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ANALYSES OF LOCATION-PRICE GAME ON NETWORKS WITH STOCHASTIC CUSTOMER BEHAVIOR AND ITS HEURISTIC ALGORITHM*

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Abstract In this paper, a two-stage model is developed to investigate the location strategy and the commodity pricing strategy for a retail firm that wants to enter a spatial market with multiple competitive facilities, where a competitor firm is already operating as a monopoly with several outlets. Expected market shares are calculated based on the stochastic customer behavior on networks. The authors provide a sufficient condition for the existence of equilibrium prices in the price game for the first time. The existence and uniqueness of the pure strategy Nash equilibrium price with a specified utility function are proved in the subgame. A metaheuristic based on tabu search is proposed to search the optimal location-price solution of the model. In addition, the authors provide two numerical examples to illustrate how to obtain the optimal solution and conduct sensitivity analysis. The analysis shows that the best location decision is robust for the follower firm, price game is more intense when incomes of consumers are lower or there are more substitution products, and neither chain retail gains from the price competition.

Key words Competitive location, heuristic, Nash equilibrium, networks, pricing, tabu search, two-stage model.

1 Introduction

Recently competitive location problem has become one of the most critical fields of operations research, regional science, and spatial economics. After Hotelling's classical model[1] on two firms competing in a linear market with consumers distributed uniformly along a line, different models with modifications on the original assumptions had got lots of interesting results[2−5]. Economide[5] put the case that product is of characteristics of position and price. It is found that a non-cooperative equilibrium in prices exists for all symmetrical locations of firms. Over the last few decades, many researchers aimed at developing insights concerning the

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equilibrium pattern of competitive location and pricing. Although these models (linear models) are rich in their theoretical insights about spatial competition and have greatly enhanced our understanding of locational interdependence, they provide very little guidance for developing practical approaches to facility location in competitive environments[6].

Mallozzi[7] investigated non-cooperative strategies for several facilities case. Santos-Peñãte[8] summarized some results for the leader-follower location model on networks in several scenarios. Discretization results are considered. Differences derived from the inelastic and elastic demand assumptions, as well as from the customer's choice rule, are also emphasized. Miller[9] formulated a dynamic facility location model for a firm locating on a discrete network. By using sensitivity analysis of variational inequalities within a hierarchical mathematical programming approach, they developed a reaction function to optimize the Stackelberg firm's location decision. In the above literature, the location space was a plane or a network, but most of them dealt exclusively with no price competition situations, and little attention was paid to the characterization of market equilibria.

In the literature of spatial economics and industrial organization, competitive location of chain retail facilities may incorporate not only location decision but also pricing decision as decision variables in the models, see, for instance, [10]. It is far-sighted to take long-term benefit into consideration when making location decision. Price competition influences firm profits directly after location decision for retail stores. The possibility of price undercutting results in a large degree of instability, as witnessed by many contributions in the economics literature, for a survey one can refer to [11–13].

When customers face a patronage choice among several facilities, it is commonly assumed that customers select the closest facility[1]. The distribution rule can be argued for it does not reflect reality unless there is a central control that forces the selection of the closest facility. A typical rational consumer would set up a utility function including all the information about himself, facility attributes and the spatial separation. Serra and ReVelle[10] assumed that customers patronize the cheapest facility. In their model, the cost includes commodity price and transport cost. Eiselt and Laporte[14] showed that if all products and facilities are truly homogeneous, a rational customer will satisfy his whole demand at the facility with the highest utility, resulting in "winner-takes-all" behavior. However, customers may not have full information about distance to all facilities and commodity prices or may not behave in a rational manner. Thus, we can safely assume that not all customers at any location patronize the same facility. When there is no central control, a probabilistic rule is the appropriate one[15]. Drezner[16] proposed a utility function that depends on the distance between the customer and the facility, as well as on other characteristics of the facility, such as prices, variety of products, environmental conditions, accessibility, parking area, and so on, which determine its quality[17]. For stochastic customer behavior, the location problem has been studied in [18]. The location problem is solved for a wide range of quality values, but no optimal solution to the location and quality problem is given. To the best of our knowledge, little research has been done on this kind of problem with simultaneous decisions on location and quality with stochastic customer choice in network space.

In this paper, we propose a two-stage model to investigate location and commodity pricing strategies of a chain retail firm to maximize the firm's profit with stochastic consumer behavior under a competitive environment. We provide a sufficient condition for the existence of equilibrium prices in the subgame. A piecewise utility function is introduced to describe the costumer preference. Factors of distance and price are hierarchical. The threshold of the costumer endurance towards distance is given. In this situation, we analytically prove the existence and uniqueness of the pure strategy NE of prices. A metaheuristic algorithm based on the tabu search is proposed to find the optimal solution. Extensive numerical analyses are conducted and managerial insights are proposed.

The rest of the paper is organized as follows. In Section 2, we formulate a two-stage model for this problem. Some definitions and properties of solutions in the model are given in Section 3. A heuristic algorithm for our model is developed in Section 4. Section 5 provides two illustrative examples of the method and further sensitivity analysis are presented. Conclusions are given in Section 6.

