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# ON CONTROLLABILITY FOR STOCHASTIC CONTROL SYSTEMS WHEN THE COEFFICIENT IS TIME-VARIANT*

**Feng LIU · Shige PENG**

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**Abstract** This paper investigates the controllability problem of time-variant linear stochastic control systems. A sufficient and necessary condition is established for stochastic exact controllability, which provides a useful algebraic criterion for stochastic control systems. Furthermore, when the stochastic systems degenerate to deterministic systems, the algebraic criterion becomes the counterpart for the complete controllability of deterministic control systems.

**Key words** Backward stochastic differential equation (BSDE), E-well-posedness, Stochastic control system, stochastic exact controllability.

## 1 Introduction

It is well-known about the study on controllability of deterministic control system. Are there similar definitions and results for stochastic control systems?

Chen[1] studied deeply stochastic controllability and stochastic observability. Recently, study on backward stochastic differential equation (BSDE) provides a new viewpoint on this research. BSDE has a completely new structure of equation. This equation can determine uniquely adapted solution of stochastic differential equation by using given random adapted terminal condition[2]. Peng[3] first defined the exact terminal-controllability and exact controllability of stochastic control system for BSDE. He proved a necessary condition of exact terminal-controllability for nonlinear stochastic control systems. Like deterministic control systems, Peng obtained an algebraic criterion which was a sufficient and necessary condition of stochastic exact controllability for linear stochastic control systems. Obviously, if stochastic systems degenerate to deterministic systems, stochastic exact controllability becomes complete controllability of deterministic systems. The algebraic criterion for stochastic exact controllability of linear stochastic control system becomes the counterpart for complete controllability of linear deterministic control system. Peng and Wu[4−7] et al. used BSDE to study some control systems problem: Probability theory and optimal control. Duffie et al.[8−10] introduced

Feng LIU
*School of Electronics and Information Engineering, Beijing Jiaotong University, Beijing* 100044, *China.*
Email: fliu2@bjtu.edu.cn.
Shige PENG
*School of Mathematics and System Sciences, Shandong University, Ji'nan* 250100, *China.*
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stochastic differential utilities in economic models. Bensoussan and Wu[11−12] studied optimal control problem of incomplete observable in forward-backward stochastic systems. To our best knowledge, few effort has been made in addressing controllability of stochastic linear time-variant systems. In this paper, we investigate the controllability problem of time-variant linear stochastic control systems. A sufficient and necessary condition is established for stochastic exact controllability, which provides a useful algebraic criterion for stochastic control systems.

The rest of the paper is organized as follows. Stochastic systems model and main results are described in Section 2. In Section 3, based on Peng's results, a necessary condition of exact terminal-controllability for nonlinear stochastic control systems which the coefficient is time-variant is obtained. A sufficient and necessary condition of stochastic exact controllability for stochastic control systems is obtained. Next, a simple stochastic linear system is studied and a controllable subspace is given in Section 4. Finally, conclusions are given in Section 5.

## 2 Main Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathcal{F}_t, t \geq 0$. Let $(W_t, t \geq 0)$ be a $d$-dimensional standard Wiener process (Brownian motion):

$$W = (W^1, W^2, \cdots, W^d).$$

In this paper, we only study one-dimensional brownian motion. We assume that the filtration $\mathcal{F}_t$ is generated by this Wiener process $W$: $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$. All process mentioned in this paper are assumed to be $\mathcal{F}_t$-adapted, the set of all $R^n$ valued square integrable process are denoted by $L^2_{\mathcal{F}}(0, T; R^n)$, i.e.,

$$E \int_0^T |V_t|^2 dt < \infty.$$

The stochastic nonlinear system considered in this work takes the following form,

$$dx_t = b(x_t, v_t, t)dt + \sigma(x_t, v_t, t)dW_t, \tag{1}$$

where $b$ and $\sigma$ are $R^n$-valued functions defined on $(x, v) \in R^{n \times k}$. We assume that $b$ and $\sigma$ are continuous with respect to $(x, v)$ and uniformly Lipschitzian in $x$. We also assume that $b$ and $\sigma$ satisfy linear growth conditions in $(x, v)$. A process $v(\cdot) \in L^2_{\mathcal{F}}(0, T; R^k)$ is called to be an admissible control if it takes values in a given subset $U \subset R^k$. We denote by $\mathcal{U}$ the set of all admissible controls. A process $x(\cdot) \in L^2_{\mathcal{F}}(0, T; R^n)$ is said to be a trajectory corresponding to an admissible control $v(\cdot) \in \mathcal{U}$ if it is a solution of (1).

