypergames into Bayesian games that we call Bayesian representation of hypergames. We also give an example in the end.

Harsanyi claims that any kinds of uncertainties about a game as well as perceptual differences among agents can be modeled in a unified way[8], which goes on as follows[9]:

1) (Agents) Whether an agent is participating in the game can be converted into what the agent's action set is, by allowing her only one action, "non-participation" (NP), when she is supposed to be out of the game.

2) (Actions) Whether a particular action is feasible for an agent can in turn be converted into what the agent's utility function is, by saying that she will get some very low utility whenever she takes the action that is supposed to be infeasible.

3) (Utility functions) This way, any uncertainty or perceptual differences about agents as well as actions can be reduced to those about utility functions, if any. Then by regarding each

7. That is, $(p_i)_{i\in I}$ is consistent iff there exists a probability distribution p^o on T such that $\forall i \in I, \forall t \in T, p_i(t_{-i}|t_i) = p^o(t)/\sum_{t_{-i}\in T_{-i}} p^o(t_i, t_{-i})$.

8. Consistency of priors still remains controversial[13–14], though we do not go into the details of the topic.

9. See also [15].

possible utility function of each agent as a type of the agent, the game can be modeled as a Bayesian game.

Let us call the above argument Harsanyi's claim. When we apply it to situations represented as hypergames, it is interpreted as follows. For example, suppose first that agent i thinks that another agent j does not participate in the game, though j actually is in the game. Then the claim argues that i's exclusion of j is game-theoretically equivalent to saying that i includes j in the set of the agents and allow j to use only one action, "non-participation". This way allows every agent to see the common set of agents, which coincides with the set of all the agents actually involved in the hypergame[10].

Next, suppose agent i thinks that another agent j's particular action, a_j, is not feasible for j, while it is actually included in j's action set. Then Harsanyi's claim argues that this is equivalent to saying that i considers that a_j is surely in j's action but gives j very low utility whenever j uses it. Consequently every agent sees in turn the same action set of a particular agent, which is the union of the agent's action set originally conceived by each agent. As a result, perceptual differences in agents as well as actions are resolved, and those only in utility functions remain in our hand.

Generally, according to Harsanyi's claim, in any hypergames, each subjective game can be "extended" as the following.

Definition 3.1 (Extended subjective games) Let $H = (I, (G^i)_{i \in I})$ be a hypergame. For any $i \in I$, a normal form game $\overline{G}^i = (\overline{I}^i, \overline{A}^i, \overline{u}^i)$ is called agent i's extended subjective game (induced from H) iff it satisfies all of the following conditions:

1) $\overline{I}^i = I$.

2) $\overline{A}^i = \times_{j \in I} \overline{A}^i_j$, where $\forall j \in I$, $\overline{A}^i_j = \bigcup_{k \in I} A^k_j$ if $j \in I^k$ for any $k \in I$, $\overline{A}^i_j = \bigcup_{k \in I} A^k_j \cup \{NP\}$ otherwise.

3) $\overline{u}^i = (\overline{u}^i_j)_{j \in I}$, where $\overline{u}^i_j : \overline{A}^i \to \Re$. For any $j \in I$ and $a = (a_j, a_{-j}) \in \overline{A}^i$, $\overline{u}^i_j(a)$ is defined as follows[11], where c is a real constant bigger than $-\infty$:

(i) if $I^i = I$,

$$\overline{u}^i_j(a) = \begin{cases} u^i_j(a), & \text{if } a \in A^i, \\ -\infty, & \text{if } a_j \notin A^i_j, \\ c, & \text{otherwise,} \end{cases}$$

(ii) if $I^i \neq I$,

$$\overline{u}^i_j(a) = \begin{cases} u^i_j((a_l)_{l \in I^i}), & \text{if } j \in I^i \wedge a_k = NP \text{ for any } k \in I \setminus I^i \wedge (a_l)_{l \in I^i} \in A^i, \\ -\infty, & \text{if } (j \in I^i \wedge a_k = NP \text{ for any } k \in I \setminus I^i \wedge a_j \notin A^i_j) \\ & \vee (j \notin I^i \wedge a_j \neq NP), \\ c, & \text{otherwise,} \end{cases}$$

Then the hypergame $\overline{H} = (I, (\overline{G}^i)_{i \in I})$ is called the extended hypergame (induced from H) when for any $i \in I$, $\overline{G}^i$ is agent i's extended subjective game induced from H. Conversely, we may say that H is the original hypergame of $\overline{H}$ and G^i is the original subjective game of $\overline{G}^i$.

10. One might think this way results in the set of agents becoming tremendously enormous because it requires one to include anyone. As is typical in the literature of Bayesian games, we ignore any agent who is regarded as a participant of the game by nobody and actually not in the game. Note also that we have assumed that the agent set in an agent's subjective game never includes anybody who is actually not involved in the game (Definition 2.3).

11. Recall $I^i \subseteq I$.

Although the definition may look complicated, the underlying idea is simple: We just follow Harsanyi's claim. In an agent's extended subjective game, the agent set includes all the agents actually involved in the situation. Then the action set for a particular agent is given as the union of the agent's action set in each agent's original subjective game (and NP (non-participation) if at least one agent thinks the agent is out of the game).