2 A Two-Stage Model for Location-Price Game

In an investment period, we consider the facility location decision and commodity pricing decision of a chain retail firm, known as Firm A. Commodity here is assumed as single for simplification of the problems. Variety of commodities is not the emphasis we want to discuss. Suppose that the other competitive chain retail, known as Firm B, has already existed in this market to sell the commodity as a monopoly. However, now the market is dominated by A and B, i.e., this is a duopoly market. Demand for this commodity is inelastic, such as in the case of essential goods may be reasonable. Thus, the whole demands here may be equal to the entire market shares. Demand points, as well as potential facility points, are restricted to the vertices of network. The facilities (the competitive facilities and the facilities of a single firm) are not permitted to assemble; otherwise assembling effect will come into being (see [19]). In addition, it is natural to assume that the commodity price is set uniformly, i.e., facilities belonged to the same firm have the same price. Customers undertake all the transport cost that is assumed to be positively correlated with the distance between the customers and the facility. It is obvious that the less gap between prices of two firms, the more influence of the distance on the customers. Besides, if the distance is greater than the threshold customers could endure, they will not choose the facilities. If the distance is less than the threshold, demands of the customers will be distributed in proportional to the facility utility. The sequence of the decision-making is as follows:

1) Firm A selects his facility locations;

2) Firms A and B determine their respective price simultaneously.

After Firm A selecting his facility locations, there is a price game between Firms A and B which is a static sub-game with perfect information. The price game between them can be forecasted by Firm A when he selects his facility locations.

The following notations are used in the model:

i, I: Index and set of indexes of demand points and potential facility points;

j, k: Indexes of facility points;

J^A: Index set of facility locations of Firm A;

J^B: Index set of facility locations of Firm B;

n^A: Number of chain retail facilities of Firm A;

d_{ij}: Distance between the demand point i and the facility point j;

a_i: Demand of customers located at point i;

g_j: Fixed set-up cost at point j;

m: Mill price of the commodity;

u_{ij}: Utility perceived by customers located at point i towards facility at point j, where $i \in I$, $j \in J^A \cup J^B$;

w_{ij}: Standardized utility of facility point j of Firm A to demand point i, i.e., the portion of the buying power generated at demand point i that patronizes facility point j of Firm A;

h_i: The total utility perceived by customers located at point i towards Firm A, i.e., the portion of the buying power generated at demand point i that patronizes Firm A;

Location variables:
$x_j = \begin{cases} 1 & \text{if Firm A locates at candidate } j, \\ 0 & \text{otherwise;} \end{cases}$
Price variables:
$p_j = \begin{cases} p^A & \text{if } j \in J^A \\ p^B & \text{if } j \in J^B \end{cases}$ Commodity price at facility point j, where p^A and p^B represent the uniform price of Firm A and B, respectively.

In fact, this is a two-stage game in which the location decision-making is followed by the commodity pricing. Thus, a two-stage model is developed, in which first-stage (I) focuses on the location decision to maximize Firm A's profit, while second-stage (II$'$ and II$''$) is used to investigate the Nash equilibrium prices of Firms A and B. The model for first-stage is formulated as follows:

$$\text{(I)} \qquad \max_{x_j} \ f_1^A = (p^A - m)\sum_{i\in I} a_i h_i - \sum_{j\in J^A} g_j x_j, \tag{1}$$

$$\text{s.t.} \ \sum_{j\in J^A} x_j = n^A \tag{2}$$

$$x_j = 0, 1 \tag{3}$$

where $h_i = \sum\limits_{j\in J^A} w_{ij}$, $w_{ij} = \dfrac{u_{ij}x_j}{\sum\limits_{k\in J^B} u_{ik} + \sum\limits_{j\in J^A} u_{ij}x_j}$ and (p^A, p^B) is the NE prices for Firms A and B of model (II$'$ and II$''$).

The model for second-stage is formulated as follows:

$$\text{(II}'\text{)} \qquad \max_{p^A|x_j} \ f_2^A = (p^A - m)\sum_{i\in I} a_i h_i - \sum_{j\in J^A} g_j x_j \tag{4}$$

$$\text{s.t.} \ \ p^A \geq m \tag{5}$$

$$\text{(II}''\text{)} \qquad \max_{p^B|x_j} \ f_2^B = (p^B - m)\sum_{i\in I} a_i (1 - h_i) \tag{6}$$

$$\text{s.t.} \ \ p^B \geq m \tag{7}$$

The objective of problem I is to maximize the total profit that Firm A can achieve with its n^A facilities for any given p^A and p^B. The first term of the objective function is the sale profit and the second is the fixed set-up cost. The objective functions of II$'$ and II$''$ are profits of Firms A and B after Firm A has selected his facility locations. Formulas (5) and (7) are the pricing constraints for Firms A and B.

3 Definitions and Properties of Solutions in the Model

To analyze further, we present some definitions and properties of solutions that will be used later in this paper.

Definition 1 The model for second-stage is a strategy game with perfect information, denoted by $G{=}\langle A, B, p^A, p^B, f_2^A, f_2^B\rangle$. If p^{A*} and p^{B*} are the best response prices of Firms A and B when p^{B*} or p^{A*} is given, i.e., $p^{A*}(p^{B*}) = \arg\max\limits_{p^A} \ f^A(p^A, p^{B*})$ and $p^{B*}(p^{A*}) =$

arg $\max\limits_{p^B} f^B(p^{A*}, p^B)$, then (p^{A*}, p^{B*}) is called the pure Nash equilibrium (NE) price of the subgame $G=\langle A, B, p^A, p^B, f_2^A, f_2^B\rangle$.

