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A NOTE ON STABILITY OF THE SPLIT-STEP BACKWARD EULER METHOD FOR LINEAR STOCHASTIC DELAY INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract In the literature (Tan and Wang, 2010), Tan and Wang investigated the convergence of the split-step backward Euler (SSBE) method for linear stochastic delay integro-differential equations (SDIDEs) and proved the mean-square stability of SSBE method under some condition. Unfortunately, the main result of stability derived by the condition is somewhat restrictive to be applied for practical application. This paper improves the corresponding results. The authors not only prove the mean-square stability of the numerical method but also prove the general mean-square stability of the numerical method. Furthermore, an example is given to illustrate the theory.

Key words General mean-square stability, mean-square stability, split-step backward Euler method, stochastic delay integro-differential equations.

1 Introduction

For the reader's convenience, throughout this paper we make use of the similar notations as in [1]. Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right-continuous while F_0 contains all P-null sets). Let $\tau > 0$ and $C([-\tau, 0]; \boldsymbol{R})$ denote the family of all continuous $\boldsymbol{R}$-valued functions on $[-\tau, 0]$. Moreover, $|\cdot|$ is the Euclidean norm in $\boldsymbol{R}$ and $||\psi||$ is defined by $||\psi|| = \sup\limits_{-\tau\leq t\leq 0} |\psi(t)|$. We assume that $\psi(t), t \in [-\tau, 0]$ is F_0-measurable and right-continuous, and $E||\psi||^2 < \infty$. Let $C^b_{F_0}([-\tau, 0]; \boldsymbol{R})$ be the family of all F_0-measurable bounded $C([-\tau, 0]; \boldsymbol{R})$-valued random variables $\psi = \{\psi(\theta) : -\tau \leq \theta \leq 0\}$.

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Consider the linear stochastic delay integro-differential equations (SDIDEs) of the form

$$\begin{cases} dx(t) = \left[Ax(t) + Bx(t-\tau) + C\int_{t-\tau}^{t} x(s)ds\right]dt + [Dx(t) + Ex(t-\tau)]dW(t), \qquad t \geq 0, \\ x(t) = \psi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{1}$$

where the initial data $x(t) = \psi(t) \in C([-\tau, 0]; \boldsymbol{R})$, $A, B, C, D, E \in \boldsymbol{R}$, and $\tau_0 = 0$. $W(t)$ is a standard one-dimensional Brownian motion.

Under the above assumptions, Equation (1) has a unique continuous solution denoted by $x(t;\psi)$ on $t \geq -\tau$. It is also easy to see that the equation admits the trivial solution $x(t;0) = 0$. To analyze the stability of SSBE method, we need a useful lemma as follows.

Lemma 1[2] *If*

$$A + |B| + \tau|C| + \frac{1}{2}(|D| + |E|)^2 < 0, \tag{2}$$

then the solution of Equation (1) *is mean-square stable, that is,*

$$\lim_{t\to\infty} E|x(t)|^2 = 0.$$

Now, the split-step backward Euler (SSBE) method applied to Equation (1) produces

$$\begin{cases} y_n^* = y_n + \left[Ay_n^* + By_{n-m} + Ch\sum_{k=1}^{m} y_{n-k}\right]h, \\ y_{n+1} = y_n^* + (Dy_n^* + Ey_{n-m})\triangle W_n, \end{cases} \tag{3}$$

where $h > 0$ is a stepsize which satisfies $\tau = mh$ for a positive integer m, and $t_n = nh$. y_n is an approximation to $x(t_n)$, when $t_n \leq 0$, we have $y_n = \psi(t_n)$. Moreover, the increments $\triangle W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0,h)$-distributed Gaussian random variables. Let $T = s\tau = N_1 h$ and $t \in [-\tau, T]$, where $N_1 = sm, s$ is a positive integer. If $1 - hA \neq 0$, we can obtain the sequences $\{y_n^*, n \geq 0\}$ and $\{y_n, n \geq 0\}$ via (3), when given $y_n = \psi(t_{-n})$ for $n \in \{0, 1, \cdots, m\}$. We assume y_n is F_{t_n}-measurable at the mesh-points t_n. To study the stability of SSBE method, in this paper we always assume that $1 - hA \neq 0$.