We now introduce some notions of controllability that measure the capacity of a stochastic control system steering a trajectory from a given initial point

$$x_0 = x \in R^n$$

to a given terminal point

$$x_T = \xi \in L^2(\Omega, \mathcal{F}_T, P; R^n).$$

**Definition 1**[3] (E-well-posedness) Stochastic differential equation (1) is called E-well-posed if for any given

$$x_T = \xi \in L^2(\Omega, \mathcal{F}_T, P; R^n),$$

there exists at least one pair of processes

$$(x(\cdot), v(\cdot)) \in L^2_{\mathcal{F}}(0, T; R^n) \times L^2_{\mathcal{F}}(0, T; R^k)$$

which is the solution of system (1).

Here, we give some necessary conditions for E-well-posed systems. As a product of this discussion, an algebraic criterion of exact controllability for linear stochastic control systems is obtained.

In [2], Pardoux and Peng used a well-known square-integrable martingale representation theorem in terms of Brown motion to prove the existence of stochastic differential equation (1). This theorem tells us that for any square-integrable $\mathcal{F}_T$-measurable random variable $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^n)$, we can find a unique process $q \in L^2_{\mathcal{F}}(0,T;R)$ such that

$$\xi = E\xi + \int_0^T q_s dW_s.$$

**Definition 2**[3] A stochastic control system (1) is called exactly terminal-controllable if for any $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^n)$, there exists at least one admissible control $v \in \mathcal{U}$, such that the corresponding trajectory $x_t$ satisfies the terminal condition $x_T = \xi$.

**Definition 3**[3] A stochastic control system (1) is called exactly controllable if for any $x \in R^n$ and $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^n)$, there exists at least one admissible control $v \in \mathcal{U}$, such that the corresponding trajectory $x_t$ satisfies the initial condition $x_0 = x \in R^n$ as well as the terminal condition $x_T = \xi$.

Throughout the paper, we only discuss the case where the control constraint is the whole space. But the conclusions may be obviously extended to more general cases.

The linear time-variant stochastic system considered in this work takes the following form,

$$dx_t = (F(t)x_t + G(t)v_t)dt + (F_1(t)x_t + G_1(t)v_t)dW_t, \tag{2}$$

where $F(\cdot), F_1(\cdot) \in R^{n\times n}$ matrices; $G(\cdot), G_1(\cdot) \in R^{n\times k}$ matrices. The following main result establishes a sufficient and necessary condition for the controllability of system (2).

**Theorem 1** *System* (2) *is exactly terminal-controllable if and only if*

$$\text{Rank } G_1(t) = n.$$

In this case, we can use a simple linear transformation

$$v(t) = M(t)\begin{pmatrix} z \\ u \end{pmatrix} + K(t)x(t)$$

to transform system (2) into an equivalent form

$$-dx_t = (A(t)x_t + A_1(t)z_t + B(t)u_t)dt - z_t dW_t, \tag{3}$$

where $M$ is the invertible $k \times k$-matrix, $z$ and $u$ are $R^n$-valued and $R^{k-n}$-valued processes, respectively. So the controllable algebraic criterion of system (3) can be obtained.