Definition 2 If $L^{A*} = (x_j^*)$, $j = 1, 2, \cdots, n^A$, is the optimal solution of the model (I) and (p^{A*}, p^{B*}) is the pure strategy NE of prices in the model (II$'$ and II$''$), then (L^{A*}, p^{A*}, p^{B*}) is called the optimal location- price solution of the two-stage model.

Assumption 1 u_{ij} is nonnegative.

The model is based on customer behavior. Thus, if customers patronize the facility, then the utility must be positive, otherwise, attraction of the facility towards customers should be zero.

Assumption 2 u_{ij} is nonincreasing with p_j.

Generally speaking, lower prices will attract more customers if other qualities, such as brand, environmental conditions, are similar. According to Assumptions 1 and 2, we know that the proportion of the demand generated at point i that patronizes Firm A is decreasing with the commodity price of Firm A, see the proof in Appendix A.

Lemma 1[20] *Suppose that for each player the strategy space is compact and convex and the profit function is continuous and quasi-concave with respect to each player's own strategy. Then there exists one pure strategy NE in the game.*

Proposition 1 *If* $\frac{\partial^2 h_i}{\partial {p^A}^2} \geq 0$*, the profit functions* f_2^A *and* f_2^B *are concave.*

Theorem 1 *If* $\frac{\partial^2 h_i}{\partial {p^A}^2} \geq 0$*, there exists one pure strategy NE price* (p^{A*}, p^{B*}) *in the subgame* $G=\langle A, B, p^A, p^B, f_2^A, f_2^B\rangle$.

Because $\frac{\partial h_i}{\partial p^A} \leq 0$ (see Appendix A), Theorem 1 tells us that much more customers would be attracted to buy lower-price commodity when the unit price is decreased in the price game of low-price commodity. While for high-price commodity, customers care brand and quality more than price, thus, fewer customers would change their mind when the same price is cut down. This fact could be interpreted as a manifestation of the law of diminishing marginal utility. The utility means the portion increased due to price reduction. Obviously, this is a reasonable phenomenon in the real world, and so is the existence of the pure strategy NE prices of Firms A and B. The sufficient conditions here for the existence of equilibrium prices is satisfied naturally in reality, thus, the equilibrium of prices is certainly existent in the price game.

Particularly, we specify a utility function. Three kinds of utility functions are introduced in location literature: multiplicative, additive, and exponential[8]. Hodgson[21] proposed an exponential utility function which was applied to locate varying numbers of motor vehicle licensing outlets of varying sizes in Edmonton, Canada. Based on his work and the psychological characteristics of customers, we propose a piecewise utility function

$$u_{ij} = \begin{cases} \mathrm{e}^{a-bp_j-cd_{ij}}, & d_{ij} \leq D, \\ 0, & d_{ij} > D, \end{cases}$$

where the positive b and c represent the customer sensitivity towards commodity price and distance, respectively; if the incomes of consumers are lower, corresponding price sensitivity parameter will be larger. D is a positive constant, representing the common distance threshold of all the costumers can endure.

Corollary 1 *There exists one pure strategy NE prices* (p^{A*}, p^{B*}) *in the subgame* $G=\langle A, B, p^A, p^B, f_2^A, f_2^B\rangle$ *with the piecewise exponential function* u_{ij}.

Lemma 2[20] *The pure strategy NE is unique in the subgame if* $\sum_{i=1, i\neq k}^{n} \left|\frac{\partial^2 f_k}{\partial x_k \partial x_i}\right| < \left|\frac{\partial^2 f_k}{\partial x_k^2}\right|, \forall k$.

Proposition 2 *In the model* (*II*′ *and* *II*″), *we have* $\left|\frac{\partial^2 f_2^A}{\partial p^A \partial p^B}\right| < \left|\frac{\partial^2 f_2^A}{\partial {p^A}^2}\right|$, $\left|\frac{\partial^2 f_2^B}{\partial p^B \partial p^A}\right| < \left|\frac{\partial^2 f_2^B}{\partial {p^B}^2}\right|$ *with the piecewise exponential function* u_{ij}.

Proposition 3 *In the model* (*II*′ *and* *II*″), *we have* $0 < \frac{dp^A}{dp^B} < 1$ *and* $0 < \frac{dp^B}{dp^A} < 1$ *respectively with the piecewise exponential function* u_{ij}.

Proposition 3 shows that the pricing strategies of Firm A and Firm B are positively related and the range of price variation of each firm is less than that of the competitor with a piecewise exponential utility function, that is, if Firm B increases commodity price, then, Firm A will also increase commodity price, but the best response of Firm A is less makeup. For the case of price reduction, the analysis is similar. In this situation, the pure strategy NE prices are unique.

Theorem 2 *The pure strategy NE price* (p^{A*}, p^{B*}) *is existent and unique in the game* $G{=}\langle A, B, p^A, p^B, f_2^A, f_2^B\rangle$ *with the exponential function* u_{ij}.

Using Corollary 1, Lemma 2 and Proposition 2, the conclusion can be deduced easily.

4 Location-Price Game Heuristic Algorithm (LPGH)

The location problems are combinational in nature. Therefore, exact algorithms exist only for small problems and heuristic procedures have to be used for large practical problems[22–23]. In this paper, a heuristic procedure, including three stages of greedy adding, vertex substitution, and tabu search (TS), is developed to solve the competitive location-price problem.

