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SAMPLE AVERAGE APPROXIMATION METHOD FOR A CLASS OF STOCHASTIC VARIATIONAL INEQUALITY PROBLEMS*

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Abstract This paper considers a class of stochastic variational inequality problems. As proposed by Jiang and Xu (2008), by using the so-called regularized gap function, the authors formulate the problems as constrained optimization problems and then propose a sample average approximation method for solving the problems. Under some moderate conditions, the authors investigate the limiting behavior of the optimal values and the optimal solutions of the approximation problems. Finally, some numerical results are reported to show efficiency of the proposed method.

Key words Convergence, gap function, sample average approximation method, stochastic variational inequality.

1 Introduction

Equilibrium is a central concept in numerous disciplines including management science, operations research, economics and engineering. It is known that the deterministic variational inequality problem is one of very important methodologies for studying equilibrium problems. There have been proposed a list of methods for solving the deterministic variational inequality problem. For details, see the monograph[1] and the references therein, for examples, [2–4].

On the other hand, since some elements in many practical problems may involve uncertain data, the stochastic variational inequalities have attracted much attention in the recent literature. In this paper, we consider the following stochastic variational inequality problem: Find a vector $x^* \in S$ such that

$$(x - x^*)^{\mathrm{T}}\mathrm{E}[F(x^*, \omega)] \geq 0, \qquad \forall x \in S, \tag{1}$$

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where S is a closed convex subset of $\Re^n$ and $\mathrm{E}[\,\cdot\,]$ denotes the expectation operator with respect to $\omega \in \Omega$, here the expectation being taken as component-wise, and $F : \Re^n \times \Omega \to \Re^n$ is a vector-valued function. This problem has been studied in [5–6]. In particular, making use of the regularized gap function[7]

$$g(x) := \max_{y\in S} \Big\{ (x-y)^{\mathrm{T}}\mathrm{E}[F(x,\omega)] - \frac{\alpha}{2}\|x-y\|_G^2 \Big\}, \tag{2}$$

where G is a given symmetric and positive definite square matrix and $\|z\|_G := \sqrt{z^{\mathrm{T}}Gz}$ is the G-norm of $z \in \Re^n$. Jiang and Xu[6] reformulated (1) as the following constrained optimization problem:

$$\min_{x\in S} g(x). \tag{3}$$

It is known that any minimizer with zero optimal value of (3) is a solution of the original problem (1), vice versa. Based on the above reformulation, some stochastic approximation methods are studied in [6]. This paper focuses on problem (3). We will propose a sample average approximation (SAA) method for solving (3).

The SAA method and its variants, known under various names such as "stochastic counterpart method", "sample-path method", "simulated likelihood method", have been discussed in the stochastic programming and statistics literature[5,8−10]. In particular, Gürkan, Özge and Robinson[5] proposed the sample path optimization (SPO) method for solving the problem and showed that under moderate conditions, a sequence of solutions to the SPO problem converges to its true counterpart. In fact, the SAA method is not new in the field of statistics. Shapiro[11] first introduced this method to solve stochastic mathematical programs with equilibrium constraints. Later, Meng and Xu[12], and Xu and Meng[13] further investigated the SAA method on stochastic mathematical programs with (nonsmooth) equality constraints.

The rest of the paper is organized as follows. We give some results on coercivity of the regularized gap function in Section 2. Then, in Section 3, we investigate the limiting behavior of the optimal values and the optimal solutions of the approximation problems generated by SAA method.

2 Coercivity of the Function $g(x)$

In this section, we provide some sufficient conditions for the coercivity of the regularized gap function $g(x)$ on the set S. These results imply that problem (3) always has an optimal solution for $\alpha > 0$ sufficiently small.

Theorem 1 *Suppose that there exists a point $x_0 \in S$ such that $\mathrm{E}[\eta(x_0,\omega)] > 0$, where*

$$\eta(x_0,\omega) = \liminf_{\substack{x\,\in\, S\\ \|x\|\,\to\,\infty}} \frac{(F(x,\omega)-F(x_0,\omega))^{\mathrm{T}}(x-x_0)}{\|x-x_0\|^2}.$$

If $\alpha \in (0, 2\|G\|^{-1}\mathrm{E}[\eta(x_0,\omega)])$, then $\lim\limits_{\substack{x\,\in\, S\\ \|x\|\,\to\,\infty}} g(x) = +\infty$.