Utility functions are determined based on the next three principles. First, any outcomes modeled in the original subjective game assign the same utilities to each agent in its extension as well. Second, when someone takes an action that is not modeled in the original subjective game, the agent always gets extremely low utility, $-\infty$. Third, in such cases, the other agents are supposed to get some utility c.

Since the extension is unique up to $c \in \Re$, we say $\overline{G}^i$ is the extended subjective game of agent i. Let us denote $\overline{I}^i$ and $\overline{A}^i$ by $\overline{I}$ and $\overline{A}$, respectively, because they are identical for all $i \in I$. Recall that Harsanyi's claim argues any perceptual differences about agents and actions can be resolved.

Henceforth, we assume that, for any $i, j \in I$ and $a \in A^i$, $u^i_j(a) > -\infty$, that is, in an original subjective game, utility one can obtain is always bigger than $-\infty$. Then the next lemma assures that Nash equilibria in each (original) subjective game are "preserved" in its extension, and vice versa.

Lemma 3.1 (Nash equilibria of extended subjective games) *Let* $H = (I, (G^i)_{i\in I})$ *be a hypergame and* $\overline{G}^i = (\overline{I}, \overline{A}, \overline{u}^i)$ *be the extended subjective game of* $i \in I$. *Then we have, for any* $i \in I$,

$$N(\overline{G}^i) = \begin{cases} N(G^i), & \text{if } I^i = I, \\ \{a \in \overline{A} | (a_j)_{j\in I^i} \in N(G^i) \wedge \forall k \in I \setminus I^i, a_k = NP\}, & \text{otherwise.} \end{cases}$$

Proof (i: case of $I^i = I$) (proof of $N(\overline{G}^i) \supseteq N(G^i)$) Suppose $a^* = (a^*_j)_{j\in I^i} \in N(G^i)$, which means, $\forall j \in I^i, \forall a_j \in A^i_j, u^i_j(a^*) \geq u^i_j(a_j, a^*_{-j})$. In $\overline{G}^i$, $\overline{I} = I^i$ and $\overline{A}_j \supseteq A^i_j$ for any $j \in \overline{I}$. Then, for any $j \in \overline{I}$, (a) $\forall a_j \in \overline{A}_j \cap A^i_j, \overline{u}^i_j(a^*)(= u^i_j(a^*)) \geq \overline{u}^i_j(a_j, a^*_{-j})(= u^i_j(a_j, a^*_{-j}))$ and (b) $\forall a_j \in \overline{A}_j \setminus A^i_j, \overline{u}^i_j(a^*)(= u^i_j(a^*)) \geq \overline{u}^i_j(a_j, a^*_{-j})(= -\infty)$ hold. Both (a) and (b) hold for any $j \in \overline{I} \Leftrightarrow \forall j \in \overline{I}, \forall a_j \in \overline{A}_j, \overline{u}^i_j(a^*) \geq \overline{u}^i_j(a_j, a^*_{-j})$. This is equivalent to $a^* \in N(\overline{G}^i)$.

(proof of $N(\overline{G}^i) \subseteq N(G^i)$) Next, suppose $a^* = (a^*_j)_{j\in\overline{I}} \in N(\overline{G}^i)$, which means, $\forall j \in \overline{I}, \forall a_j \in \overline{A}_j, \overline{u}^i_j(a^*) \geq \overline{u}^i_j(a_j, a^*_{-j})$. Now, suppose $\exists j \in \overline{I}, a^*_j \notin A^i_j$. Then $\overline{u}^i_j(a^*) = -\infty$, therefore, $\exists a_j \in \overline{A}_j, \overline{u}^i_j(a_j, a^*_{-j})(\geq c) > \overline{u}^i_j(a^*)$. Thus, $\forall j \in \overline{I}, a^*_j \in A^i_j$. Then $\forall j \in I^i, \forall a_j \in A^i_j, u^i_j(a^*)(= \overline{u}^i_j(a^*)) \geq u^i_j(a_j, a^*_{-j})(= \overline{u}^i_j(a_j, a^*_{-j}))$. This is equivalent to $a^* \in N(G^i)$.

Hence, we have $N(\overline{G}^i) = N(G^i)$.