**Theorem 2** *System* (3) *is exactly controllable if and only if*

$$\text{Rank}\left[E\int_0^{\mathrm{T}} (Y(t)B(t))(Y(t)B(t))^{\mathrm{T}} dt\right] = n,$$

*where $Y(t)$ is $R^{n\times n}$-valued stochastic process and satisfied the following differential equation,*

$$\begin{cases} dY(t) = Y(t)[A(t)dt + A_1(t)dW_t], \\ Y(0) \;\; = I. \end{cases} \tag{4}$$

## 3 Exact Controllability for Stochastic Control Systems

For System (1), we have the following assumptions,

**Assumption 1** For each $(x, v) \in R^m \times R^k$, processes $b(x, v, \cdot)$ and $\sigma(x, v, \cdot)$ are in $L^2_{\mathcal{F}}(0, T; R^n)$, and

$$\lim_{t \to T} E|\sigma(x, v, t) - \sigma(x, v, T)|^2 = 0.$$

$b(x, v, t)$ and $\sigma(x, v, t)$ satisfy the linear growth conditions with respect to $(x, v)$ uniformly in $t \in [0, T]$. $\sigma$ is Lipschitzian with respect to $y$ uniformly in $(v, t)$.

Based on the Assumption 1, we consider a special case which $\sigma(x, v, t)$ is linear on $v$, i.e.,

$$\sigma(x, v, t) = G_1(t)v + \sigma_1(x, t).$$

To prove the main results, it is worth noting the following fundamental Lemma.

**Lemma 1** *Let $b$ and $\sigma_1$ be uniform Lipschitzian in $(x, v)$. Then stochastic differential equation* (1) *is E-well-posed if and only if*

$$\text{Rank } G_1(t) = n.$$

*Proof* If system (1) is E-well-posed, from solution properties of stochastic differential equation, any pair of solutions $(x, v)$ in system Equation (1) is satisfied with convergence of Assumption 1, so

$$\lim_{t \to T} E|\sigma(x_t, a, t) - \sigma(x_T, a, T)|^2 = 0,$$

and $\sigma(x(\cdot), v(\cdot), \cdot) \in L^2_{\mathcal{F}}(0, T; R^n)$, where $a \in R^k$ is arbitrary vector. If Rank $G_1(t) < n$, that is to say, there at least exists one point $s \in [0, T]$ s.t. $G_1(s) < n$, then we can find a vector $b \in R^n; |b| = 1$ such that $b \cdot G_1(s) = 0$ at $s$, for any pair of solution $(x, v)$ at $s$,

$$b \cdot [G_1(s)(v - a)] = 0, \forall (x, v) \in R^n \times R^k.$$

It contradict with the Proposition 2.2 in [3], then Rank $G_1(t) = n$.

If Rank $G_1(t) = n$, let

$$x_T = \xi \in L^2(\Omega, \mathcal{F}_T, P; R^n),$$

$G_1(t)$ is nonsingular, does there exist nonsingular $k \times k$-dimensional matrix $M(t)$ such that $G_1(t)M(t) = [I_n, 0]$? That is to say, the following set

$$\{G_1(\cdot) \in R^{n \times k} : \exists k \times k \text{ inverible matrix } M(t), \text{ s.t.} G_1(t)M(t) = [I_n, 0]\} \tag{5}$$

is null set or not? According to the construction of the set, it is easy to observe that the set is not null, because a lot of time-invariant matrix included in the set. Next, when matrix is time-variant, we will construct some matrix which satisfied the set (5). Consider the following $k \times k$-dimensional stochastic differential equations:

$$\begin{cases} d\Phi(t) = A(t)\Phi(t)dt + B(t)\Phi(t)dW_t \\ \Phi(0) \ \ = I, \end{cases} \tag{6}$$

$$\begin{cases} d\Psi(t) = (\Psi(t)B(t)B(t) - \Psi(t)A(t))dt - \Psi(t)B(t)dW_t \\ \Psi(0) \ \ = I. \end{cases} \tag{7}$$

Applying Ito formula to $\Phi(t)\Psi(t)$, yields

$$d\Psi(t)\Phi(t) = (d\Psi(t))\Phi(t) + \Psi(t)d(\Phi(t)) - \Psi(t)B(t)B(t)\Phi(t)dt. \tag{8}$$

