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AN AUGMENTED BV SETTING FOR FEEDBACK SWITCHING CONTROL*

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Abstract This paper considers dynamical systems under feedback with control actions limited to switching. The authors wish to understand the closed-loop systems as approximating multi-scale problems in which the implementation of switching merely acts on a fast scale. Such hybrid dynamical systems are extensively studied in the literature, but not much so far for feedback with partial state observation. This becomes in particular relevant when the dynamical systems are governed by partial differential equations. The authors introduce an augmented BV setting which permits recognition of certain fast scale effects and give a corresponding well-posedness result for observations with such minimal regularity. As an application for this setting, the authors show existence of solutions for systems of semilinear hyperbolic equations under such feedback with pointwise observations.

Key words Feedback, functions of bounded variation, hybrid dynamical systems, partial state observation, switching control.

1 Introduction

Interaction among components operating at distinct time scales is a challenging and important area of research and — though having great practical consequences — is not yet understood in its full complexity. One approach to such multi-scale problems is the theory of hybrid dynamical systems which, as far as possible, suppresses consideration of unmodeled details of the fast scale dynamics. One important scenario in this context is a continuous time dynamical process on our (slow) scale coupled with an observation based feedback controller acting on a much faster time-scale which we will then be approximating as instantaneous. The effect of control decisions on the fast scale then largely shows up as switching, selecting a discrete mode from a finite set by appropriate switching rules: a paradigmatic example is a thermostat, 'instantaneously' switching a furnace ON or OFF depending on the temperature observed at a single point. We do also allow for the possibility that a control action may take place entirely

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on the fast time scale, showing up as a jump in the state or may be a combination of these. We do not describe these possibilities explicitly, but that description should be clear.

When the continuous dynamics are governed by ordinary differential equations and full state information is assumed, well-posedness of such closed-loop systems has been addressed by many authors in this area — for example, [1–6].

Note that one typically considers switching structures in which the modal transitions are restricted to specified edges of a transition graph — see Figure 2 — so the sequencing of these control decisions (e. g., with respect to the fast scale) remains significant, even when one may have a cascade of several such actions which could be viewed as 'simultaneous' on the slow scale. For this reason it has seemed necessary to model the time domain of the closed loop system as an augmented version of the 'normal time' interval in $I \subset \mathbb{R}$ — a lexicographically ordered set

$$\mathcal{T}_* = \{(t,0) : t \in I\} \cup \{(t_k, j) : 0 \le j \le n_k \text{ for } k = 1, 2, \cdots, K\} \subset \mathbb{R} \times \mathbb{Z}, \tag{1}$$

with the real value t modeling the 'normal time' and the integer value j indexing the cascade of discrete control actions at the same nominal time, see, e. g., [3, 5, 7–9]. A difficulty with this for sets of functions defined on such augmented intervals is that the relevant $\mathcal{T}_*$ will vary with the function. Note that we leave specification of the set $\{t_k\}$ somewhat ambiguous by allowing $n_k = 0$ so we can always compare functions on augmented intervals $\mathcal{T}_*$ and $\mathcal{T}'_*$ (with the same underlying normal time interval) by working with the union of these sets for each of them.

We will be considering functions of bounded variation ($\boldsymbol{BV}$) in a slight modification $\boldsymbol{BV}_*$ of the Jordan sense, defined on such augmented domains $\mathcal{T}_*$. Certainly the multi-scale perspective we have in mind is entirely consistent with fast scale effects other than the control decisions and we therefore accept the possibility of concomitant jumps in the state (and so possibly in the observation) as the slow scale recognition of fast scale dynamics. This is one reason we will wish to extend known results to such a $\boldsymbol{BV}$ context.

As an application we will here be considering switching control of a distributed parameter system so the scenario above couples the evolution of the system, governed by a partial differential equation, with modal switching based on sensor observations: $y(s)$ taking values in an observation space $\mathcal{Y}$ as described in [10]. The provision of partial state observation becomes necessary for such infinite dimensional hybrid systems because the assumption of full state observation is unrealistic in many cases. One wishes to characterize switching rules (5) giving admissible switching controls with a minimum of regularity assumptions for this observations, making up another reason for a $\boldsymbol{BV}$ context.