The TS metaheuristic[24] is used in the procedure to guide the search process when one solution moving to a better one. The procedure is efficient because the search space is only a small subset of all feasible solutions that contains the optimal solution. Local optimum can be avoided by accepting inferior solution when the better solution is tabu and freeing tabu solution when the adapter value is the best so far.

Next, we start with the key parameters designed for TS procedure and then describe the components of the LPGH.

4.1 Key Parameters in TS Procedure

We use 0-1 coding since the location decision variable is 0-1. The objective of the first-stage model is to maximize Firm A's profit, so the Firm A's net profit function can be used as the adapter function naturally. Because the TS algorithm is based on the neighborhood search, the initial solution is critical to the searching performance, such as the total time, the near-globally optimal solution, and so on. Greedy adding algorithm and vertex substitution algorithm are proposed to obtain the initial locations. Through swapping the checked with the unchecked, we have the neighborhood structure. Tabu length $\sqrt{n^A}$ remains fixed. For it is less than the number of candidates n^A the case is impossible to appear that all candidates are free but none of them is better than the current optimal solution. Swapping operation is the tabu object, i.e., if and are swapped in this step, then they are not permitted to exchange until step $\sqrt{n^A}$.

4.2 Procedures of Heuristic Algorithm

Phase 1 Greedy Adding

Step 1 Set $L_0^A = \phi$, $p_0^A = p_0^B$, $k = 1$.

Step 2 Set $L_k^A = L_{k-1}^A \cup v_k$, $v_k \in \left\{I - J^B - L_{k-1}^A\right\}$. Compute the NE prices $\left(p_k^{A*}, p_k^{B*}\right)$in the model (II' and II"). Select v_k s.t. max $\left[\Delta \Pi_k^A = f_1^A\left(L_{k-1}^A \cup v_k, p_k^{A*}, p_k^{B*}\right) - f_1^A\left(L_{k-1}^A, p_k^{A*}, p_k^{B*}\right)\right]$.

Step 3 Set $k = k + 1$. Go to Step 2 if $k < n^A$. Go to Phase 2 until $k = n^A$.

Phase 2 Vertex Substitution

Step 1 Set $L^{A*} = L^A_{n^A}$, $\varPi^A_\wedge = f_1^A\left(L^A_{n^A}, p^{A*}_{n^A}\right)$.

Step 2 Set $L^A_0 = L^A_{n^A}$, $p^A_0 = p^B_0$, $\varPi^A_0 = \varPi^A_\wedge$, $t = 1$.

Step 3 Set $L^A_t = L^A_{t-1} - v_k + v_l$, where $v_k \in L^A_{t-1}$, $v_l \in \left\{I - J^B - L^A_{t-1}\right\}$. Compute $\left(p^{A*}_t, p^{B*}_t\right)$.

Step 4 If $\varPi^A_t\left(L^A_t, p^{A*}_t\right) > \varPi^A_\wedge$, set $L^{A*} = L^A_t$, $p^{A*} = p^{A*}_t$, and $\varPi^A_\wedge = \varPi^A_t$. Go to Step 3 until v_k has exchanged with all points in $\left\{I - J^B - L^A_{t-1}\right\}$. Set $t = t+1$. Go to Step 3. Go to Phase 3 until $t = n^A$.

Phase 3 Tabu Search

Step 1 Reset $t = 0$.

Step 2 Set $L^A_t = L^{A*}$, $p^A_0 = p^B_0$, $\varPi^A_0 = \varPi^A_\wedge$. Initialize the tabu list empty.

Step 3 Consider the solution set $L^{A,i}_t$ produced by neighborhood of L^A_t. If there is no facility at $v_i{}''$ we swap $v_i{}'' \in \left\{I - J^B - L^A_t\right\}$ with $v_i{}' \in L^A_t$. Array $L^{A,i}_t$ decreasingly and choose the top n^A as the candidates.

Step 4 Set $i = 1$. If $\varPi^A_t\left(L^{A,i}_t, p^{A,i}_t\right) > \varPi^A_\wedge$ or $userswap(v_i{}', v_i{}'')$ are not tabuwe set $L^A_{t+1} = L^{A,i}_{\mathrm{t}}$ and update the current optimal adapter value to $\varPi^A_\wedge = \varPi^{A,i}_t(L^{A,i}_t, p^{A,i*}_t)$. Add $userswap(v_i{}', v_i{}'')$ into tabu list and fix tabu length as $\sqrt{n^A}$. Go to Step 5. Otherwise, set $i = i+1$. Go to Step 3.

Step 5 Set $t = t+1$. T is the upper limit of iterations. If $t < T$, go to Step 3. Otherwise, Stop.