Definition 1 Under condition (2), a numerical method is said to be mean-square stable (MS-stable), if there exists a constant $h_0(A, B, C, D, E) > 0$ such that any application of the method to problem (1) generates numerical approximations $\{y_n\}$, which satisfy $\lim_{n\to\infty} E|y_n|^2 = 0$ for all $h \in (0, h_0(A, B, C, D, E))$ with $h = \frac{\tau}{m}$.

Definition 2 Under condition (2), a numerical method is said to be general mean-square stable (GMS-stable), if any application of the method to problem (1) generates numerical approximations $\{y_n\}$, which satisfy $\lim_{n\to\infty} E|y_n|^2 = 0$ for every stepsize $h = \frac{\tau}{m}$.

The following theorem is the main result of Tan and Wang[1].

Theorem 1 *Under condition* (2)*, the SSBE method applied to Equation* (1) *is MS-stable.*

Unfortunately, the above result is somewhat restrictive to be applied for practical application, which is not enough to reflect the stable features. In the next section we state and prove our theorem which improves Theorem 1.

2 Stability of SSBE Method

In this section, the MS-stability and GMS-stability of SSBE method are testified.

For simplicity, we set

$$\mu = B^2D^2 + 2|BC|D^2\tau - 2ABDE + (|CD|\tau - A|E|)^2,$$
$$nu = 2|B|D^2 + 2|C|D^2\tau - 2A|DE| + B^2 + 2|BC|\tau - 2AE^2$$
$$+2BDE + 2|CDE|\tau + C^2\tau^2 - A^2,$$

and

$$\triangle = \nu^2 - 4\mu(2A + 2|B| + 2|C|\tau + (|D| + |E|)^2).$$

Theorem 2 *Assume that condition* (2) *holds. Then*

i) *if* $\mu = 0$ *and* $\nu \leq 0$; *or* $\mu > 0$ *and* $\frac{-\nu+\sqrt{\triangle}}{2\mu} \geq 1$; *or* $\mu < 0$ *and* $\triangle < 0$, *then the SSBE method is GMS-stable;*

ii) *if* $\mu = 0$ *and* $\nu > 0$, *and* $h_0(A,B,C,D,E) = -\frac{2A+2|B|+2|C|\tau+(|D|+|E|)^2}{\nu}$, *then the SSBE method is MS-stable;*

iii) *if* $\mu > 0$ *and* $\frac{-\nu+\sqrt{\triangle}}{2\mu} < 1$; *or* $\mu < 0$, $\triangle \geq 0$, *and* $\nu > 0$, *and* $h_0(A,B,C,D,E) = \frac{-\nu+\sqrt{\triangle}}{2\mu}$, *then the SSBE method is MS-stable;*

iv) *if* $\mu < 0, \triangle \geq 0$, $\nu \leq 0$, *and* $h_0(A,B,C,D,E) = \frac{-\nu-\sqrt{\triangle}}{2\mu}$, *then the SSBE method is MS-stable.*

Proof Note $1 - hA \neq 0$ and (2) implies $A < 0$. Then we can see from (3) that

$$y_n^* = \frac{1}{1-Ah}\left(y_n + Bhy_{n-m} + Ch^2\sum_{k=1}^{m} y_{n-k}\right) \tag{4}$$

and

$$y_{n+1} = \frac{1+D\triangle W_n}{1-Ah}\left(y_n + Bhy_{n-m} + Ch^2\sum_{k=1}^{m} y_{n-k}\right) + E\triangle W_n y_{n-m}. \tag{5}$$