(ii : case of $I^i \neq I$) (proof of $N(\overline{G}^i) \supseteq \{a \in \overline{A} | (a_j)_{j\in I^i} \in N(G^i) \wedge \forall k \in I \setminus I^i, a_k = NP\}$) Suppose $a^* = (a^*_j)_{j\in\overline{I}} \in \{a \in \overline{A} | (a_l)_{l\in I^i} \in N(G^i) \wedge \forall k \in I \setminus I^i, a_k = NP\}$. In $\overline{G}^i$, $\overline{I} \supset I^i$ and $\overline{A}_j \supseteq A^i_j$ for any $j \in \overline{I} \cap I^i$. Then (a) $\forall j \in \overline{I} \cap I^i, \forall a_j \in \overline{A}_j \cap A^i_j, \overline{u}^i_j(a^*)(= u^i_j((a^*_l)_{l\in I^i})) \geq \overline{u}^i_j(a_j, a^*_{-j})(= u^i_j(a_j, (a^*_l)_{l\in I^i\setminus\{j\}}))$ because $(a^*_l)_{l\in I^i} \in N(G^i)$, (b) $\forall j \in \overline{I} \cap I^i, \forall a_j \in \overline{A}_j \setminus A^i_j, \overline{u}^i_j(a^*)(= u^i_j((a^*_l)_{l\in I^i})) \geq \overline{u}^i_j(a_j, a^*_{-j})(= -\infty)$, and (c) $\forall j \in \overline{I} \setminus I^i, \forall a_j \in \overline{A}_j, \overline{u}^i_j(a^*)(= c) > \overline{u}^i_j(a_j, a^*_{-j})(= -\infty)$, where $c \in \Re$. All of (a), (b) and (c) hold for any $j \in \overline{I}$ $\Leftrightarrow \forall j \in \overline{I}, \forall a_j \in \overline{A}_j, \overline{u}^i_j(a^*) \geq \overline{u}^i_j(a_j, a^*_{-j}) \Leftrightarrow a^* \in N(\overline{G}^i)$.

(proof of $N(\overline{G}^i) \subseteq \{a \in \overline{A} | (a_j)_{j\in I^i} \in N(G^i) \wedge \forall k \in I \setminus I^i, a_k = NP\}$) Suppose $a^* = (a^*_j)_{j\in\overline{I}} \in N(\overline{G}^i)$, which means, $\forall j \in \overline{I}, \forall a_j \in \overline{A}_j, \overline{u}^i_j(a^*) \geq \overline{u}^i_j(a_j, a^*_{-j})$. Now, suppose $\exists k \in I \setminus I^i, a^*_k \neq$

NP. Then $\overline{u}^i_k(a^*) = -\infty$, which is smaller than $\overline{u}^i_k(NP, a^*_{-k})(= c)$. Thus, $\forall k \in I \setminus I^i, a^*_k = NP$. Next, suppose $\exists j \in \overline{I} \cap I^i, a^*_j \notin A^i_j$. Then $\overline{u}^i_j(a^*) = -\infty$, therefore, $\exists a_j \in \overline{A}_j, \overline{u}^i_j(a_j, a^*_{-j})(\geq c) \geq \overline{u}^i_j(a^*)$. Thus, $\forall j \in \overline{I} \cap I^i, a^*_j \in A^i_j$. Then $\forall j \in I^i, \forall a_j \in A^i_j, u^i_j((a^*_k)_{k \in I^i})(= \overline{u}^i_j(a^*)) \geq u^i_j(a_j, (a^*_l)_{l \in I^i \setminus \{j\}})(= \overline{u}^i_j(a_j, a^*_{-j}))$. This is equivalent to $(a^*_k)_{k \in I^i} \in N(G^i)$. Consequently, $a^* \in \{a \in \overline{A} | (a_l)_{l \in I^i} \in N(G^i) \wedge \forall k \in I \setminus I^i, a_k = NP\}$.

Hence, we have $N(\overline{G}^i) = \{a \in \overline{A} | (a_l)_{l \in I^i} \in N(G^i) \wedge \forall k \in I \setminus I^i, a_k = NP\}$.

By (i) and (ii), the lemma holds. ∎

The lemma says, when the agent set in an agent's original subjective game includes all the agents involved, the set of Nash equilibria in it coincides with that in its extension. Otherwise an outcome is a Nash equilibrium in an agent's extended subjective game if and only if the choices of those agents included in the agent set in her original subjective game constitute a Nash equilibrium of it and the others all choose NP.

Then an extended hypergame can be transformed into a Bayesian game by regarding each possible view as a type. We call the transformed game, or the reformulation process itself, the Bayesian representation of the hypergame.

Definition 3.2 (Bayesian representation of hypergames) Let $H = (I, (G^i)_{i \in I})$ be a hypergame and $\overline{G}^i = (\overline{I}, \overline{A}, \overline{u}^i)$ be the extended subjective game of $i \in I$. A Bayesian game $G^b(H) = (I^b, A, T, p, u)$ is called the Bayesian representation of H iff each elements of $G^b(H)$ satisfies the following conditions:

1) $I^b = \overline{I}$, where I^b is the agent set in $G^b(H)$.

2) $A = \overline{A}$.

3) $T = \times_{i \in I^b} T_i$. For all $i, j \in I^b$, $T_i = \{t^j_i | j \in I^b\}$. $t^j_i \in T_i$ is a type of agent i whose view is associated with $\overline{G}^j$.

4) $p = (p_i)_{i \in I^b}$, where $p_i(\cdot | t_i)$ is agent i's subjective prior, which is a joint probability distribution on T_{-i} for each $t_i \in T_i$ such that for any $j \in I^b, p_i(t_{-i} | t^j_i) = 1$ if $t_{-i} = (t^j_k)_{k \in I^b \setminus \{i\}}$, $p_i(t_{-i} | t^j_i) = 0$ otherwise.