Proof Since $x_0 \in S$, we have from the definition of $g(x)$ that, for any $x \in S$,

$$\begin{aligned} g(x) &\ge (x-x_0)^{\mathrm{T}}\mathrm{E}[F(x,\omega)] - \frac{\alpha}{2}\|x-x_0\|_G^2 \\ &\ge (x-x_0)^{\mathrm{T}}\mathrm{E}[F(x_0,\omega)] + (x-x_0)^{\mathrm{T}}\mathrm{E}[F(x,\omega)-F(x_0,\omega)] - \frac{\alpha}{2}\|G\|\cdot\|x-x_0\|^2 \\ &\ge \|x-x_0\|^2\bigg(\frac{(x-x_0)^{\mathrm{T}}\mathrm{E}[F(x,\omega)-F(x_0,\omega)]}{\|x-x_0\|^2} - \frac{\|\mathrm{E}[F(x_0,\omega)]\|}{\|x-x_0\|} - \frac{\alpha}{2}\|G\|\bigg). \end{aligned} \tag{4}$$

By Fatou Lemma, one has that

$$\liminf_{\substack{x \in S \\ \|x\| \to \infty}} \mathrm{E}\left[\frac{(F(x,\omega)-F(x_0,\omega))^{\mathrm{T}}(x-x_0)}{\|x-x_0\|^2}\right] \ \geq\ \mathrm{E}[\eta(x_0,\omega)] \ >\ 0.$$

Noting that $\alpha\|G\| < 2\mathrm{E}[\eta(x_0,\omega)]$, one has that

$$\liminf_{\substack{x \in S \\ \|x\| \to \infty}} \left(\frac{(x-x_0)^{\mathrm{T}}\mathrm{E}[F(x,\omega)-F(x_0,\omega)]}{\|x-x_0\|^2} - \frac{\|\mathrm{E}[F(x_0,\omega)]\|}{\|x-x_0\|} - \frac{\alpha}{2}\|G\|\right) \geq \mathrm{E}[\eta(x_0,\omega)] - \frac{\alpha}{2}\|G\| > 0.$$

It follows from (4) that $\liminf_{\substack{x \in S \\ \|x\| \to \infty}} g(x) \geq +\infty$. This completes the proof. ∎

Definition 1 The function F is said to be uniformly monotone over S with probability 1 if there exists an integrable function $C(\omega) \geq 0$ such that

$$\Pr\{(x-y)^{\mathrm{T}}[F(x,\omega)-F(y,\omega)] \geq C(\omega)\|x-y\|^2\} = 1, \qquad \forall x, y \in S.$$

Corollary 1 *Suppose that F is uniformly monotone over S with probability 1 and $C(\omega)$ is such that* $\mathrm{E}[C(\omega)] > 0$. *Then* $\lim_{\substack{x \in S \\ \|x\| \to \infty}} g(x) = +\infty$.

Proof Take a vector x_0 from S. Then one has that

$$\Pr\{(x-x_0)^{\mathrm{T}}[F(x,\omega)-F(x_0,\omega)] \geq C(\omega)\|x-x_0\|^2\} = 1,$$

which implies

$$\mathrm{E}[\eta(x_0,\omega)] \geq \mathrm{E}[C(\omega)] > 0.$$

Here $\eta(x_0,\omega)$ is the same as in Theorem 1. The conclusion follows from Theorem 1 immediately. The proof is completed. ∎

Note that, if the set S is bounded, then the above results hold trivially. Therefore, the main utility of these results is for the case that S is unbounded.

3 SAA Method and Its Convergence

In this section, we present an SAA method for solving (3) and investigate its convergence. Let $\omega^1, \omega^2, \cdots, \omega^N$ be independently and identically distributed samples drawn from Ω. Then, we obtain the following sample average approximation of (3):

$$\begin{aligned} \min\ \ & g_N(x) = \max_{y \in S}\left\{(x-y)^{\mathrm{T}}\left[\frac{1}{N}\sum_{j=1}^{N}F(x,\omega^j)\right] - \frac{\alpha}{2}\|x-y\|_G^2\right\} \\ \text{s.t.}\ \ & x \in S. \end{aligned} \tag{5}$$

Before implementing our analysis further, we need the following definition given by Facchinei and Pang[1].