Squaring both sides of the above equality, then simplifying and rearranging the equality obtained, we have

$$\begin{aligned}
y_{n+1}^2 = {} & \frac{1+D^2(\triangle W_n)^2}{(1-Ah)^2}\left(y_n^2 + B^2h^2y_{n-m}^2 + C^2h^4\left(\sum_{k=1}^{m} y_{n-k}\right)^2 + 2Bhy_ny_{n-m}\right.\\
& \left.+2Ch^2y_n\sum_{k=1}^{m} y_{n-k} + 2BCh^3y_{n-m}\sum_{k=1}^{m} y_{n-k}\right)\\
& +\frac{2D\triangle W_n}{(1-Ah)^2}\left(y_n^2 + B^2h^2y_{n-m}^2 + C^2h^4\left(\sum_{k=1}^{m} y_{n-k}\right)^2 + 2Bhy_ny_{n-m}\right.\\
& \left.+2Ch^2y_n\sum_{k=1}^{m} y_{n-k} + 2BCh^3y_{n-m}\sum_{k=1}^{m} y_{n-k}\right)\\
& +\frac{2E\triangle W_n}{1-Ah}\left(y_ny_{n-m} + Bhy_{n-m}^2 + Ch^2y_{n-m}\sum_{k=1}^{m} y_{n-k}\right)\\
& +\frac{2DE(\triangle W_n)^2}{1-Ah}\left(y_ny_{n-m} + Ch^2y_{n-m}\sum_{k=1}^{m} y_{n-k}\right)\\
& +\frac{2DE(\triangle W_n)^2}{1-Ah}(Bhy_{n-m}^2) + E^2(\triangle W_n)^2y_{n-m}^2.
\end{aligned}$$

It follows from $2\beta\gamma xy \le |\beta\gamma|(x^2+y^2)$, where $\beta,\gamma \in \boldsymbol{R}$, $\sum_{k=1}^m y_{n-k} \le m \max\limits_{1\le k\le m} y_{n-k}$ and $mh=\tau$, we get

$$\begin{aligned} y_{n+1}^2 \le\ & \frac{1+D^2(\triangle W_n)^2}{(1-Ah)^2}\Big(y_n^2 + B^2h^2y_{n-m}^2 + C^2h^2\tau^2 \max_{1\le k\le m} y_{n-k}^2 + |B|h(y_n^2+y_{n-m}^2) \\ & +|C|\tau h\Big(y_n^2 + \max_{1\le k\le m} y_{n-k}^2\Big) + |BC|\tau h^2\Big(y_{n-m}^2 + \max_{1\le k\le m} y_{n-k}^2\Big)\Big) \\ & +\frac{2D\triangle W_n}{(1-Ah)^2}\Big(y_n^2 + B^2h^2y_{n-m}^2 + C^2h^4\Big(\sum_{k=1}^m y_{n-k}\Big)^2 + 2Bhy_ny_{n-m} \\ & +2Ch^2y_n\sum_{k=1}^m y_{n-k} + 2BCh^3y_{n-m}\sum_{k=1}^m y_{n-k}\Big) \\ & +\frac{2E\triangle W_n}{1-Ah}\Big(y_ny_{n-m} + Bhy_{n-m}^2 + Ch^2y_{n-m}\sum_{k=1}^m y_{n-k}\Big) \\ & +\frac{(\triangle W_n)^2}{1-Ah}\Big(|DE|\big(y_n^2+y_{n-m}^2\big) + |DEC|\tau h\Big(y_{n-m}^2 + \max_{1\le k\le m} y_{n-k}^2\Big)\Big) \\ & +\frac{2DE(\triangle W_n)^2}{1-Ah}\big(Bhy_{n-m}^2\big) + E^2(\triangle W_n)^2y_{n-m}^2. \end{aligned}$$