5) $u = (u_i)_{i \in I^b}$, where $u_i : A \times T \to \Re$ such that for any $a \in A, t^j_i \in T_i$ and $t_{-i} \in T_{-i}$, $u_i(a, (t^j_i, t_{-i})) = \overline{u}^j_i(a)$.

Recall that, in Bayesian games, a type is characterized by both subjective prior and utility function. Intuitively, we define $t^j_i \in T_i$ as agent i's type who believes it is common knowledge that the game they play is agent j's extended subjective game, $\overline{G}^j$. Hence t^j_i assigns probability 1 to a combination of types of the others each of which also perceives $\overline{G}^j$ while assigning probability 0 to any other combinations, and has the same utility function as in $\overline{G}^j$, i.e. $\overline{u}^j_i$. The subjective priors reflect the basic assumption of hypergames that every agent believes each own subjective game is common knowledge.

Since the agent set in a hypegame is always identical with that in its Bayesian representation, that is, I in a hypergame H is same as I^b in its Bayesian representation $G^b(H)$, henceforth we use the same symbol I to denote the agent set in the Bayesian representation as well and write $G^b(H) = (I, A, T, p, u)$.

Bayesian games can be formulated with objective priors instead of subjective priors. Since, in the original hypergame, agent i considers the game is $\overline{G}^i$, we particularly say t^i_i is agent i's actual type for any $i \in I$. Based on the idea, we define objective priors of Bayesian representations of hypergames as follows.

Definition 3.3 (Objective priors) Let $G^b(H) = (I, A, T, p, u)$ be the Bayesian representation of a hypergame H. Then p^o is called the objective prior of $G^b(H)$ iff for any $t = (t_i, t_{-i}) \in T$,

$$p^o(t) = \begin{cases} 1, & \text{if } \forall i \in I, t_i = t_i^i, \\ 0, & \text{otherwise.} \end{cases}$$

An objective prior reflects each agent's actual view in the original hypergame. It assigns probability 1 to a combination of types each of which is the actual type of each agent, while probability 0 to any other combinations. We may also write the Bayesian representation of a hypergame H as $G^b(H) = (I, A, T, p^o, u)$. It is easy to see that Bayesian representations of any simple hypergames do not allow the agents to have consistent subjective priors.

We illustrate two examples of Bayesian representations of hypergames.

Example 3.1 Reconsider the two-agent hypergame of Example 2.1. According to Definition 3.1, each agent's subjective game in the hypergame is extended as shown in Tables 5 and 6, respectively. Let $\overline{G}^1$ and $\overline{G}^2$ denote the extended subjective games for each and $\overline{H}$ be the extended simple hypergame. They now have the common action sets. With respect to utility values, for example, in $\overline{G}^1$, since agent 2's action s is not modeled in its original, G^1, whenever 2 takes it, 2 gets utility $-\infty$ while 1 gets utility c. It is easy to see that Lemma 3.1 holds, that is, each extended subjective game has the same set of Nash equilibria as its original.

Table 5 1's extended subjective game, $\overline{G}^1$

1	2			
	p	q	r	s
a	3, 3	0, 0	0, 0	c, $-\infty$
b	0, 0	2, 2	0, 0	c, $-\infty$
c	0, 0	0, 0	1, -1	c, $-\infty$
d	1, 1	0, 0	0, 0	c, $-\infty$

Table 6 2's extended subjective game, $\overline{G}^2$

1	2			
	p	q	r	s
a	3, 3	0, 0	-5, 0	0, 5
b	0, 0	2, 2	0, 0	1, 0
c	5, 0	0, 0	-1, 1	0, 0
d	$-\infty$, c	$-\infty$, c	$-\infty$, c	$-\infty$, c

Then its Bayesian representation $G^b(H) = (I, A, T, p, u)$ is formulated as follows:
1) $I = \{1, 2\}$.
2) $A = A_1 \times A_2$, where $A_1 = \{a, b, c, d\}$ and $A_2 = \{p, q, r, s\}$.
3) $T = T_1 \times T_2$, where $T_1 = \{t_1^1, t_1^2\}$ and $T_2 = \{t_2^1, t_2^2\}$.
4) $p = (p_1, p_2)$, where for any $i, j \in I, p_i(t_{-i}|t_i^j) = 1$ if $t_{-i} = (t_{-i}^j)$, $p_i(t_{-i}|t_i^j) = 0$ otherwise.
5) $u = (u_1, u_2)$, where for each $i \in I$, $u_i : A \times T \to \Re$ such that for any $a \in A, t_i^j \in T_i$ and $t_{-i} \in T_{-i}$, $u_i(a, (t_i^j, t_{-i})) = \overline{u}_i^j(a)$.