Definition 2 Let S be a convex subset of $\Re^n$ and $G \times \Re^{n\times n}$ a symmetric positive definitive matrix. The projection operator $\mathit{\Pi}_{S,G} : \Re^n \to S$ is called the skewed projection mapping in $\Re^n$ into S if for every fixed $x \in \Re^n$, and $\mathit{\Pi}_{S,G}(x)$ is the solution of the following convex optimization problem:

$$\begin{cases} \min_y \dfrac{1}{2}\|x-y\|_G^2 = \dfrac{1}{2}(y-x)^{\mathrm{T}}G(y-x) \\ \text{s.t.} \quad y \in S \end{cases}$$

It follows from [7] that

$$g(x) = (x - H(x))^{\mathrm{T}} f(x) - \frac{\alpha}{2}\|x - H(x)\|_G^2,$$
$$g_N(x) = (x - H_N(x))^{\mathrm{T}} f_N(x) - \frac{\alpha}{2}\|x - H_N(x)\|_G^2,$$

where

$$f(x) = \mathrm{E}[F(x,\omega)], \qquad f_N(x) = \frac{1}{N}\sum_{j=1}^{N} F(x,\omega^j),$$

and

$$H(x) = \mathrm{Proj}_{S,G}(x - \alpha^{-1}G^{-1}f(x)), \qquad H_N(x) = \mathrm{Proj}_{S,G}(x - \alpha^{-1}G^{-1}f_N(x)).$$

Furthermore, we make the following assumptions throughout this section:

A1) The set S is nonempty and compact.

A2) The function $F(x,\omega)$ is continuous with respect to x for every $\omega \in \Omega$.

A3) There exists an integrable function $\phi(\omega)$ such that

$$\sup_{x\in S} \|F(x,\omega)\| \le \phi(\omega), \qquad \text{a.e. } \omega \in \Omega,$$

where notation "a.e." denotes almost everywhere.

From [9], we obtain the following result.

Lemma 1 *Assume that* A1)–A3) *hold. Then the following results hold:*

1) $f(x)$ *is finite and continuous on* S*;*

2) $\{f_N(x)\}$ *uniformly converges to* $f(x)$ *with probability* 1 *on* S*, that is,*

$$\lim_{N\to\infty} \max_{x\in S} \|f_N(x) - f(x)\| = 0.$$

The following theorem provides the uniform convergence of the objective functions of the approximation problems (3).

Theorem 2 *Assume that* A1)–A3) *hold. Then* $\{g_N(x)\}$ *uniformly converge to the function* $g(x)$ *with probability* 1 *on* S.

Proof By the definitions of $g(x)$ and $g_N(x)$, one has

$$\begin{aligned} |g(x) - g_N(x)| \le& \left|(x - H(x))^{\mathrm{T}} f(x) - (x - H_N(x))^{\mathrm{T}} f_N(x)\right| \\ &+ \left|\frac{\alpha}{2}\|x - H_N(x)\|_G^2 - \frac{\alpha}{2}\|x - H(x)\|_G^2\right| \\ =& |(x - H(x))^{\mathrm{T}}(f_N(x) - f(x)) - (H_N(x) - H(x))^{\mathrm{T}} f_N(x)| \\ &+ \frac{\alpha}{2}\left|\|x - H_N(x)\|_G^2 - \|x - H(x)\|_G^2\right| \\ \le& \|x - H(x)\| \cdot \|f_N(x) - f(x)\| + \|H_N(x) - H(x)\| \cdot \|f_N(x)\| \\ &+ \frac{\alpha}{2}\left|\|x - H_N(x)\|_G^2 - \|x - H(x)\|_G^2\right|. \end{aligned}$$

Therefore, one has

$$\begin{aligned}&\sup_{x\in S}|g(x)-g_N(x)|\\ &\le \sup_{x\in S}\|x-H(x)\|\cdot\sup_{x\in S}\|f_N(x)-f(x)\|+\sup_{x\in S}\|H_N(x)-H(x)\|\cdot\sup_{x\in S}\|f_N(x)\|\\ &\quad+\frac{\alpha}{2}\sup_{x\in S}\Big|\|x-H_N(x)\|_G^2-\|x-H(x)\|_G^2\Big|. \end{aligned}\tag{6}$$

Since $g(x)\ge 0$, one has

$$\frac{\alpha}{2}\|x-H(x)\|_G^2\le (x-H(x))^{\mathrm T}f(x)\le\frac{1}{\sqrt{\lambda_{\min}}}\|x-H(x)\|_G\cdot\|f(x)\|,$$

where $\lambda_{\min}$ denotes the smallest eigenvalue of G. One furtherly has

$$\|x-H(x)\|\le\frac{1}{\sqrt{\lambda_{\min}}}\|x-H(x)\|_G\le\frac{2}{\alpha\sqrt{\lambda_{\min}}}\|f(x)\|.$$