It is important to consider the possibility of switching rules leading to Zeno phenomena, i. e., accumulation points of control decisions (switching times); this is one of the main technical difficulties in obtaining global existence results. In $\boldsymbol{BV}_*$ one does allow infinitely many small jumps in the state, but 'non-Zenoness' refers to finiteness of the set of control actions and so means not only that there are at most finitely many slow scale switching times, but also that there are no infinite cascades. We note, of course that there are physical systems, e. g., a buzzer, whose hybrid idealization would involve an infinite cascade, and our present theory would not cover these unless the time scaling separates these discrete transitions (treating the sliding mode as chattering) or one uses some averaging (homogenization) to redefine this behavior as a single mode.

We apply our results for the augmented $\boldsymbol{BV}_*$ setting to show existence of solutions for systems of semi-linear hyperbolic equations under such switching feedback control with pointwise observations. Since our example is a first order transport system, we will also be led to the use of $\boldsymbol{BV}_*$ for the spatial domain.

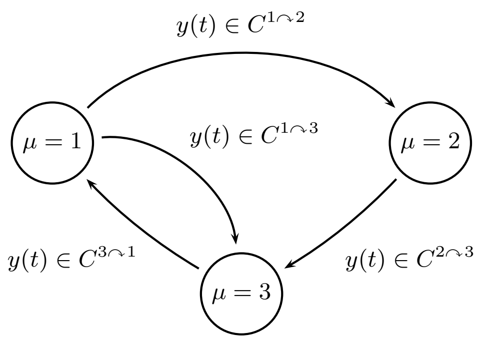

Figure 1 Example of a switching structure

2 $\boldsymbol{BV}_*$ and Switching Rules

As anticipated in the introduction, we wish to distinguish 'points' in time with the same 'normal time', but with well-defined ordering. This distinction is to be viewed as permitting phenomena on a finer scale (in time) to rise to our attention as necessary. We think of this as introducing an augmented version $\mathcal{T}_*$ of the normal (slow scale) control interval as in (1).

To consider $\boldsymbol{BV}$ for functions defined on an augmented time domain $\mathcal{T}_*$, we introduce

$$\mathrm{Var}(y, \mathcal{T}_*) = \sup_S \left\{ \sum_k |y(s_n) - y(s_{n-1})| \right\} \tag{2}$$

with the sup taken over all (finite) sequences $S = (s_0 \prec s_1 \prec \cdots)$ in $\mathcal{T}_*$. Here, '$\prec$' means the lexicographic order in $\mathbb{R} \times \mathbb{Z}$. We set

$$\boldsymbol{BV}_* = \boldsymbol{BV}(\mathcal{T}_*) = \{y : \mathcal{T}_* \to \mathbb{R} : \mathrm{Var}(y, \mathcal{T}_*) < \infty\} \tag{3}$$

with the subscript $*$ to be understood as a reminder of the augmention introduced above.

We will be addressing the Zeno phenomenon and some implications of that time-domain modeling when the feedback uses partial state information based on sensor data y. Given an augmented time domain $\mathcal{T}_* \subset \mathbb{R} \times \mathbb{Z}$ as explained above, an observation-trajectory is a mapping $y : \mathcal{T}_* \to \mathcal{Y}$ with each sensor value $y(s)$ given by the system state at the (augmented) time $s \in \mathcal{T}_*$. Accordingly, we consider the set of mappings $\{\mu : \mathcal{T}_* \to M\}$, where M is the finite set of available modes and $\mu(s) \in M$ for $s \in \mathcal{T}_*$. Observe that, since the switching times are initially unknown, this modeling implies that the time domain $\mathcal{T}_*$ of the closed-loop system is not given a-priori, but must be constructed causally during the system's evolution. A feedback law then assigns a set of admissible mode-trajectories to a given observation-trajectory and is, therefore, of the form

$$\Phi : [\mathcal{T}_* \to Y] \to 2^{[\mathcal{T}_* \to M]}, \tag{4}$$

which we make precise by assuming that the feedback Φ is given by a set of switching rules of the form:

$$\left\{ \begin{array}{l} \text{If one is in the mode } \mu \text{ at any given event time } s \in \mathcal{T}_*, \text{ then:} \\ \quad \text{switching } \mu \curvearrowright \mu' \text{ is permitted (only) if } y(s) \in C(\mu \curvearrowright \mu'), \\ \quad \text{staying in mode } \mu \text{ is permitted (only) if } y(s) \in \mathcal{A}(\mu). \end{array} \right. \tag{5}$$

where the sets $\mathcal{A}(\mu)$, $\{C(\mu \curvearrowright \mu') : \mu' \neq \mu\}$ cover $\mathcal{Y}$ for each $\mu \in M$. Such switching rules encompass a very broad class of feedback laws.