We interpret each type t_i^j (with $i, j \in \{1, 2\}$) as a type of agent i who believes $\overline{G}^j$ is common knowledge. Therefore, for example, t_1^1 assigns probability 1 to t_2^1 while probability 0 to t_2^2, and has the same utility function as $\overline{u}_1^1$. In this case, the actual types of each agent are t_1^1 and t_2^2. Thus the objective prior p^o is defined as, for any $t = (t_1, t_2) \in T$,

$$p^o(t) = \begin{cases} 1, & \text{if } t_1 = t_1^1 \text{ and } t_2 = t_2^2, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.2 As with typical Bayesian games, a Bayesian representation can be illustrated as a game tree. Reconsider the example used in Section 1, namely, a two-agent hypergame of agents 1 and 2 in which 1's subjective game is prisoners' dilemma (Table 1) while that of 2 is chicken game (Table 2). Since they both perceive correctly the set of agents as well as actions for each agent, the extended hypergame induced it is same as the original.

The Bayesian representation of the hypergame can be described as Figure 1. Nature moves first and determines types of the agents. For example, 21 means that agent 1's type is t_1^2, that is, agent 1 whose view is associated with chicken game, while that of agent 2 is t_2^1, that is, agent 2 whose view is associated with prisoners' dilemma. The objective prior describes probabilities of nature's move. In this case, it assigns probability 1 to 12 while probability 0 to any other combinations of types.

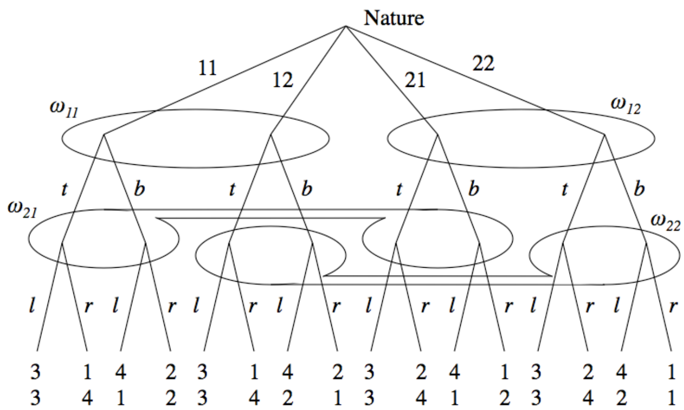

Figure 1 The Bayesian representation

Following the nature's move, each agent knows her own type but does not know the opponent's type. The fact is expressed with the information sets described in the game tree: ω_{11} and ω_{12} are agent 1's information sets while ω_{21} and ω_{22} are for agent 2. For example, if agent 1's type is t_1^1, 1 faces the information set ω_{11} which contains two decision nodes. This means, as usual, that there agent 1 cannot tell at which node out of the two 1 actually is. But 1 has a belief about it which is represented by the subjective prior. In this case, t_1^1 assigns probability 1 to agent 2's type being t_2^1, while probability 0 to t_2^2. Likewise, in any other information set, the agent who faces it has a belief about the opponent's type. Finally, utilities for each agent are determined by actions and types of the agents: The upper value is for agent 1 and the lower is for agent 2.

4 Comparisons of Equilibrium Concepts and Results

Next we investigate relationship between hypergames and those Bayesian representations, particularly by examining relations of their equilibria concepts. We shall show that hyper Nash equilibrium and Bayesian Nash equilibrium are highly related with each another in Section 4.1, and so are best response equilibrium and Nash equilibrium of Bayesian games in Section 4.2. The implications of those results will be discussed in the next section.

4.1 Hyper Nash Equilibria and Bayesian Nash Equilibria

Our first result describes a deep relation between hyper Nash equilibrium and Bayesian Nash equilibrium. Recall that by introducing strategies that are functions which associate types of each agent with her actions, i.e. $s_i : T_i \to A_i$ for each $i \in I$, we can conduct equilibrium analysis of Bayesian games.

Proposition 4.1 (Equilibria of hypergames and their Bayesian representations (with subjective priors)) *Let $H = (I, (G^i)_{i\in I})$ be a hypergame and $G^b(H) = (I, A, T, p, u)$ be its Bayesian representation. Then $a^* = (a_i^*, a_{-i}^*) \in HN(H)$ iff there exists $s^* = (s_i^*, s_{-i}^*) \in BN(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$.*

Proof Suppose $(s_i^*, s_{-i}^*) \in BN(G^b(H))$. This means, $\forall i \in I, \forall t_i \in T_i, \forall s_i \in S_i, \sum_{t_{-i}\in T_{-i}} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i}))p_i(t_{-i}|t_i) \geq \sum_{t_{-i}\in T_{-i}} u_i((s_i(t_i), s_{-i}^*(t_{-i})), (t_i, t_{-i}))p_i(t_{-i}|t_i)$. In $G^b(H)$, this is equivalent to the fact, $\forall i, j \in I, \forall s_i \in S_i$,

$$u_i((s_i^*(t_i^j), (s_k^*(t_k^j))_{k\in I\setminus\{i\}}), (t_i^j, (t_k^j)_{k\in I\setminus\{i\}})) \geq u_i((s_i(t_i^j), (s_k^*(t_k^j))_{k\in I\setminus\{i\}}), (t_i^j, (t_k^j)_{k\in I\setminus\{i\}})).$$

Let $\overline{G}^i = (\overline{I}, \overline{A}, \overline{u}^i)$ be the extended subjective game for each $i \in I$. Then the above statement is equivalent to, $\forall i, j \in I, \forall s_i \in S_i$,

$$\overline{u}_i^j(s_i^*(t_i^j), (s_k^*(t_k^j))_{k\in I\setminus\{i\}}) \geq \overline{u}_i^j((s_i(t_i^j), (s_k^*(t_k^j))_{k\in I\setminus\{i\}})),$$

which is equivalent to, $\forall j \in I, (s_i^*(t_i^j))_{i\in I} \in N(\overline{G}^j)$.