Since S is nonempty and compact, there exists a positive scalar M such that

$$\sup_{x\in S}\|f(x)\|<M\quad\text{and}\quad\sup_{x\in S}\|f_N(x)\|<M$$

hold with probability 1. Moreover, it is not difficult to show that

$$\sup_{x\in S}\|x-H(x)\|<\frac{2}{\alpha\sqrt{\lambda_{\min}}}M\quad\text{and}\quad\sup_{x\in S}\|x-H_N(x)\|<\frac{2}{\alpha\sqrt{\lambda_{\min}}}M$$

hold with probability 1. In addition, by the nonexpansive property of projection operator, one has

$$\begin{aligned}\|H_N(x)-H(x)\|&=\|\mathrm{Proj}_{S,G}(x-\alpha^{-1}G^{-1}f_N(x))-\mathrm{Proj}_{S,G}(x-\alpha^{-1}G^{-1}f(x))\|_G\\ &\le\|\alpha^{-1}G^{-1}f_N(x)-\alpha^{-1}G^{-1}f(x)\|_G\\ &\le\alpha^{-1}\|G^{-1}\|\cdot\|f_N(x)-f(x)\|,\end{aligned}$$

with probability 1. On the other hand, one has

$$\begin{aligned}&\sup_{x\in S}\Big|\|x-H_N(x)\|_G^2-\|x-H(x)\|_G^2\Big|\\ &=\sup_{x\in S}\Big|(x-H(x))^{\mathrm T}G(H_N(x)-H(x))+(x-H_N(x))^{\mathrm T}G(H_N(x)-H(x))\Big|\\ &\le\Big(\sup_{x\in S}\|x-H(x)\|+\sup_{x\in S}\|x-H_N(x)\|\Big)\cdot\|G\|\cdot\sup_{x\in S}\|H_N(x)-H(x)\|\\ &\le\alpha^{-1}\Big(\sup_{x\in S}\|x-H(x)\|+\sup_{x\in S}\|x-H_N(x)\|\Big)\cdot\|G\|\cdot\|G^{-1}\|\cdot\sup_{x\in S}\|f_N(x)-f(x)\|.\end{aligned}$$

From Lemma 1, $\{f_N(x)\}$ uniformly converges to $f(x)$ with probability 1 on S. Then, for any $\varepsilon>0$, there exists some positive scalar N_0 such that, when $N>N_0$,

$$\sup_{x\in S}\|f_N(x)-f(x)\|\le\frac{\varepsilon}{\alpha^{-1}M\big(\frac{2}{\sqrt{\lambda_{\min}}}+\|G^{-1}\|+\frac{2}{\sqrt{\lambda_{\min}}}\|G\|\cdot\|G^{-1}\|\big)}$$

with probability 1. Thus, by (6), one has

$$\sup_{x\in S}|g(x)-g_N(x)|<\alpha^{-1}M\left(\frac{2}{\sqrt{\lambda_{\min}}}+\|G^{-1}\|+\frac{2}{\sqrt{\lambda_{\min}}}\|G\|\cdot\|G^{-1}\|\right)\cdot\sup_{x\in S}\|f_N(x)-f(x)\|\le\varepsilon.$$

This indicates that $\{g_N(x)\}$ uniformly converges to $g(x)$ with probability 1 on S. ∎

Definition 3[14] Assume that every function of the sequence $\{f_n\}_{n=1}^{\infty}$ is lower semicontinuous and the function f is lower semicontinuous. We say that $\{f_n\}$ epi-converges to f if for any x,

i) for every sequence $\{x_n\}$ converging to x, there holds $\liminf_{n\to\infty} f_n(x_n)\ge f(x)$;

ii) there exists a sequence $\{x_n\}$ converging to x such that $\limsup_{n\to\infty} f_n(x_n)\le f(x)$.

The following theorem shows the epi-convergence of $\{g_N\}$.

Theorem 3 *Assume that* A1)–A3) *hold. Then* $\{g_N\}$ *epi-converges to* g.