5 Numerical Examples

5.1 Example 1

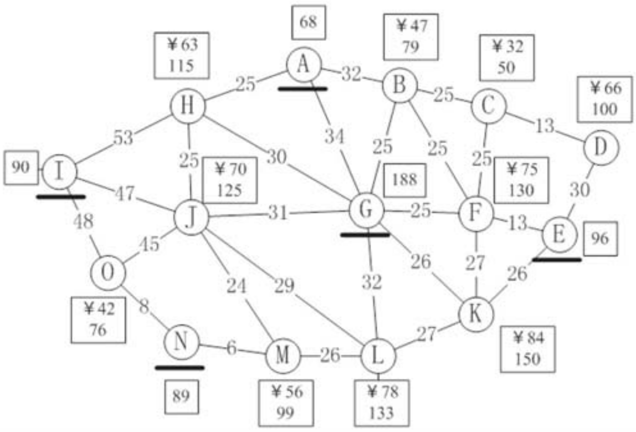

Figure 1 A network for numerical example 1

In Figure 1, the numbers by line are distances between two points. Fixed set-up cost is represented by "¥" and the other is demand. The existing Firm B owns facilities at points A, E, G, I, and N. Now Firm A intends to find five suitable locations to maximize profit, i.e., $n^A = 5$. Mill price of the commodity is 40¥. Results are showed in Table 1.

The curves of optimal response prices of Firms A and B are showed in Figure 2. Point of intersection is the pure strategy NE prices. The reaction curve graphs provide strong evidence that the equilibrium we compute is unique up to firm labels.

Table 1 Solution via LPGH algorithm

Initial conditions	Firm A's optimal locations and NE prices, profits for two firms(¥)				
$p_0^B = 50,\ D = 70,\ b = 1$					
$J^B = [A, E, G, I, N]$	L^{A*}	p^{A*}	f_2^{A*}	p^{B*}	f_2^{B*}
$u_{ij} = \begin{cases} e^{100-p_j-0.1d_{ij}} & d_{ij} \leq 70 \\ 0 & d_{ij} > 70 \end{cases}$	D, H, J, K, L	44.5229	3152.9	44.7221	3830.1

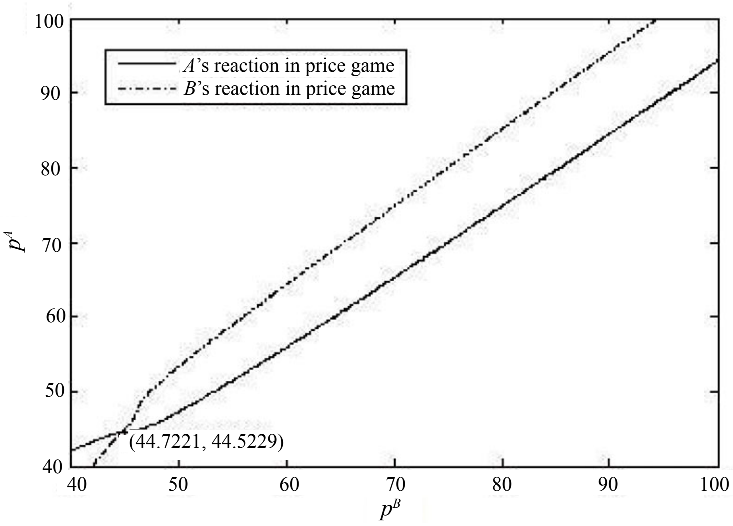

Figure 2 Curves of optimal response prices of Firm A and B

Next, we investigate the influence on location-pricing results due to the changes of initial parameters, such as Firm B's monopolistic price, locations and sensitivity parameter b to price in the utility function. The sensitivity analysis is conducted for managerial insights by numerical experiment because of the analytical complexity of expression.

1) If monopolistic price p_0^B of Firm B is changed, observe the corresponding change of the optimal location-pricing strategy of Firm A. Set the profit function value of Firm A equal to zero, i.e., $f_1^A = (p^A - m) \sum\limits_{i \in I} a_i h_i - \sum\limits_{j \in J^A} g_j x_j = 0$, we get $p_0^B = 40.3807$. At this situation, there is no incentive for Firm A to enter this market, that is to say, the pricing of Firm B sets price barrier for Firm A. When $p_0^B > 40.3807$, the results are showed in Table 2.

Table 2 The influence of p_0^B

p_0^B	L^{A*}	p^{A*}	f_2^{A*}	p^{B*}	f_2^{B*}
45	D, H, J, K, L	44.5229	3152.9	44.7221	3830.1
55	D, H, J, K, L	44.5229	3152.9	44.7221	3830.1
60	D, H, J, K, L	44.5229	3152.9	44.7221	3830.1
70	D, H, J, K, L	44.5229	3152.9	44.7221	3830.1

The table shows that the change of monopolistic pricing of Firm B acts no influence on the optimal location-pricing strategy of Firm A. Firm B cannot master the market price unless in the absolute monopoly market. In the competitive market, customers could enjoy more substantial benefit. What's more, the location strategy of Firm A is robust when the market is stable. Locations of Firm B are immovable and the information of utility function is fixed.

2) If the locations of Firm B are changed, observe the corresponding change of the optimal location-pricing strategy of Firm A. The results are showed in Table 3.

Table 3 The influence of J^B

J^B	L^{A*}	p^{A*}	f_2^{A*}	p^{B*}	f_2^{B*}
A, F, G, I, N	D, H, J, K, L	42.2853	1407.5	42.4043	1957.5
A, F, G, L, N	C, E, H, I, K	42.1271	1292.7	42.3395	1945.9
D, G, J, L, O	B, F, H, I, K	42.6811	1756.7	42.8290	2306.6
C, F, H, J, O	A, G, I, K, L	42.3862	1490.3	42.5649	2110

We find that the change of locations of Firm B influences the optimal solution to a great extent. So Firm B may set up facilities with foresight of the best response of followers, as studied by many models in [25]. Capture can be maximized using the Leader-Follower model.