Hence, $\forall i \in I, a_i^* \in N_i(\overline{G}^i)$ iff there exists $s^* = (s_i^*, s_{-i}^*) \in BN(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$. Since $N_i(\overline{G}^i) = N_i(G^i)$ for any $i \in I$ (due to Lemma 3.1), $\forall i \in I, a_i^* \in N_i(\overline{G}^i) \Leftrightarrow \forall i \in I, a_i^* = N_i(G^i)$, which is equivalent to, $(a_i^*)_{i\in I} \in HN(H)$. ∎

Proposition 4.1 refers to a relation between hyper Nash equilibrium and Bayesian Nash equilibrium. Precisely, it claims that if an outcome is a hyper Nash equilibrium in a hypergame, then its Bayesian representation has some Bayesian Nash equilibrium in which the actual type of each agent chooses the same action as in the hyper Nash equilibrium, and conversely, if a Bayesian representation of a hypergame has a Bayesian Nash equilibrium, then the actions chosen by the actual type of each agent in it must be a hyper Nash equilibrium in the original hypergame.

4.2 Best Response Equilibria and Nash Equilibria of Bayesian Games

Next, let us consider Bayesian representations with objective priors (instead of subjective priors). Then we have the next proposition.

Proposition 4.2 (Equilibria of extended hypergames and Bayesian representations (with objective priors)) 1) *Let $H = (I, (G^i)_{i\in I})$ be a hypergame, $\overline{H}$ be the extended hypergame induced from it, and $G^b(H) = (I, A, T, p^o, u)$ be its Bayesian representation. Then $a^* = (a_i^*, a_{-i}^*) \in BE(\overline{H})$ iff there exists $s^* = (s_i^*, s_{-i}^*) \in N(G^b(H))$ such that $\forall i \in I, s_i^*(t_i^i) = a_i^*$.*

Proof Suppose $(s_i^*, s_{-i}^*) \in N(G^b(H))$. This means, $\forall i \in I, \forall s_i \in S_i$,

$$\sum_{t\in T} u_i((s_i^*(t_i), s_{-i}^*(t_{-i})), t)p^o(t) \geq \sum_{t\in T} u_i((s_i(t_i), s_{-i}^*(t_{-i})), t)p^o(t).$$

In $G^b(H)$, this is equivalent to the fact that $\forall i \in I, \forall s_i \in S_i$,

$$u_i((s_i^*(t_i^i), (s_j^*(t_j^j))_{j\in I\setminus\{i\}}), (t_i^i, (t_j^j)_{j\in I\setminus\{i\}})) \geq u_i((s_i(t_i^i), (s_j^*(t_j^j))_{j\in I\setminus\{i\}}), (t_i^i, (t_j^j)_{j\in I\setminus\{i\}})).$$

Let $\overline{G}^i = (\overline{I}, \overline{A}, \overline{u}^i)$ be the extended subjective game of each $i \in I$ induced from H. Then the statement above is equivalent to, $\forall i \in I$, $\forall s_i \in S_i, \overline{u}_i^i(s_i^*(t_i^i), (s_j^*(t_j^j))_{j\in I\setminus\{i\}}) \geq$

$\overline{u}^i_i(s_i(t^i_i), (s^*_j(t^j_j))_{j\in I\setminus\{i\}})$. Thus, $\forall i \in I$, $\forall a_i \in \overline{A}_i, \overline{u}^i_i(a^*_i, a^*_{-i}) \geq \overline{u}^i_i(a_i, a^*_{-i})$ iff there exists $s^* = (s^*_i, s^*_{-i}) \in N(G^b(H))$ such that $\forall i \in I, s^*_i(t^i_i) = a^*_i$. The former of the statement is equivalent to, $(a^*_i, a^*_{-i}) \in BE(\overline{H})$. Hence, we have the proposition. ∎

The proposition in turn refers to the relation between best response equilibrium of extended hypergames and Nash equilibrium of Bayesian games. Precisely, it says that an outcome is a best response equilibrium in an extended hypergame if and only if the Bayesian representation (with objective prior) has some Nash equilibrium in which the actual type of each agent chooses the same action as in the outcome.

Since Proposition 4.2 refers only to extended hypergames, we then try to specify how original hypergames and those Bayesian representations relate to each other. The following lemma would be useful for that purpose.