Proof By A2), both $f_N(x)$ and $f(x)$ are continuous. From law of large number, one has

$$\lim_{N\to\infty} f_N(x)=f(x)$$

with probability 1. Let the sequence $\{x_N\}\subset S$ converge to $x\in S$. We first show that

$$\lim_{N\to\infty} f_N(x_N)=f(x)$$

with probability 1. In fact, for any $\varepsilon>0$, one has from Lemma 1 that there exist $\delta>0$ and $\overline{N}>0$ such that when $N>\overline{N}$ and $\|x_N-x\|\le\delta$,

$$\|f_N(x_N)-f(x_N)\|<\varepsilon,\qquad \|f(x_N)-f(x)\|<\varepsilon.$$

It follows that

$$\|f_N(x_N)-f(x)\|\le\|[f_N(x_N)-f(x_N)]\|+\|[f(x_N)-f(x)]\|\le 2\varepsilon.$$

This means that the sequence $\{f_N\}$ epi-converges to f with probability 1. Furthermore, since both $H(\cdot)$ and $H_N(\cdot)$ are continuous, one has

$$\begin{aligned}\|H_N(x_N)-H(x)\|&=\|\mathrm{Proj}_{S,G}(x_N-\alpha^{-1}G^{-1}f_N(x_N))-\mathrm{Proj}_{S,G}(x-\alpha^{-1}G^{-1}f(x))\|_G\\&\le\|(x_N-x)-\alpha^{-1}G^{-1}(f_N(x_N))-f(x))\|_G\\&\le\|(x_N-x)\|+\alpha^{-1}\|G^{-1}\|\cdot\|(f_N(x_N))-f(x))\|,\end{aligned}$$

and hence $\lim_{N\to\infty} H_N(x_N)=H(x)$ with probability 1. Recall that

$$\begin{aligned}g(x)&=(x-H(x))^{\mathrm{T}}f(x)-\frac{\alpha}{2}\|x-H(x)\|_G^2,\\g_N(x)&=(x-H_N(x))^{\mathrm{T}}f_N(x)-\frac{\alpha}{2}\|x-H_N(x)\|_G^2.\end{aligned}$$

We then obtain

$$\lim_{N\to\infty} g_N(x_N)=g(x)$$

with probability 1, that is, the sequence $\{g_N\}$ almost surely epi-converges to g. ∎

The following theorem shows the convergence of optimal values of the approximation problem (3).

Theorem 4 *If* A1)–A3) *hold, then we have* $\lim\limits_{N\to\infty}\min\limits_{x\in S} g_N(x) = \min\limits_{x\in S} g(x)$ *with probability* 1.

Proof Note that, by A1) and A2), for every N, both $\min\limits_{x\in S} g_N(x)$ and $\min\limits_{x\in S} g(x)$ are finite. Let ε be an arbitrary positive number. Then there exists a vector $x_\varepsilon \in S$ such that $g(x_\varepsilon) \le \min\limits_{x\in S} g(x)+\varepsilon$. By Theorem 3, the sequence $\{g_N\}$ epi-converges to g with probability 1, which means that there almost surely exists a sequence $\{x_N\}$ converging to x_ε such that $\limsup\limits_{N\to\infty} g_N(x_N) \le g(x_\varepsilon)$. Therefore, we have

$$\limsup_{N\to\infty}\min_{x\in S} g_N(x) \le \limsup_{N\to\infty} g_N(x_N) \le g(x_\varepsilon) \le \min_{x\in S} g(x) + \varepsilon$$

with probability 1. By the arbitrariness of ε, we have that, with probability 1, there holds

$$\limsup_{N\to\infty}\min_{x\in S} g_N(x) \le \min_{x\in S} g(x). \tag{7}$$

On the other hand, for any $\varepsilon > 0$, there exists a sequence $\{x_N\} \subset S$ such that

$$0 \le g_N(x_N) \le \min_{x\in S} g_N(x) + \varepsilon.$$

Then there exists a subsequence $\{x_{N_k}\}$ converging to some vector $x_\varepsilon \in S$ and satisfying

$$\lim_{k\to\infty} g_{N_k}(x_{N_k}) = \liminf_{N\to\infty} g_N(x_N).$$

Since, by Theorem 3, $\{g_N\}$ epi-converges to g with probability 1, one has that

$$\liminf_{N\to\infty}\min_{x\in S} g_N(x) \ge \liminf_{k\to\infty} g_{N_k}(x_{N_k}) - \varepsilon \ge g(x_\varepsilon) - \varepsilon \ge \inf_{x\in S} g(x) - \varepsilon$$

with probability 1. It follows from the arbitrariness of ε that

$$\liminf_{N\to\infty}\min_{x\in S} g_N(x) \ge \min_{x\in S} g(x) \tag{8}$$

with probability 1.