Note that $E(\triangle W_n)=0, E[(\triangle W_n)^2]=h$ and y_n, y_{n-k} are all F_{t_n}-measurable, where $k=1,2,\cdots,m$. Hence,

$$E(\triangle W_n y_s y_j) = E(y_s y_j E(\triangle W_n | F_{t_n})) = 0,\ \ s,j\in\{n,n-k\}\ (k=1,2,\cdots,m)$$

and

$$E((\triangle W_n)^2 y_s^2) = E(y_s^2 E((\triangle W_n)^2|F_{t_n})) = hE(y_s)^2,\ \ s\in\{n,n-k\}\ (k=1,2,\cdots,m).$$

Let $Y_n = E|y_n|^2$ and taking expectations on both sides of the above inequality. Then we have

$$\begin{aligned} Y_{n+1} &\le P(A,\cdots)Y_n + Q(A,\cdots)Y_{n-m} + R(A,\cdots)\max_{1\le k\le m} Y_{n-k} \\ &\le (P(A,\cdots)+Q(A,\cdots)+R(A,\cdots))\max_{1\le k\le m}\{Y_n, Y_{n-k}\}, \end{aligned}$$

where $P(A,\cdots), Q(A,\cdots), R(A,\cdots)$ are dependent on A,B,C,D,E,h and

$$P(A,\cdots) = \frac{1+D^2h}{(1-hA)^2}(1+|B|h+|C|\tau h) + \frac{|DE|h}{1-Ah},$$

$$Q(A,\cdots) = \frac{1+D^2h}{(1-hA)^2}(|B|^2h^2+|B|h+|BC|\tau h^2) + E^2h + \frac{|DE|h}{1-Ah} + \frac{|CDE|\tau h^2}{1-Ah} + \frac{2BDEh^2}{1-Ah},$$

and

$$R(A,\cdots) = \frac{1+D^2h}{(1-hA)^2}(C^2\tau^2h^2+|C|\tau h+|BC|\tau h^2) + \frac{|CDE|\tau h^2}{1-Ah}.$$

It is clear that $P(A,\cdots)+Q(A,\cdots)+R(A,\cdots)\ge 0$. By recursive calculation we conclude that $Y_n\to 0 (n\to\infty)$ if

$$P(A,\cdots)+Q(A,\cdots)+R(A,\cdots) < 1, \tag{6}$$

which is equivalent to

$$\begin{aligned}&(B^2D^2+2|BC|D^2\tau-2ABDE+(|CD|\tau-A|E|)^2)h^2\\&+(2|B|D^2+2|C|D^2\tau-2A|DE|+B^2+2|BC|\tau-2AE^2+2BDE\\&\quad+2|CDE|\tau+C^2\tau^2-A^2)h\\&+2A+2|B|+2|C|\tau+(|D|+|E|)^2<0.\end{aligned}$$

Let

$$f(h)=\mu h^2+\nu h+2A+2|B|+2|C|\tau+(|D|+|E|)^2,$$

where

$$\mu:=B^2D^2+2|BC|D^2\tau-2ABDE+(|CD|\tau-A|E|)^2$$

and

$$\begin{aligned}\nu:=\ &2|B|D^2+2|C|D^2\tau-2A|DE|+B^2+2|BC|\tau-2AE^2\\&+2BDE+2|CDE|\tau+C^2\tau^2-A^2.\end{aligned}$$

Case 1 If $\mu=0$, the function $f(h)$ reduces to $f(h)=\nu h+2A+2|B|+2|C|\tau+(|D|+|E|)^2$.

i) When $\nu\le 0$, consider (2), we can obtain $f(h)<0$ for any $h>0$. Thus, (6) holds for any $h>0$ with $h=\tau/m$, that is, the SSBE method is GMS-stable.

ii) When $\nu>0$, noting that (2), we can see that if $h<-\frac{2A+2|B|+2|C|\tau+(|D|+|E|)^2}{\nu}$, $f(h)<0$ holds. Therefore, (6) holds for $h\in(0,h_1(A,B,C,D,E))$, that is, the SSBE method is MS-stable, where $h_1(A,B,C,D,E)=-\frac{2A+2|B|+2|C|\tau+(|D|+|E|)^2}{\nu}$.