Lemma 4.3 (Best response equilibria of extended hypergames) *Let $H = (I, (G^i)_{i\in I})$ with $G^i = (I^i, A^i, u^i)$ for each $i \in I$ be a hypergame and $\overline{H}$ be the extended hypergame induced from it. Then we have $BE(H) \subseteq BE(\overline{H})$. Particularly, the equality holds if* (i) *$\forall i \in I, I^i = I$, and* (ii) *$\forall i, j \in I, A^j_j \subseteq A^i_j$.*

Proof Let $\overline{G}^i = (\overline{I}, \overline{A}, \overline{u}^i)$ be the extended subjective game of $i \in I$ induced from H. Suppose $(a^*_i, a^*_{-i}) \in BE(H)$, which means, $\forall i \in I, \forall a_i \in A^i_i, u^i_i(a^*) \geq u^i_i(a_i, a^*_{-i})$. Then $\forall i \in I, \forall a_i \in \overline{A}_i, \overline{u}^i_i(a^*) \geq \overline{u}^i_i(a_i, a^*_{-i})$ because $\overline{u}^i_i(a^*) = u^i_i(a^*)$, and $\overline{u}^i_i(a_i, a^*_{-i}) = u^i_i(a_i, a^*_{-i})$ if $a_i \in A^i_i$, $\overline{u}^i_i(a_i, a^*_{-i}) = -\infty$ otherwise. This is equivalent to $a^* \in BE(\overline{H})$. Hence, we have $BE(H) \subseteq BE(\overline{H})$.

Next, let us assume H satisfies (i) $\forall i \in I, I^i = I$, and (ii) $\forall i, j \in I, A^j_j \subseteq A^i_j$. Suppose $a^* = (a^*_i, a^*_{-i}) \in BE(\overline{H})$, which means, $\forall i \in I, \forall a_i \in \overline{A}_i, \overline{u}^i_i(a^*) \geq \overline{u}^i_i(a_i, a^*_{-i})$. Since $\forall i \in I, \forall a_i \in \overline{A}^i_i \setminus A^i_i, \forall a_{-i} \in A_{-i}, \overline{u}^i_i(a_i, a_{-i}) = -\infty, \forall i \in I, a^*_i \in A^i_i$. By (ii), then $\forall i, j \in I, a^*_j \in A^i_j$. Thus,

$$\forall i \in I, \forall a_i \in A^i_i, u^i_i(a^*)(= \overline{u}^i_i(a^*)) \geq u^i_i(a_i, a^*_{-i})(= \overline{u}^i_i(a_i, a^*_{-i})).$$

This is equivalent to $a^* \in BE(H)$. Under the conditions (i) and (ii), we have both $BE(H) \subseteq BE(\overline{H})$ and $BE(\overline{H}) \subseteq BE(H)$, hence $BE(H) = BE(\overline{H})$. ∎

Lemma 4.3 claims that if an outcome is the best response equilibrium in a hypergame, then it is so in its extension as well. Moreover it refers to a sufficient condition under which the converse is also true.

Consequently, we have the next proposition.

Proposition 4.4 (Equilibria of hypergames and Bayesian representations (with objective priors)) *Let H be a hypergame and $G^b(H) = (I, A, T, p^o, u)$ be its Bayesian representation. If $a^* = (a^*_i, a^*_{-i}) \in BE(H)$, then there exists $s^* = (s^*_i, s^*_{-i}) \in N(G^b(H))$ such that $\forall i \in I, s^*_i(t^i_i) = a^*_i$. Particularly, if H satisfies the sufficient condition of Lemma* 4.3*, the converse also holds.*

Proof The proposition is straightforward from Proposition 4.2 and Lemma 4.3. ∎

Proposition 4.4 says that if an outcome is a best response equilibrium in a hypergame, then the Bayesian representation (with objective prior) has some Nash equilibrium in which the actual type of each agent chooses the same action as in the best response equilibrium. Particularly, if the hypergame satisfies the sufficient condition of Lemma 4.3, the converse is also true.

By using Examples 2.1 and 3.1, let us illustrate the theoretical results presented above.

First, consider Proposition 4.1 that refers to the relation between hyper Nash equilibrium and Bayesian Nash equilibrium. Recall, in our example, $HN(H) = \{(a, q), (b, q)\}$. On the other hand, $G^b(H)$ has two Bayesian Nash equilibria, i.e., $((s_1(t^1_1), s_1(t^2_1)), (s_2(t^1_2), s_2(t^2_2)) = ((a, b), (p, q))$ and $((b, b), (q, q))$. Since the actual types are t^1_1 and t^2_2, we can see the proposition certainly holds.

Next, Proposition 4.2 describes the relation between best response equilibrium and Nash equilibrium of Bayesian games. With the objective prior above, $G^b(H)$ has 48 Nash equilibria, i.e., $((s_1(t_1^1), s_1(t_1^2)), (s_2(t_2^1), s_2(t_2^2)) = ((b, x_1), (x_2, q)), ((c, x_1), (x_2, r))$ and $((d, x_1), (x_2, s))$, where x_i can be any of agent i's action. Since $BE(\overline{H}) = \{(b, q), (c, r), (d, s)\}$, the proposition holds here. Furthermore, given that $BE(H) = \{(b, q), (c, r)\}$, Proposition 4.4 also holds. In this case, H does not satisfy the sufficient condition of Lemma 4.3, and $BE(H)$ is a proper subset of $BE(\overline{H})$.