The conclusion follows from (7) with (8) immediately. ∎

Definition 4[14] Let $\{C_n\}$ be a sequence of closed sets in $\Re^n$. The outer limit of $\{C_n\}$ are defined as follows:

$$\text{ls } C_n = \left\{x \,\middle|\, \exists\{x_{n_k}\} \text{ s.t. } x_{n_k} \in C_{n_k}, x = \lim_{k\to\infty} x_{n_k}\right\}.$$

Denote by $\arg\inf g$ and $\arg\inf g_N$ the optimal solution sets of problems (3) and (5), respectively.

Theorem 5 *Suppose that* A1)–A3) *hold. Then we almost surely have* $\text{ls}\{\arg\inf g_N\} \subseteq \arg\inf g$.

Proof Let $x_{N_k} \in \arg\inf g_{N_k}$ and $\lim\limits_{k\to\infty} x_{N_k} = x \in S$. It is sufficient to prove $x \in \arg\inf g$ almost surely.

By Theorem 3, the sequence $\{g_N\}$ epi-converges to g with probability 1. As a result, we have

$$\liminf_{k\to\infty} g_{N_k}(x_{N_k}) \ge g(x).$$

In a similar way as in the last theorem, we can show

$$\limsup_{N\to\infty} \inf_{u\in S} \{g_N(u)\} \leq \inf_{u\in S} g(x)$$

with probability 1. It follows from the above two inequalities that

$$g(x) \leq \inf_{u\in S} g(u)$$

with probability 1. Since $x \in S$, we have $x \in \arg\inf g$ almost surely. ∎

Theorem 6 *Let* A1)–A3) *hold and $F(x,\omega)$ be uniformly monotone with respect to x for almost every $\omega \in \Omega$, that is, there exists a nonnegative integrable function $C(\omega)$ such that*

$$(x-y)^{\mathrm{T}}(F(x,\omega) - F(y,\omega)) \geq C(\omega)\|x-y\|^2, \qquad \forall x, y \in \Re^n.$$

Suppose that $\mathrm{E}[C(\omega)] > 0$. *Let x_N be an optimal solution of problem* (5) *for each N. Then, the sequence $\{x_N\}$ converges to the unique solution of the original problem* (1) *with probability* 1.

Proof First of all, it is easy to see from the assumptions that $\mathrm{E}[F(\cdot,\omega)]$ is strongly monotone with modulus $\mathrm{E}[C(\omega)] > 0$. This indicates that the original problem (1) has a unique solution[1]. We denote by x^* the solution. It is obvious that x^* is also a unique solution of problem (3), that is, $\arg\inf g = \{x^*\}$. From Theorem 5, the bounded sequence $\{x_N\}$ has a unique accumulation point x^* and so the conclusion is valid. ∎

4 Sample Average Approximation Scheme

In this section, we detail a sample average approximation method scheme for solving stochastic variational inequality.

In the SAA scheme, a random sample $\omega^1, \omega^2, \cdots, \omega^N$ of N realization (scenarios) of the random vector ω is generated, and the expectation $\mathrm{E}[F(x,\omega)]$ is approximated by the sample average function $N^{-1}\sum_{n=1}^{N} F(x,\omega^n)$. Consequently, the original problem is approximated by the problem

$$\min_{x\in S} g_N(x), \tag{9}$$

where

$$g_N(x) = \max_{y\in S}\left\{(x-y)^{\mathrm{T}}\left[\frac{1}{N}\sum_{n=1}^{N} F(x,\omega^n)\right] - \frac{\alpha}{2}\|x-y\|_G^2\right\}. \tag{10}$$

In (10), we know that $g_N(x)$ is the maximum value of the optimization problem, and can be solved or approximated by deterministic optimization method.

Let v_N and $\widehat{x}$ be the optimal value and an optimal solution vector, respectively, of the SAA problems (9). Note that v_N and $\widehat{x}$ are random in the sense that they are functions of the corresponding random sample. However, for a particular realization $\omega^1, \omega^2, \cdots, \omega^N$ of the random sample, the problem (9) is deterministic and can be solved by appropriate optimization techniques. It is possible to show that under mild regularity conditions, as the sample size N increases, v_N and $\widehat{x}$ converge with probability on their true value counterparts.