Example 1 For $\mathcal{Y} = \mathbb{R}$, $M = \{0, 1\}$ and thresholds $\rho_1 < \rho_2$ in $\mathbb{R}$, setting

$$\mathcal{A}(0) = \{y \le \rho_2\} \text{ and } C(0 \curvearrowright 1) = \{y \ge \rho_2\},$$
$$\mathcal{A}(1) = \{y \ge \rho_1\} \text{ and } C(1 \curvearrowright 0) = \{y \le \rho_1\}$$

defines the well-known (closed) non-ideal relay, the elementary hysteron of [11].

We further remark that, as in this Example, we do not require the sets $A(\mu)$ and $C(\mu \curvearrowright \mu')$ to be disjoint in general and thus permit situations with, e. g., $\mathcal{A}(\mu) \cap C(\mu \curvearrowright \mu') \cap C(\mu \curvearrowright \mu'') \ne \emptyset$ where staying in mode μ, switching to mode μ' or switching to mode μ'' are all feasible according to (5). Of course one cannot expect unique solutions of the closed-loop system in the case of such switching rules and an appropriate theory must handle such non-determinism.

Now observe that the nonempty sets $C(\mu \curvearrowright \mu')$ in (5) imply a switching structure in the form of an underlying modal transition graph; see Figure 1 for an illustration. Also observe that the switching rules (5) do permit cascades, i. e., compound jumps $\mu \curvearrowright \mu' \curvearrowright \cdots \curvearrowright \mu''^{\cdots\prime}$ (abbreviated as $\mu \curvearrowright\!\!\curvearrowright \mu''^{\cdots\prime}$) occuring at the same 'normal time' t. Indexing so $\mu_0 = \mu, \cdots, \mu_N = \mu''^{\cdots\prime}$ (where the length of the cascade is $N = \#\{\mu, \cdots, \mu''^{\cdots\prime}\}$) and setting $s_n = (t, n)$ for $n = 0, 1, \cdots, N$, this requires

$$y(s_n) \in C(\mu_n \curvearrowright \mu_{n+1}) \text{ for } 0 \le n < N, \qquad y(s_N) \in \mathcal{A}(\mu_N) \tag{6}$$

with the next control event, if any, being a switch to some mode $\mu_{N+1} \ne \mu_N$ at a time $t' > t$. We will continue to use this terminology and notation even for $N = 1$, when this would not represent a true cascade.

Independently of any feedback structure, we will call switching sequences $\mu(\cdot) : \mathcal{T}_* \to M$ admissible, if

i) for any two consecutive (distinct) modes μ, μ' of the sequence, the directed edge $[\mu \to \mu']$ is in the modal transition graph (feasibility),

ii) there are only finitely many switches $\mu \curvearrowright \mu'$ in each finite interval in 'normal time' (non-Zenoness).

With this in mind we consider possible paths $[\mu_0 \to \mu_1 \to \cdots \to \mu_{N+1}]$ in the transition graph and the corresponding sets

$$\mathcal{B}[\mu_0, \mu_1, \cdots, \mu_{N+1}] = C(\mu_0 \curvearrowright \mu_1) \times \cdots \times C(\mu_{N-1} \curvearrowright \mu_N) \times \mathcal{A}(\mu_N) \times C(\mu_N \curvearrowright \mu_{N+1}).$$

We can then define $\Delta[\mu_0, \mu_1, \cdots, \mu_{N+1}]$ by

$$\Delta[\cdots] = \inf \left\{ \sum_{n=0}^{N} |\eta_{n+1} - \eta_n| : (\eta_0, \cdots, \eta_{N+1}) \in \mathcal{B}[\mu_0, \cdots, \mu_{N+1}] \right\}. \tag{7}$$

We do impose the following assumptions on the switching rules.

Hypothesis 1 1) Each $C(\mu \curvearrowright \mu')$ is closed (and empty unless $[\mu \to \mu']$ is an edge of the modal transition graph).

2) For each $\mu \in M$ we have $\left(\mathcal{Y} \setminus \left[\bigcup_{\mu' \ne \mu} C(\mu \curvearrowright \mu')\right]\right) \subset \mathcal{A}(\mu)$.

3) There exists $\Delta_* > 0$ such that $\Delta[\mu_0, \mu_1, \cdots, \mu_{N+1}] \ge \Delta_*$ for each path.