Case 2 If $\mu>0$, in view of (2), then

$$\triangle:=\nu^2-4\mu(2A+2|B|+2|C|\tau+(|D|+|E|)^2)>0$$

always holds.

i) When $\frac{-\nu+\sqrt{\triangle}}{2\mu}\ge 1$. We can easily obtain that $f(h)<0$ holds for any $0<h<1$. Then the SSBE method is GMS-stable.

ii) When $\frac{-\nu+\sqrt{\triangle}}{2\mu}<1$. We can easily obtain that $f(h)<0$ holds for $h\in(0,h_2(A,B,C,D,E))$ where $h_2(A,B,C,D,E)=\frac{-\nu+\sqrt{\triangle}}{2\mu}$. Then the SSBE method is MS-stable in this case.

Case 3 If $\mu<0$, from (2) we have

i) When $\triangle<0$, we can easily see that $f(h)<0$ always holds for any $h>0$ with $h=\tau/m$. Consequently, the SSBE method is MS-stable in this case.

ii) When $\triangle\ge 0$ and $\nu\le 0$, we can easily see that $f(h)<0$ holds for $h\in(0,h_3(A,B,C,D,E))$, where $h_3(A,B,C,D,E)=\frac{-\nu-\sqrt{\triangle}}{2\mu}$. Then the SSBE method is MS-stable in this case.

iii) When $\triangle\ge 0$ and $\nu>0$, we can easily obtain that $f(h)<0$ holds for $h\in(0,h_3(A,B,C,D,E))$, where $h_3(A,B,C,D,E)=\frac{-\nu+\sqrt{\triangle}}{2\mu}$. Consequently, the SSBE method is MS-stable in this case. The proof is completed. ∎

3 Numerical Experiments

In this section, we give an example to illustrate our theory. Our objective is to illustrate intuitively the stability obtained in the previous section. Furthermore, we compare the restrictions on stepsize of the stable SSBE method with that of the Euler method and Milstein method (see [1, 3–4]).

Let $W(t)$ be a scalar Brownian motion. Consider a one-dimensional linear stochastic delay integro-differential equations

$$dx(t) = \left[Ax(t) + Bx(t-1) + C\int_{t-1}^{t} x(s)ds\right]dt + [Dx(t) + Ex(t-1)]dW(t) \tag{7}$$

on $t \geq 0$ with initial data $\psi(t) = t + 1$ for $t \in [-1, 0]$.

In the following tests, we show the influence of stepsize h on mean-square stability of the SSBE method and compare our results with that obtained in [1, 3–4]. The data used in all figures are obtained by the mean square of data by 100 trajectories, that is, $\Omega : 1 \leq i \leq 100, Y_n = \frac{1}{100}\sum_{i=1}^{100} |Y_n(\Omega)|^2$.

Case 1 Let $A = -9, B = 4, C = 3, D = 0, E = 0.8$.

This test equation has been investigated in [3]. It is easily verified that conditions (2) is satisfied. By Theorem 2 we know that the SSBE method is MS-stable for $h \in (0, 0.5219)$. The result in [1] narrows down the range to $h \in (0, 0.4757)$, then it is restrictive to be applied for practical application. Thus, we improve the stable result of [1]. Figure 1 illustrates the MS-stability of numerical solution obtained by the SSBE method when $h = 0.5$. However, applied to the same test equation, the Milstein method is unstable when the stepsize $h = 0.5$[3] and is MS-stable when the stepsize $0 < h < 1/9$. This shows that in this sense the SSBE method is superior to the Milstein method.