3) If parameter b in utility function is changed, observe the corresponding change of optimal location-pricing strategy of Firm A. The results are showed in Table 4.

Table 4 The influence of parameter b

b	L^{A*}	p^{A*}	f_2^{A*}	p^{B*}	f_2^{B*}
0.2	D, H, J, K, L	62.6147	17208	63.6102	19150
0.5	D, H, J, K, L	49.0459	6666.8	49.4441	7660.1
1.5	D, H, J, K, L	43.0153	1981.6	43.1481	2553.4
2	D, H, J, K, L	42.2615	1395.9	42.3610	1915

Observed from Table 4, we see that the NE prices and profits of two firms continue declining sharply as the increasing sensitivity of consumers towards price, maybe when the incomes of consumers are lower or there are more substitution products and so on. Neither side gains from the price competition.

5.2 Example 2

The 55-node network was used several times in the literature to test algorithms, see [10, 26]. In order to show how efficient the LPGH used in our model, we also compute the location-price problem on the network.

Demands at each point are generated by uniform distribution in [50, 150]. The existing Firm B owns facilities at points $1, 5, 7, 9, 14$. Other conditions are the same as Example 1.

We choose parameters as follows: tabu length: 3; iteration: 20 times; searching 250 neighborhoods at each iteration. Total computing time is 276s. Algorithm experimental results are given in Table 5.

For this problem, the average time complexity of complete enumeration is $O\left(\left(\frac{m}{n}\right)^n \frac{1}{\sqrt{n}}\mathrm{e}^n\right)$, where $\left(m = |I| - \left|J^B\right|, n = n^A\right)$. The required time increases exponentially with the scale (n^A), while the average time complexity of LPGH is polynomial-time $O\left(mn\right)$ Thus it shows that the heuristic algorithm is efficient with a fast convergence rate, especially for a large-scale target assignment.

6 Conclusions

In this paper, a two-stage model is developed to investigate facility location strategy and commodity pricing strategy for a chain retail firm. Given the assumptions on the characteristics of the retail stores, the objective is to maximize the firm's profit in a competitive environment.

Table 5 Solution via LPGH algorithm

Initial conditions	Firm A's optimal locations and NE prices, profits for two firms(¥)				
$p_0^B = 50$, $D = 30$, $b = 1$					
$J^B = [1, 5, 7, 9, 14]$	L^{A*}	p^{A*}	f_2^{A*}	p^{B*}	f_2^{B*}
$u_{ij} = \begin{cases} e^{100-p_j-0.1d_{ij}} & d_{ij} \leq 30 \\ 0, & d_{ij} > 30 \end{cases}$	16,36,33,23,8	42.2774	5786.8	42.4383	2605.8

First-stage focuses on the location decision, while second-stage is used to obtain the pure strategy NE of prices. The problem is solved with stochastic customer behavior to divide market shares. We provide a sufficient condition for the existence of equilibrium prices in the price game.

A piecewise utility function is used to specify customers' patronizing preference. The fixed demands are splitted over facilities proportionally with the facility attraction within the endurable distance threshold. Compared with the utility functions in the literature, the utility function in our model conveys two new ideas: Factors of distance and price are hierarchical; Attractiveness is determined by the two factors simultaneously in some kind of situation, which is undoubtedly more practical. We also analytically prove the existence and uniqueness of the pure strategy NE price in the asymmetric network case.

The problem is a nonlinear fractional programming problem that is difficult to obtain exact solution. A metaheuristic based on tabu search algorithm for it is proposed. In our model, the number of facilities n^A of Firm A is specified, but this assumption could be relaxed by applying the algorithm for $n^A = 1, 2, \cdots\cdots$ and choose n^A to give the highest profit. Two numerical examples are illustrated using the method. Judging from the computing time, the algorithm is efficient. Further sensitivity analysis has been conducted. It is found that the location decision is robust for the follower firm; price game is more intense when incomes of consumers are lower and neither side gains from the price competition. Future research can be conducted in exploring the facility location and commodity pricing strategies when demand is assumed to be elastic.

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Appendix A Proof of Proposition 1

Firstly, we prove f_2^A is a concave function.

$$f_2^A = f_2^A(p^A) = (p^A - m)\sum_{i\in I} a_i h_i(p^A) - \sum_{j\in J^A} g_j x_j.$$

Recall the properties of concave functions. f_2^A is concave if only if for any $x_1 < x_2 < x_3$,

$$\frac{f_2^A(x_2) - f_2^A(x_1)}{x_2 - x_1} \geq \frac{f_2^A(x_3) - f_2^A(x_2)}{x_3 - x_2}. \tag{8}$$

Without loss of generality, we may take $x_2 = 0$, then (8) can be rewritten as

$$\frac{f_2^A(0) - f_2^A(x_1)}{-x_1} \geq \frac{f_2^A(x_3) - f_2^A(0)}{x_3},$$

i.e.,

$$\frac{-x_1 \sum\limits_{i\in I} a_i h_i(x_1) - m \sum\limits_{i\in I} a_i(h_i(0) - h_i(x_1))}{-x_1} \geq \frac{x_3 \sum\limits_{i\in I} a_i h_i(x_3) - m \sum\limits_{i\in I} a_i(h_i(x_3) - h_i(0))}{x_3}.$$