Taking (6) and (7) into differential equation (8), we have

$$\begin{cases} d\Psi(t)\Phi(t) = (d\Psi(t))\Phi(t) + \Psi(t)d(\Phi(t)) - \Psi(t)B(t)B(t)\Phi(t)dt \\ \qquad = ((\Psi(t)B(t)B(t) - \Psi(t)A(t))dt - \Psi(t)B(t)dW_t)\Phi(t) \\ \qquad\quad + \Psi(t)(A(t)\Phi(t)dt + B(t)\Phi(t)dW_t) - \Psi(t)B(t)B(t)\Phi(t)dt = 0, \end{cases} \tag{9}$$

based on the initial condition of $\Phi(t)$ and $\Psi(t)$, we get

$$\Phi(t)\Psi(t) = I, \ \forall t \in [0, T].$$

In equations (6) and (7), let matrix $B(t) = 0$, then systems (6) and (7) degenerate to deterministic systems, and conclusion also holds. Let $G_1(t)$ be ahead $n$ lines of matrix which is solution of equation (6), and $M(t) = \Psi(t)$, so

$$G_1(t)M(t) = [I_n, 0].$$

From above we know the set (5) which we defined is not null. Let

$$M^{-1}(t)v = \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \quad \mu \in R^n, \quad \nu \in R^{k-n}.$$

By this transform, system (2) can be rewritten as follows:

$$\begin{cases} dx_t = h(x_t, \mu_t, \nu_t, t)dt + [F_1(t)x_t + \mu_t]dW_t, \\ x_T \ = \xi, \ 0 \le t \le T, \end{cases}$$

where $h(x_t, \mu_t, \nu_t, t) = [F(t)x_t + G(t)M(t)\binom{\mu}{\nu}]$ is satisfied Lipschitzian condition with $x, \mu, \nu$, so there exists unique a pair of solution $(x, \mu) \in L^2_{\mathcal{F}}(0, T; R^{2n})$ satisfied

$$\begin{cases} dx_t = h(x_t, \mu_t, 0, t)dt + [F_1(t)x_t + \mu_t]dW_t, \\ x_T \ = \xi, \ 0 \le t \le T, \end{cases} \tag{10}$$

then $(x_t, v_t) = (x_t, M(t)\binom{\mu_t}{0})$, $t \in [0, T]$ is solution of system(10), from above all we know system (1) is E-well-posed. ∎

**Remark 1** The necessary condition of Lemma 1 implies automatically that the dimension of $v$ must be larger than or equal to that of $x$: $k \ge n$. The sufficient's proof does not imply uniqueness, the uniqueness holds if and only if $k = n$.

From Definition 1 and 2, we can get the definition of exactly terminal controllability is just equivalent to the E-well-posedness of stochastic differential equation (1), so Rank $G_1(t) = n$ is also the exactly terminal-controllable sufficient and necessary condition of system (1).

*Proof of Theorem* 1 Now, we consider the linear system (2), from Lemma 1, it is obviously that Theorem 1 is holds. ∎

In order to prove Theorem 2, we can use a simple linear transformation

$$v(t) = M(t)\begin{pmatrix} z \\ u \end{pmatrix} + K(t)x(t),$$

where $z$ and is $u$ are $n$-dimension and $(k-n)$-dimension vectors, respectively, and $M$ is the invertible $k \times k$-matrix which satisfied $G_1(t)M(t) = [I_n, 0]$.

Taking the transformation into system (2), let $\widehat{z_t} = F_1(t)x_t + z_t + G_1K(t)x_t$, system (2) is changed into an equivalent form

$$-dx_t = (A(t)x_t + A_1(t)\widehat{z_t} + B(t)u_t)dt - \widehat{z_t}dW_t, \tag{11}$$

where

$$-A(t) = F(t) + G(t)K(t) - G(t)MF_1(t) - G(t)MG_1K(t),$$
$$-A_1(t) = G(t)M, -B(t) = G(t)M.$$

system (11) is a classical BSDE, because its coefficient matrices satisfied the condition in [2], we can direct obtain that BSDE (11) exist a unique solution $x(t)$. Here we only consider terminal point $x(T) = 0$, whereas the other terminal value, according to the superposition of linear systems, solution $x(t)$ can achieve. Applying Ito formula to $Y(t)x(t)$, integral it from 0 to $T$, we have

$$x(0) = -E\int_0^{\mathrm{T}} Y(t)B(t)u(t)dt. \tag{12}$$

From Theorem 1 we know system (11) is exactly terminal-controllable. Is it exactly controllable condition? To answer this question, we give Theorem 2. Based on the above analysis, here we only consider terminal point $x(T) = 0$. In order to prove Theorem 2, a classical adjoint equation should be introduce as:

$$\begin{cases} dY(t) = Y(t)[A(t)dt + A_1(t)dW_t], \\ Y(0) \ \ = I. \end{cases}$$