4) For any $V > 0$, there exists $N_* = N_*(V)$ such that $\Delta[\mu_0, \mu_1, \cdots, \mu_{N+1}] > V$ whenever $N > N_*$.

These are, of course, purely geometric verifiable conditions on the sets $\mathcal{A}(\mu)$ and $C(\mu \curvearrowright \mu')$ which define the switching rules.

With the Hypotheses 1 at hand, we have the following Theorem.

Theorem 1 *Suppose, for some T and $\delta > 0$, $y \in \boldsymbol{BV}_*([T, T+\delta]_*)$. Then, $\mu(\cdot)$ given by* (5) *is an admissible switching signal on some $[T, T+\delta]_*$.*

Proof Consider any time interval $\mathcal{I}_* = [T, T+\delta]_*$ and let $V = \mathrm{Var}(y(\cdot), \mathcal{I}_*)$. The Hypothesis 1.4 ensures, provided one has non-Zenoness, that the switching signal can always be constructed causally on $\mathcal{I}_*$ by (5) with no cascade longer than $N_*(V)$. Note that the domain $\mathcal{I}'_*$ of $\mu(\cdot)$ for the normal time $\mathcal{I} = [T, T+\delta]$ is here determined by this signal, as it is constructed. Clearly, from 3) of Hypothesis 1 and (6), the separation distance between y at the switching event starting any cascade and at the start of the next cascade must be at least Δ_* — i. e., if t_k and t_{k+1} are consecutive switching times in $\mathcal{I}$, then

$$\mathrm{Var}(y, [t_k, t_{k+1}]_*) \geq \Delta_*.$$

By subadditivity of the variation, $V \leq \sum_k \mathrm{Var}(y, [t_k, t_{k+1}]_*) > K\Delta_*$, where K is the number of such completed cascades during $[T, T+\delta]$. Thus, we must have $K \leq V/\Delta_*$ and so at most $(K+1)N_*$ switchings altogether. ∎

3 Semilinear Hyperbolic Systems in $\boldsymbol{BV}_*$

As an example for a $\boldsymbol{BV}_*$ setting, we consider semilinear transport/reaction problems and, as in [12] will be interested in feedback-control of the kind described by switching rules of the form (5). We wish to extend the results of [12] in two ways: first to be able to handle matrix problems (in particular, systems of equations that can be regarded as the linearization of the shallow-water equations, the Euler-equations for gas-flow in pipes, equations of traffic flow, multi-commodity flow, etc.) and, second, to the BV setting (which paves the way to a potential treatment of the fully nonlinear problems).

We, therefore, consider a ν-component family (parameterized by μ) of reaction/transport systems

$$u_t = A^\mu u_x + f^\mu(u), \quad 0 < x < 1, \ t > \underline{t} \tag{8}$$

with sufficiently regular $A^\mu = A^\mu(t,x)$ and $f^\mu(u) = f^\mu(t,x,u)$; we assume the input data provided at the ends $x = 0, 1$ as appropriate will be suitable, e. g., in $\boldsymbol{BV}_*$; we also assume the initial data $\bar{u}$ provided at $\underline{t}$ will be suitable.

Further, we consider partial state observation, determined by point observations on some finite set $(x_1, x_2, \cdots, x_N)$ of sensor locations, chosen in the interior of $[0,1]$, so

$$y(t) = \mathcal{P}u(t,\cdot) = [u(t,x_1), \cdots, u(t,x_N)] \in \mathbb{R}^n \quad \text{with } n = N\nu. \tag{9}$$

Note that this assumes we observe each component at each observation point, but that is not necessary. We could also have included observation of the input data if desired, but that, too, is not required. Note, finally, that we are assuming, without further mention, that $0 < x_1 < \cdots < x_N < 1$.

The combined evolution of (5) and (8) for given initial data $(\bar{\mu}, \bar{u})$ at $t = 0$ will then be given by a sequence

$$(\bar{\mu}, 0, \bar{u}) \to (\mu_1, \delta_1, u_1) \to (\mu_2, \delta_2, \mu_2) \to \cdots \tag{10}$$

with each u_k solving (8) with $\mu = \mu_k$, $\underline{t} = \delta_k$, and $\bar{u}_{k+1} = u_k(\delta_{k+1})$. Note that the evolving state is given by u_k on the time interval $[\delta_k, \delta_{k+1}]$ so there is no evolution on intervals of length 0, when $\delta_{k+1} = \delta_k$ as part of a cascade; we then have $u_{k+1}(\delta_{k+1}) = u_k(\delta_k)$. To have $\delta_{k+1} > \delta_k$ we must have $y(t) \in \mathcal{A}(\mu_k)$ on the time interval (δ_k, δ_{k+1}) while at the event times we must have $y(\delta_k) \in C(\mu_{k-1} \curvearrowright \mu_k)$ for each $k = 1, 2, \cdots$.