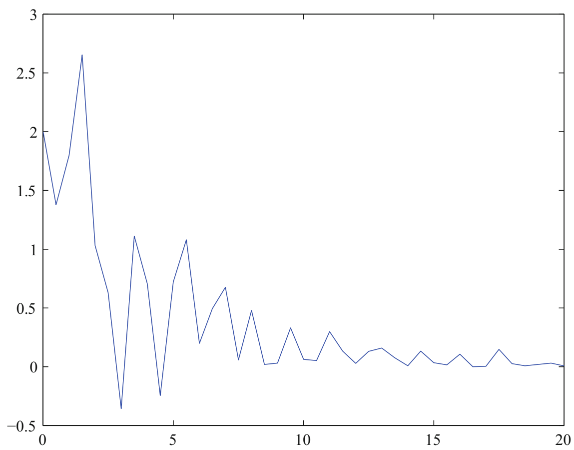

Figure 1 Numerical simulation of Case 1 with $h = 1/2$

Case 2 We choose the coefficients of Equation (7) $A = -4, B = 0, C = 1, D = 0, E = 2$. By Theorem 2 the SSBE method is MS-stable when the stepsize $h \in (0, 0.0883)$. Figure 2 shows that the SSBE method is MS-stable when $h = 1/100$ and the SSBE method is stable when $h = 1/3$. From [4] we can see the θ-Euler method is MS-stable when $h = 1/100$ and $h = 1/3$. This implies that the stability bound obtained by Theorem 2 may not be optimal. The stability of the SSBE method may be uncertain for $h > 0.0883$ in this case.

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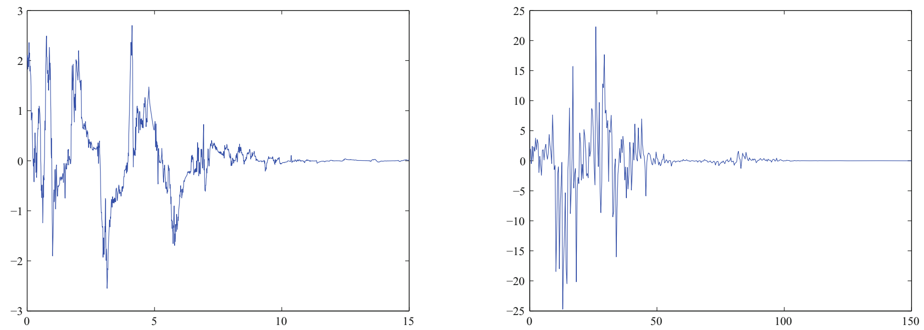

Figure 2 Numerical simulation of Case 2 with upper: left $h = 1/100$ and right $h = 1/3$

Case 3 Let $A = -0.4, B = 0.2, C = 0.6, D = 0.1, E = 4$. The condition (2) is not satisfied. To carry out the numerical simulation we choose the step size $h = 1/128$. The computer simulation result is shown in Figure 3. Clearly, the SSBE method reveals the unstable property of the solution.

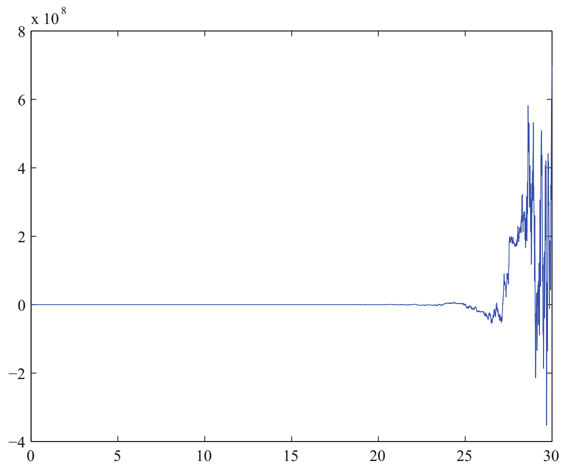

Figure 3 Numerical simulation of Case 3 with $h = 1/128$

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