Then, it is sufficient to verify

$$\sum_{i\in I} a_i h_i(x_1) \geq \sum_{i\in I} a_i h_i(x_3) \tag{9}$$

and

$$\frac{\sum\limits_{i\in I} a_i(h_i(0)-h_i(x_1))}{x_1} \geq \frac{\sum\limits_{i\in I} a_i(h_i(0)-h_i(x_3))}{x_3}. \tag{10}$$

Since

$$\frac{\partial h_i}{\partial p^A} = \frac{\sum\limits_{j\in J^A} \frac{\partial u_{ij}x_j}{\partial p^A} \sum\limits_{j\in J^B} u_{ij}}{\left(\sum\limits_{j\in J^A} u_{ij} + \sum\limits_{j\in J^B} u_{ij}x_j\right)^2} \leq 0,$$

(9) is trivial. We only need to prove (10).

Applying the Median Theory to (10), there exists $\xi_1^{(i)} \in (x_1, x_2)$ and $\xi_2^{(i)} \in (x_2, x_3)$, such that

$$-h_i'(\xi_1^{(i)}) = \frac{h_i(0)-h_i(x_1)}{x_1} \quad \text{and} \quad -h_i'(\xi_2^{(i)}) = \frac{h_i(0)-h_i(x_3)}{x_3}.$$

We can rewrite (10) as

$$-h_i'(\xi_1(i)) \geq -h_i'(\xi_2^{(i)}). \tag{11}$$

Noticing that $\xi_1^{(i)} \leq \xi_2^{(i)}$, (11) is valid. We see that f_2^A is a concave function.

The property for f_2^B can be proved in a similar way.

Appendix B Proof of Theorem 1

In the subgame, the strategy spaces are denoted by $P^A = [m, M]$ and $P^B = [m, M]$, where $M > m$. Obviously, P^A and P^B are bounded close sets, i.e., non-empty compact and convex sets in Euclidean space. According to Proposition 1, profit functions f_2^A and f_2^B are concave. Furthermore, profit functions f_2^A and f_2^B are continuous for the continuity of the utility function. According to Lemma 1, there exists one pure strategy NE price (p^{A*}, p^{B*}) in the subgame $G =< A, B, P^A, P^B, f_2^A, f_2^B >$.

Appendix C Proof of Corollary 1

Notice that $u_{ij} = \mathrm{e}^{a-bp_j-cd_{ij}}$ is nonnegativeand nonincreasing with p_j.

Note that

$$h_i(p^A, p^B) = \frac{\sum\limits_{j\in J^A} u_{ij}x_j}{\sum\limits_{j\in J^B} u_{ij} + \sum\limits_{j\in J^A} u_{ij}x_j} = \frac{\mathrm{e}^{a-bp^A} \sum\limits_{j\in J^A} \mathrm{e}^{-cd_{ij}}x_j}{\mathrm{e}^{a-bp^B} \sum\limits_{j\in J^B} \mathrm{e}^{-cd_{ij}} + \mathrm{e}^{a-bp^A} \sum\limits_{j\in J^A} \mathrm{e}^{-cd_{ij}}x_j}.$$

Differentiating $h_i(p^A, p^B)$ with respect to p^A, we get

$$\frac{\partial h_i(p^A,p^B)}{\partial p^A} = -\frac{b\alpha_i\beta_i \mathrm{e}^{a-bp^B}}{(\alpha_i \mathrm{e}^{a-bp^A}+\beta_i \mathrm{e}^{a-bp^B})^2}\mathrm{e}^{a-bp^A},$$

where

$$\alpha_i = \sum_{j\in J^A} \mathrm{e}^{-cd_{ij}}x_j, \quad \beta_i = \sum_{j\in J^B} \mathrm{e}^{-cd_{ij}}.$$

According to $h_i(p^A,p^B)$ and $\frac{\partial h_i(p^A,p^B)}{\partial p^A}$, we have

$$\frac{\partial^2 h_i}{\partial p^{A^2}} \geq 0.$$

Using Lemma 1, Proposition 1 and Theory 1, it is now obvious that the conclusion holds.

Appendix D Proof of Proposition 2

Note that

$$h_i(p^A,p^B) = \frac{\sum\limits_{j\in J^A} u_{ij}x_j}{\sum\limits_{j\in J^B} u_{ij} + \sum\limits_{j\in J^A} u_{ij}x_j} = \frac{\mathrm{e}^{a-bp^A}\sum\limits_{j\in J^A}\mathrm{e}^{-cd_{ij}}x_j}{\mathrm{e}^{a-bp^B}\sum\limits_{j\in J^B}\mathrm{e}^{-cd_{ij}} + \mathrm{e}^{a-bp^A}\sum\limits_{j\in J^A}\mathrm{e}^{-cd_{ij}}x_j}.$$