Applying Ito formula to $Y(t)x(t)$, upon integration from 0 to $T$, an algebraic criterion can be obtained. Now we give the proof of Theorem 2.

*Proof of Theorem* 2 If system (3) is exactly controllable, from Definition 3 we have

$$\mathrm{span}\{x_0^u; u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^{n-k})\} = n.$$

Let

$$\mathrm{Gram} = E\int_0^{\mathrm{T}} (Y(t)B(t))(Y(t)B(t))^{\mathrm{T}}dt.$$

Using reductio ad absurdum, if Rank (Gram) $< n$, then there exists a vector $\beta \in R^n$ which is not zero such that $\beta^{\mathrm{T}}\mathrm{Gram}\beta = 0$, that is to say, $\beta^{\mathrm{T}}(Y(t)B(t)) = 0$, a.e. For any given initial value $x_0$, from (12) and above, we get $< x_0, \beta >= 0$, it contradict with the fact that

$$\mathrm{span}\{x_0^u; u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^{n-k})\} = n,$$

so Rank (Gram) $= n$.

If Rank (Gram) $= n$, for any initial value $x(0)$, we construct controller

$$u(t) = (Y(t)B(t))^{\mathrm{T}}(\mathrm{Gram})^{-1}x(0). \tag{13}$$

Taking (13) into (12), it is obviously that (12) holds, so initial value $x(0)$ belongs to controllable space, system (3) is exactly controllable. ∎

**Remark 2** By the conditions of Theorem 2, we know that the exact controllability of the system is needed to enough controllable information, that is to say, the dimension of controller is greater than the dimension of state, $k-n > 0$. In finance markets, this is also condition between the expected return of financial investment and initial investment. But the general sense, even if the controller dimension is greater than the dimension of the state, the system controllability does not necessarily guarantee, this is the financial risk. Corresponds to the solutions of the equations are not unique in Remark 1, it may eventually gain the same in finance markets even if control strategy (investment strategy) is different.

## 4 A Controllable Subspace for an Example

We now consider a special time-variant stochastic system, a continues-time finance system

$$\begin{cases} -dx_t = (A_1(t)x_t + A_2(t)\widehat{z}_t + Bu_t)dt - \widehat{z}_t dW_t, \\ x_T = 0, \quad 0 \le t \le T, \end{cases} \tag{14}$$

where $A_1(t), A_2(t) : [0,T] \to R^{n\times n}$ are $C^\infty$-functions.

In order to give controllable subspace structure in system (14), the following theorem is obtained.

**Theorem 3** *For system* (14), *the following relation holds,*

$$\begin{aligned}&\text{span}\,\{x^u_{t_0}; u(\cdot) \in L^2_{\mathscr{F}}(t_0, T; R^{k-n})\} \supset \\ &\text{span}\,\{\{I; \{A^*_i(t)^{\mathrm{T}}\}_{i=1,2}; \{A^*_{ij}(t)^{\mathrm{T}}\}_{i,j=1,2}; \{A^*_{ijk}(t)^{\mathrm{T}}\}_{i,j,k=1,2}; \cdots\}B\}|_{t=0}.\end{aligned}$$