In the following, we present a sample average approximation scheme for solving the stochastic variational inequality problem

Step 1 Generate M independent samples each of size N, i.e., $(\omega^1, \omega^2, \cdots, \omega^N)$ for $j = 1, 2, \cdots, M$. For each sample solve the corresponding SAA problem

$$\min_{x \in S} g_N^j(x),$$

where $g_N^j(x) = \max\limits_{y \in S}\{(x-y)^{\mathrm{T}}[\frac{1}{T}\sum_{n=1}^{N} F(x, \omega_j^n)] - \frac{\alpha}{2}\|x-y\|_G^2\}$.

Step 2 Compute $\overline{v}_{N,M} := \frac{1}{M}\sum_{j=1}^{M} v_N^j$ and $\sigma^2_{v_{N,M}} = \frac{1}{(M-1)M}\sum_{j=1}^{M}(v_N^j - \overline{v}_{N,M})^2$.

Step 3 Choose a feasible solution $\widetilde{x}$ of the true problem. Estimate the true objective function value $g(\widetilde{x})$ as

$$\widetilde{g}_{N'}(\widetilde{x}) = \max_{y \in S}\left\{(x-y)^{\mathrm{T}}\left[\frac{1}{N'}\sum_{n=1}^{N'} F(x, \omega^n)\right] - \frac{\alpha}{2}\|x-y\|_G^2\right\},$$

where $\{\omega^1, \omega^2, \cdots, \omega^{N'}\}$ is a sample of size N' generated independently of the sample used to obtain $\widetilde{x}$, and N' is taken much bigger than the sample size N used in solving the SAA problems.

Step 4 Compute $g_{N'}(\widetilde{x})$ and $\sigma^2_{N'}(\widetilde{x}) := \frac{1}{(N'-1)N'}\sum_{n=1}^{N'}(h(\widetilde{x}, \omega^n) - \widetilde{g}_{N'}(\widetilde{x}))$, where $h(\widetilde{x}, \omega^n) = \max\limits_{y \in S}\{(x-y)^{\mathrm{T}} F(x, \omega^n) - \frac{\alpha}{2}\|x-y\|_G^2\}$.

5 Numerical Results

In this section, we describe numerical experiments using SAA method.

In our experiments, we assume that the matrix G is the identity matrix and $\alpha = 1$. All tests are implemented in the same PC with system memory 2.0G and CPU 2.0G. In each experiment, we implement the test 20 or 50 times independently, respectively. The sample size N is taken as 100 and 400, respectively.

The numerical results shown in Tables 1–4 reveal that our proposed method was able to successfully solve the problems considered.

Example 1 Consider the stochastic variational inequality problem (1) in which ω is uniformly distributed on $\Omega = [0, 1]$, $S = \Re_+ \times \Re_+ \times \Re_+$ and $F : \Re^3 \times \Omega \to \Re^3$ is given by

$$F(x, \omega) = \begin{pmatrix} x_1 - \omega x_2 + 3 - 2\omega \\ -\omega x_1 + 2x_2 + \omega x_3 - 2 - \omega \\ \omega x_2 + 3x_3 - 3 - \omega \end{pmatrix}.$$

This problem has a unique solution $x^* = (0, 1, 1)^{\mathrm{T}}$ for each $\omega \in \Omega$. Numerical results are shown in Table 1.

Table 1 The computational results for Example 1

N	M	N'	$\overline{z}_{N,M}$	$\widetilde{x}$	$g_{N'}(\widetilde{x})$	$\sigma^2_{\overline{z}_{N,M}}$
100	20	600	0.005	(0.00159289, 1.02312, 0.985538)	3.60934e-003	1.1491e-005
400	20	600	0.0045	(0.000732467, 1.03854, 0.959161)	2.16543e-003	8.8789e-006
100	50	600	0.0035	(0.00140442, 0.994995, 1.00216)	2.35591e-003	9.8808e-006
400	50	600	0.004	(3.05325e-005, 1.00145, 0.999906)	6.38546e-005	1.0720e-005

Example 2 Consider the stochastic variational inequality problem (1), in which ω is uniformly distributed on $\Omega = [0, 1]$, $S = \Re_+ \times \Re_+ \times \Re_+$ and $F : \Re^3 \times \Omega \to \Re^3$ is given by

$$F(x, \omega) = \begin{pmatrix} x_1^2 - \omega x_2 + 3 - 2\omega \\ -\omega x_1 + 2x_2^2 + \omega x_3 - 2 - \omega \\ \omega x_2 + 3x_3^2 - 3 - \omega \end{pmatrix}.$$

It is easy to prove that the function $\mathrm{E}[F(x,\omega)]$ is strongly monotone. So this stochastic variational inequality problem has a unique solution $x^* = (0,1,1)^{\mathrm{T}}$. The numerical results are shown in Table 2.