According to $h_i(p^A,p^B)$ and $\frac{\partial h_i(p^A,p^B)}{\partial p^A}$, we have

$$\frac{\partial^2 h_i}{\partial p^B \partial p^A} < 0$$

and

$$\begin{aligned}\frac{\partial^2 f_2^B}{\partial p^A\partial p^B} &= \frac{\partial^2 f_2^B}{\partial p^B\partial p^A} = \frac{\partial}{\partial p^B}\left(\frac{\partial f_2^B}{\partial p^A}\right)\\ &= \frac{\partial}{\partial p^A}\left(\sum_{i\in I} a_i(1-h_i) - p^B\sum_{i\in I} a_i\frac{\partial h_i}{\partial p^B} + m\sum_{i\in I} a_i\frac{\partial h_i}{\partial p^B}\right)\\ &= -\sum_{i\in I} a_i\frac{\partial h_i}{\partial p^A} - (p^B-m)\sum_{i\in I} a_i\frac{\partial^2 h_i}{\partial p^B\partial p^A} > 0.\end{aligned}$$

$\frac{\partial^2 f_2^A}{\partial p^A\partial p^B}$ and $\frac{\partial^2 f_2^B}{\partial p^A\partial p^B}$ have the same sign because of the symmetry between f_2^A and f_2^B. Thus,

$$\frac{\partial^2 f_2^A}{\partial p^A\partial p^B} = \frac{\partial^2 f_2^A}{\partial p^B\partial p^A} = \frac{\partial}{\partial p^B}\left(\frac{\partial f_2^A}{\partial p^A}\right) = \sum_{i\in I} a_i\frac{\partial h_i}{\partial p^B} + (p^A-m)\sum_{i\in I} a_i\frac{\partial^2 h_i}{\partial p^B\partial p^A} > 0.$$

As proposition 1 points out, profit function f_2^A is concave, i.e., $\frac{\partial^2 f_2^A}{\partial p^{A^2}} \leq 0$. Now we only need to prove

$$\frac{\partial^2 f_2^A}{\partial p^A\partial p^B} < -\frac{\partial^2 f_2^A}{\partial p^{A^2}}. \tag{12}$$

Because

$$\frac{\partial^2 f_2^A}{\partial p^A\partial p^B}$$
$$= b\sum_{i\in I} a_i\left(\frac{\alpha_i\beta_i\mathrm{e}^{a-bp^A}\mathrm{e}^{a-bp^B}}{\left(\alpha_i\mathrm{e}^{a-bp^A}+\beta_i\mathrm{e}^{a-bp^B}\right)^2} + b(p^A-m)\frac{\alpha_i\beta_i\mathrm{e}^{a-bp^A}\mathrm{e}^{a-bp^B}(\alpha_i\mathrm{e}^{a-bp^A}-\beta_i\mathrm{e}^{a-bp^B})}{\left(\alpha_i\mathrm{e}^{a-bp^A}+\beta_i\mathrm{e}^{a-bp^B}\right)^3}\right)$$

and

$$-\frac{\partial^2 f_2^A}{\partial p^{A^2}}$$
$$= b\sum_{i\in I} a_i\left(\frac{2\alpha_i\beta_i \mathrm{e}^{a-bp^A}\mathrm{e}^{a-bp^B}}{\left(\alpha_i \mathrm{e}^{a-bp^A}+\beta_i \mathrm{e}^{a-bp^B}\right)^2} - b(p^A-m)\frac{\alpha_i\beta_i \mathrm{e}^{a-bp^A}\mathrm{e}^{a-bp^B}(\beta_i \mathrm{e}^{a-bp^B}-\alpha_i \mathrm{e}^{a-bp^A})}{\left(\alpha_i \mathrm{e}^{a-bp^A}+\beta_i \mathrm{e}^{a-bp^B}\right)^3}\right),$$
thus, (12) is equal to

$$\frac{\alpha_i\beta_i \mathrm{e}^{a-bp^A}\mathrm{e}^{a-bp^B}}{\left(\alpha_i \mathrm{e}^{a-bp^A}+\beta_i \mathrm{e}^{a-bp^B}\right)^2} > 0, \tag{13}$$

(13) is valid obviously.

The property can be proved in a similar way for $\left|\frac{\partial^2 f_2^B}{\partial p^B \partial p^A}\right| < \left|\frac{\partial^2 f_2^B}{\partial p^{B^2}}\right|$.

Appendix E Proof of Proposition 3

For given p^B, the first order condition for best response of Firm A is

$$\frac{\partial f_2^A}{\partial p^A} = 0.$$

Take derivative with respect to p^B, we obtain

$$\frac{\partial^2 f_2^A}{\partial p^{A^2}}\frac{\partial p^A}{\partial p^B} + \frac{\partial^2 f_2^A}{\partial p^A \partial p^B} = 0.$$

Thus,

$$\frac{dp^A}{dp^B} = -\frac{\frac{\partial^2 f_2^A}{\partial p^A \partial p^B}}{\frac{\partial^2 f_2^A}{\partial p^{A^2}}}.$$

In addition,

$$\left|\frac{\partial^2 f_2^A}{\partial p^A \partial p^B}\right| < \left|\frac{\partial^2 f_2^A}{\partial p^{A^2}}\right|.$$

According to the proof of Proposition 2,

$$\frac{\partial^2 f_2^A}{\partial p^A \partial p^B} > 0, \quad \frac{\partial^2 f_2^A}{\partial p^{A^2}} \leq 0.$$

Thus, in the model II$'$,

$$0 < \frac{dp^A}{dp^B} = -\frac{\frac{\partial^2 f_2^A}{\partial p^A \partial p^B}}{\frac{\partial^2 f_2^A}{\partial p^{A^2}}} < 1.$$

Similarly, in the model II$''$, $0 < \frac{dp^B}{dp^A} < 1$.