As already in [12], we wish to consider data for which the appropriate treatment of $s \in [0,1]$ allows us to distinguish 'points' with the same nominal position (changing in time), but with well-defined ordering. This distinction is now to be viewed as permitting phenomena on a finer spatial scale: compare (1) in Section 1. We think of this also as introducing an augmented version $[0,1]_*$ of the 'normal interval' $[0,1]$ by taking

$$[0,1]_* \subset \mathbb{R} \times \mathbb{Z} \tag{11}$$

much as for temporal intervals. We will use the same notation and definition (2) as for augmented temporal intervals, noting that the nature of one-dimensional transport systems is that the treatments of time and space should correspond through the characteristics.

We will impose the following assumptions, holding for each μ.

Hypotheses 2 1) The matrix functions A^μ depend smoothly on (t,x) and each $A^\mu(t,x)$ has distinct non-zero eigenvalues: $\lambda_k^\mu = \lambda_k^\mu(t,x) \neq 0$.

2) The reaction term f^μ is bounded ($|f^\mu| \leq \beta$) and is uniformly Lipschitzian in u (with a Lipschitz-constant L).

These are not minimal hypotheses: for example, the bound on $|f^\mu|$ is deducible from the Lipschitz condition and a bound for initial data.

Assuming that 1) of Hypotheses 2 holds, we can set

$$D^\mu = D^\mu(t,x) = \operatorname{diag}\{\lambda_k^\mu\} = P^\mu A^\mu (P^\mu)^{-1}, \quad \widehat{u} = P^\mu u$$

and

$$\widehat{f}^\mu(t,x,\widehat{u}) = \widehat{f}^\mu(\widehat{u}) = P^\mu f^\mu((P^\mu)^{-1}\widehat{u}) + D^\mu P_x^\mu (P^\mu)^{-1}\widehat{u} \tag{12}$$

to get a system

$$\widehat{u}_t + D^\mu \widehat{u}_x = \widehat{f}^\mu(\widehat{u}). \tag{13}$$

In order to use the method of characteristics we let

$$t \mapsto \sigma(t) = \sigma_k(t; t_*, x_*)$$

satisfy the ordinary differential equation

$$\dot{\sigma} = -\lambda_k^\mu(t,\sigma), \qquad \sigma(t_*) = x_*, \tag{14}$$

so (13) becomes a coupled system of ODEs for the components ω_k of $\widehat{u}$

$$\frac{d}{dt}\omega_k(t,\sigma_k(t)) = \widehat{f}_k^\mu(t,\sigma_k(t),\omega_1,\cdots,\omega_K) \tag{15}$$

with the components $\widehat{f}_k^\mu$ of (12). Observe, from (15), that singularities of each ω_k can propagate only along the characteristics σ_k.

In the following we will drop the $\widehat{\ }$ and simply assume A^μ was given as diagonal in (8) so we actually start with (13), but note both the regularity required to include P_x^μ in $\widehat{f}^\mu$ and the necessity of re-interpreting the results if we had really needed to make the change of variables.

For our present purposes in this section we assume the families of characteristics $\sigma_k(\cdot)$ are already given for each mode μ and will then actually start with the integral equation form of (15) — see (17) below — with only minimal concern for the regularity needed to derive this from previous forms. Our only significant assumptions here are following.

Hypotheses 3 1) For each k and for each $t > t_0$ we have $\sigma_k(\tau) = \sigma_k(\tau; t, x)$ defined and monotone in τ for $t_* \leq \tau \leq t$ with $\sigma_k(t; t, x) = x$. We assume each σ_k is either increasing

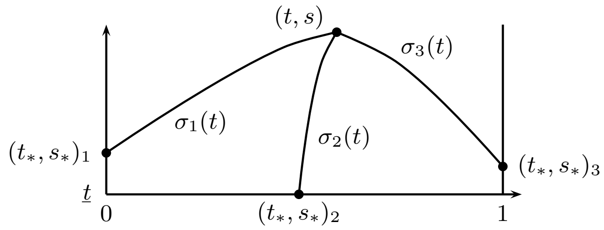

Figure 2 Typical set of characteristics $\sigma_k(t)$

(corresponding to $\lambda_k > 0$ in the setting of a smooth matrix problem, as earlier) or decreasing (corresponding to $\lambda_k < 0$). Here $p_* = (t_*, x_*)$ denotes the 'starting point': $\sigma_k(\cdot)$, going backward in τ from $p = (t, x)$, always hits either the initial time ($t_* = t_0$ with $x_* = \sigma_k(t_*)$ in $[0, 1]$) or the appropriate input boundary ($t_0 \leq t_* \leq t$ with $x_* = \sigma_k(t_*) = 0$ or 1, depending on whether σ_k is increasing or decreasing). See Figure 2 for an illustration.