5 Discussions

5.1 Implications of the Results

Our interpretations of the results presented in the previous section go as follows. If we are interested in hyper Nash equilibrium of hypergames, we would say that investigating Bayesian Nash equilibrium of those Bayesian representations leads to same implications. It is because, as Proposition 4.1 claims, an action profile that constitutes a hyper Nash equilibrium is always that taken by the actual types of each agent in some Bayesian Nash equilibrium, and vice versa. Likewise, if one wants to know best response equilibrium in a hypergame, Nash equilibrium of its Bayesian representation leads to the same implication as suggested by Proposition 4.4.

Eventually, we conclude that any hypergames can be analyzed in terms of Bayesian games with those existing equilibria as long as our interest is in the two equilibrium concepts for hypergames[12].

5.2 Epistemic Foundations of the Models

Following Harsanyi's claim, Bayesian games allow agents to see the common set of all the possibilities about the game structure, namely, types. But when we consider real interactive situations, the assumption often seems to be hard to accept, and in fact it still remains such controversial issues epistemic game theory focuses on[13]. And more importantly to us, it appears incompatible with the idea of hypergames. How does it affect our analyses?

Our answer to the question is that we can ignore the problem as long as we analyze the two equilibrium concepts for hypergames. That is, even when it is not natural to accept Harsanyi's claim, once we conduct the procedure of Bayesian representation as if we accept it, the analysis of the transformed game leads us to the same insight as discussed in Section 5.1.

We however emphasize that the problem might become critical in some cases. For example, suppose that, in a hypergame, a particular action available to agent i, a_i, is not included in i's action set in another agent j's subjective game, that is, $a_i \in A_i^i$ but $a_i \notin A_i^j$. Harsanyi's claim argues that j's exclusion of a_i is equivalent to j's thought that i will never use the action because it always gives i a very low utility. Here j is aware of the fact that a_i is feasible to i but consciously ignore it from the action set. But in real situations, people often may be unaware of something[14]. Agent j may be purely unaware of the existence of such an action in the first place. If so, once agent i uses a_i, agent j acquires completely new knowledge ("I

12. The converse obviously does not hold true: There are Bayesian games that cannot be transformed into and analyzed in terms of hypergames. Furthermore, even for a Bayesian game that is transformed from a hypergame, there might exist other hypergames that lead to the same Bayesian game as a result of those Bayesian representations. That is, although the transformation from a hypergame to a Bayesian game is unique, the converse may not.

13. See, e.g., [17–18]

14. Unawareness has recently been studied by several authors in terms of formal decision theory. For example, see [19–21].

didn't think of that!") and might change the subjective game[15]. The updated subjective view might be different with the extended subjective game under Harsanyi's claim, for j might now consider a_i can lead i to a relatively high utility. Hence, if such a dynamic situation is of interest, one cannot ignore the problem whether an agent is truly aware of something, that is, whether Harsanyi's claim is surely reasonable.

Note that hypergame framework itself tells us nothing about an agent's awareness: When $a_i \notin A_i^j$ in a hypergame, we cannot know whether agent j consciously exclude a_i or is purely unaware of it. Although hypergames are said to be models of misperceptions, misperceptions with which they deal might include unawareness, or not.

5.3 Uniqueness of Hypergame Analysis

One might think Bayesian games are general enough, and thus we do not need any other models like hypergames for games with incomplete information. But we here would like to emphasize two points about uniqueness of hypergame analyses.

First, since hypergames are much simpler than Bayesian games, it would be a good choice to use the former in order to analyze such a situation that can be captured in terms of it. In such cases, the results in Section 4 assure that we can use equilibrium concepts for hypergames when our interest is in equilibrium analysis of Bayesian games. Furthermore, recall that when we calculate equilibria for Bayesian games, we need to specify actions for all the types of each agent. On the other hand, study on equilibria in hypergames requires us less tasks.

Second, as discussed in Section 5.1, we can use Bayesian games and those existing equilibrium concepts as long as we focus on hyper Nash equilibrium or best response equilibrium, but this may not be the case when we want to analyze some other equilibrium concept. For example, stable hyper Nash equilibrium[16], defined as an outcome that is a Nash equilibrium in every agent's subjective game[1]6, cannot be captured by existing concepts of Bayesian games. Hence, if we are interested in its implications, hypergames can provide us with unique insights.

6 Conclusions

We have compared the two theoretical frameworks of games with incomplete information. Our contributions are mainly two-fold. First, we presented a general way of transforming hypergames into Bayesian games, called Bayesian representation. Second, we showed that hyper Nash equilibria and best response equilibria of hypergames lead us to same implications of Bayesian Nash equilibria and Nash equilibria those Bayesian representations, respectively. Based on the analyses, we also discussed how each model should be used according to the analyzer's purpose.

Although we have analyzed only simple hypergames, we consider the results would be applied to other hypergame models as well. We hope that, with those analyses as theoretical bases, researches of games with incomplete information would make further progress.

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15. This kind of perceptual change is discussed in [22].
16. Hence the set of stable hyper Nash equilibria in a hypergame is always subset of that of hyper Nash equilibria, and similarly that of best response equilibria.

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