Table 2 The computational results for Example 2

N	M	N'	$\overline{z}_{N,M}$	$\widetilde{x}$	$g_{N'}(\widetilde{x})$	$\sigma^2_{\overline{z}_{N,M}}$
100	20	600	304523e-004	(3.82171e-005, 0.997423, 1.00112)	5.1842e-005	3.9642e-008
400	20	600	3.1984e-004	(1.67235e-006, 1.00221, 0.996471)	3.4463e-005	4.4121e-008
100	50	600	2092122e-004	(1.91344e-005, 1.00023, 1.00057)	3.7229e-005	3.9723e-008
400	50	600	2.0089e-004	(1.7312e-006, 1.00134, 0.994871)	1.4245e-005	2.8649e-008

Example 3 Consider the stochastic variational inequality problem (1), in which ω is uniformly distributed on $\Omega = [0,1]$, $S = [0,4] \times [0,4] \times [0,4]$ and $F : \Re^3 \times \Omega \to \Re^3$ is given by

$$F(x,\omega) = \begin{pmatrix} x_1 - \omega x_2 + 3 - 2\omega \\ -\omega x_1 + 2x_2 + \omega x_3 - 2 - \omega \\ \omega x_2 + 3x_3 - 3 - \omega \end{pmatrix}.$$

This problem has a solution $x^* = (0,1,1)^{\mathrm{T}}$ for each $\omega \in \Omega$. The numerical results are shown in Table 3.

Table 3 The computational results for Example 3

N	M	N'	$\overline{z}_{N,M}$	$\widetilde{x}$	$g_{N'}(\widetilde{x})$	$\sigma^2_{\overline{z}_{N,M}}$
100	20	600	0.0046	(0.0015765, 1.02134, 0.99256)	3.70121e-003	1.1592e-005
400	20	600	0.0031	(0.00085621, 1.02978, 0.97387)	2.53124e-003	8.2567e-006
100	50	600	0.0036	(0.0020789, 0.993575, 1.00145)	2.37812e-003	9.7834e-006
400	50	600	0.003011	(3.1624e-005, 1.00267, 0.998991)	6.45671e-005	1.0023e-005

Example 4 Consider the stochastic variational inequality problem (1), in which ω is uniformly distributed on $\Omega = [0,1]$, $S = [0,4] \times [0,4] \times [0,4]$ and $F : \Re^3 \times \Omega \to \Re^3$ is given by

$$F(x,\omega) = \begin{pmatrix} x_1^2 - \omega x_2 + 3 - 2\omega \\ -\omega x_1 + 2x_2 + \omega x_3 - 2 - \omega \\ \omega x_2 + 3x_3^2 - 3 - \omega \end{pmatrix}.$$

It is easy to prove that the function $\mathrm{E}[F(x,\omega)]$ is strongly monotone. So this stochastic variational inequality problem has a unique solution $x^* = (0,1,1)^{\mathrm{T}}$. The numerical results are shown in Table 4.

Table 4 The computational results for Example 4

N	M	N'	$\overline{z}_{N,M}$	$\widetilde{x}$	$g_{N'}(\widetilde{x})$	$\sigma^2_{\overline{z}_{N,M}}$
100	20	600	3.0841e-004	(3.80427e-005, 0.999781, 1.0003)	6.07436e-005	3.9706e-008
400	20	600	2.4201e-004	(1.56462e-006, 1.00123, 0.998771)	3.475e-005	4.3964e-008
100	50	600	2.9498e-004	(1.80155e-005, 1.00052, 1.00057)	3.7444e-005	3.9326e-008
400	50	600	2.0797e-004	(0.00000, 1.00111, 1.0004)	1.50538e-005	2.8568e-008

From the above analysis for Examples 1–4, our preliminary numerical results for these examples indicate that the proposed SAA method yield a reasonable and better solution of the stochastic variational inequality problem (1).

6 Conclusions

We studied the stochastic variational inequality problem (1). By using the regularized gap function, we formulated the problem as the constrained optimization problem (3). Then, we proposed an SAA method for solving (3). We also investigated the limiting behavior of the optimal values and the optimal solutions of the approximation problems. On the other hand, one may use the so-called D-gap function[1] to replace the regularized gap function to get some unconstrained optimization problems as approximations of (1).

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