*We define:*

$$\begin{aligned}&A^*_i(t) = A^{\mathrm{T}}_i(t); \quad i = 1, 2,\\ &\begin{cases} A^*_{ij}(t) = \dfrac{dA^*_i(t)}{dt} + A^*_i(t)A^*_j(t), \quad i = 1, 2; \quad j = 1, \\ A^*_{ij}(t) = A^*_i(t)A^*_j(t), \quad i = 1, 2; \quad j = 2, \end{cases}\\ &\begin{cases} A^*_{ijk}(t) = \dfrac{dA^*_{ij}(t)}{dt} + A^*_{ij}(t)A^*_k(t), \quad i, j = 1, 2; \quad k = 1, \\ A^*_{ijk}(t) = A^*_{ij}(t)A^*_k(t), \quad i, j = 1, 2; \quad k = 2. \end{cases}\end{aligned}$$

*In general,*

$$\begin{cases} A^*_{i_1\cdots i_l}(t) = \dfrac{dA^*_{i_1\cdots i_{l-1}}(t)}{dt} + A^*_{i_1\cdots i_{l-1}}(t)A^*_{i_l}(t), \\ \qquad\qquad i_1, \cdots, i_{l-1} = 1, 2; \quad i_l = 1 \\ A^*_{i_1\cdots i_l}(t) = A^*_{i_1\cdots i_{l-1}}(t)A^*_{i_l}(t), \\ \qquad\qquad i_1, \cdots, i_{l-1} = 1, 2; \quad i_l = 2. \end{cases}$$

*The above with the criterion for* $t = 0$, *and also hold for* $\forall t = t_0 \in [0, T]$.

*Proof* From system (14) we know $(x^u, z^u)$ depend linearly on controller $u$, then we have

$$\text{span}\{x^u_0; u(\cdot) \in L^2_{\mathscr{F}}(0, T; R^{k-n})\} = \{x^u_0; u(\cdot) \in L^2_{\mathscr{F}}(0, T; R^{k-n})\}.$$

Let vector $\beta \in L^2(\Omega, \mathscr{F}_0, P; R^n)$ be not zero, such that

$$\beta \cdot x^u_0 = 0, \quad \forall u \in L^2_{\mathscr{F}}(0, T; R^{k-n}),$$

then $E[\beta \cdot x^u_0] = 0$.

Now, we give the adjoint equation of Equation (14),

$$\begin{cases} dy_t = A_1^{\mathrm{T}}(t)y_t dt + A_2^{\mathrm{T}}(t)y_t dW_t \\ y_0 \ = \beta, \quad 0 \le t \le T. \end{cases} \tag{15}$$

Applying Ito formula to $y_t x_t^u$, yields

$$E\int_0^{\mathrm{T}} y_t \cdot Bu(t)dt = \beta \cdot x_0^u = 0, \quad \forall u \in L^2_{\mathscr{F}}(0,T;R^{k-n})$$

consequently,

$$E\int_0^{\mathrm{T}} B^{\mathrm{T}} y_t u(t)dt = 0.$$

Because of arbitrary controller, we have

$$B^{\mathrm{T}} y_t \equiv 0,\ 0 \le t \le T.$$

Let $t = 0$, then $B^{\mathrm{T}}\beta = 0$. Next, Derivation of $B^{\mathrm{T}}y_t \equiv 0$, $dB^{\mathrm{T}}y_t = B^{\mathrm{T}}A_1^{\mathrm{T}}(t)y_t dt + B^{\mathrm{T}}A_2^{\mathrm{T}}(t)y_t dW_t \equiv 0$, noticing differentness between $dt$ and $dW_t$, we have

$$B^{\mathrm{T}}A_1^{\mathrm{T}}(t)y_t \equiv 0; \quad B^{\mathrm{T}}A_2^{\mathrm{T}}(t)y_t \equiv 0. \tag{16}$$

Let $t = 0$, we get $B^{\mathrm{T}}A_1^{\mathrm{T}}(0)\beta \equiv 0$; $\quad B^{\mathrm{T}}A_2^{\mathrm{T}}(0)\beta \equiv 0$. Next, we define $A_1^*(t) = A_1^{\mathrm{T}}(t); A_2^*(t) = A_2^{\mathrm{T}}(t)$, the (16) can be rewritten as: $B^{\mathrm{T}}A_1^*(t)y_t \equiv 0$; $\quad B^{\mathrm{T}}A_2^*(t)y_t \equiv 0$. Derivation of $B^{\mathrm{T}}A_1^*(t)y_t \equiv 0$, $B^{\mathrm{T}}\frac{A_1^*(t)}{dt}y_t dt + B^{\mathrm{T}}A_1^*(t)dy_t \equiv 0$, so

$$\begin{cases} B^{\mathrm{T}}\dfrac{A_1^*(t)}{dt}y_t + B^{\mathrm{T}}A_1^*(t)A_1^*(t)y_t \equiv 0, \\ B^{\mathrm{T}}A_1^*(t)A_2^*(t)y_t \equiv 0. \end{cases} \tag{17}$$

Let $t = 0$, we can get

$$B^{\mathrm{T}}\frac{A_1^*(t)}{dt} + B^{\mathrm{T}}A_1^*(t)A_1^*(t)|_{t=0} \perp \beta; \quad B^{\mathrm{T}}A_1^*(t)A_2^*(t)|_{t=0} \perp \beta.$$

Derivation of $B^{\mathrm{T}}A_2^{\mathrm{T}}(t)y_t \equiv 0$, we can get the similar result. Repeating the above derivation, we have

$$\begin{cases} A^*_{i_1\cdots i_l}(t) = \dfrac{dA^*_{i_1\cdots i_{l-1}}(t)}{dt} + A^*_{i_1\cdots i_{l-1}}(t)A^*_{i_l}(t), \quad i_1,\cdots,i_{l-1} = 1,2; \quad i_l = 1, \\ A^*_{i_1\cdots i_l}(t) = A^*_{i_1\cdots i_{l-1}}(t)A^*_{i_l}(t), \quad i_1,\cdots,i_{l-1} = 1,2; \quad i_l = 2, \end{cases} \tag{18}$$

then

$$\mathrm{span}\{\{I; \{A_i^*(t)^{\mathrm{T}}\}_{i=1,2}; \quad \{A_{ij}^*(t)^{\mathrm{T}}\}_{i,j=1,2}; \quad \{A_{ijk}^*(t)^{\mathrm{T}}\}_{i,j,k=1,2}; \cdots\}B\}|_{t=0} \perp \beta$$

is a controllable subspace.

Next, we study for any $t_0 \in [0,T]$.

Let $\beta \in L^2(\Omega, \mathscr{F}_{t_0}, P; R^n)$ such that $\beta \cdot x_{t_0}^u = 0$, then $E[\beta \cdot x_{t_0}^u] = 0$. We introduce classical adjoint equation

$$\begin{cases} dy_t = A_1^{\mathrm{T}}(t)y_t dt + A_2^{\mathrm{T}}(t)y_t dW_t \\ y_{t_0} = \beta, \quad t_0 \le t \le T. \end{cases} \tag{19}$$

Applying Ito formula to $y_t x_t^u$, we have

$$E^{\mathscr{F}_{t_0}} \int_{t_0}^{\mathrm{T}} B^{\mathrm{T}} y_t u(t) dt = 0, \quad \forall u \in L^2_{\mathscr{F}}(t_0, T; R^{k-n}).$$

Similarly, we can obtain the corresponding results. ∎

**Remark 3** The controllable subspace given by Theorem 3 is a subset of whole controllable space. Under what conditions, controllable subspace expansion for the whole space is an interesting question. In Theorem 3, if parameter matrix is time-invariant, then

$$\mathrm{span}\{\{I; \{A_i^*(t)^{\mathrm{T}}\}_{i=1,2}; \quad \{A_{ij}^*(t)^{\mathrm{T}}\}_{i,j=1,2}; \quad \{A_{ijk}^*(t)^{\mathrm{T}}\}_{i,j,k=1,2}; \cdots\}B\}$$

is the whole space.

## 5 Conclusions

In this paper, the exactly terminal-controllable and the exactly controllable sufficient and necessary condition for time-variant stochastic linear systems is given and proved. Algebraic criterion of the exactly controllability is given when the stochastic system is linear. When the stochastic systems degenerate to deterministic systems, the algebraic criterion becomes the counterpart for the complete controllability of deterministic control systems.

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