2) For $x' \prec x$ in $[0, 1]$ we have

$$\sigma_k(\tau; t, x') \prec \sigma_k(\tau; t, x) \quad \text{for each } \tau \leq t \tag{16}$$

provided $t_*, t'_* \leq \tau$; similarly, when the characteristic hits the relevant input boundary, we require $t'_* \prec t_*$ for increasing σ_k and $t_* \prec t'_*$ for decreasing σ_k.

Note that we do not insist that $\sigma_k(\tau; t, x)$ should depend continuously on (t, x) and if x, x' correspond to the same position (so, in our regular notation we have $x' = x$, say with $x' \prec x$), then we need not assume that $\sigma_k(\tau; t, x') = \sigma_k(\tau; t, x)$ although we do assume (16). In particular, if switching might occur during a time interval under consideration, we assume one can 'restart' each characteristic across the switching time.

At this point we are ready for our main concern of this paper: to prove the following existence result.

Theorem 2 *Under the Hypotheses* 1, 2, *and* 3 *the transport/reaction problem* (8) *with feedback* (5) *has solutions.*

Proof As usual, one works with (15) as a system of integral equations:

$$\omega_k(t, s) = \omega_k(p_*) + \int_{t^*}^{t} f_k^\mu(t, \sigma_k(\tau; t, x), \omega(\tau, \sigma_k(\tau; t, x)))\, d\tau \tag{17}$$

for $k = 1, 2, \cdots, \nu$ — coupled through the evaluation of f_k^μ at $u = [\omega_1, \omega_2, \cdots]$ in the integral. We will henceforth take (17) as defining our notion of a 'solution' of what we continue to write in the form (15) or (13) and so as defining our notion of solutions of (8).

Our strategy is to construct $\mu(\cdot)$ and $\omega = [\omega_1, \omega_2, \cdots]$ on short time intervals $[T, T + \delta]$, proceeding recursively. The key to this is the observation that, for the problem (8), one can choose $\delta > 0$ such that the observation $y(\cdot)$ on any $[T, T + \delta]_*$ can depend on switching prior to T. Our main concerns, then, will be to show, firstly that the constructed state evolution is such that $y(\cdot) \in \boldsymbol{BV}_*[T, T + \delta]$ and, second, that this ensures that any $\mu(\cdot)$ consistent with (5) will be in $\boldsymbol{BV}_*[T, T + \delta]$.

Thus, we begin by considering (17) on a time interval $[T, T + \delta]$ for which we assume the switching signal $\mu(\cdot)$ has been given (so this is really $[T, T + \delta]_*$) and we also have given the initial and input data. It is then standard to see that the right hand side of (17) defines a contraction mapping on the space $L^\infty([T, T + \delta]; L^1([0, 1]; \mathbb{R}^\nu))$ (with a suitable, exponentially weighted, norm), so a solution exists there. What is missing in that for our present purposes is a priori estimate for $\mathrm{Var}(u(t,\cdot);[0,1]_*)$. As in (2), we let

$$S = [0 = x_0 \prec x_1 \prec \cdots \prec x_N = 1]$$

and, temporarily fixing k, consider (17) for each $x = x_n$ with corresponding characteristics $\sigma_k(\tau; t, x_n)$. For exposition, we will assume for this k that the characteristic curves σ_k are right-moving with increasing t so the input boundary is at $x_* = 0$. In this case we note that if $x' \prec x''$ with $x'_* = x''_* = 0$, then $t'_* \succ t''_*$. Without loss of generality, we take the 'initial' time as T. We assume that $\delta = t - T$ is small enough that $\sigma(\cdot; t, 1)$ hits the initial time so $t_* = T$ for that characteristic. Again without loss of generality, we may assume that $x_{\bar{n}} \in S$ is such that $\sigma(T; t, x_{\bar{n}}) = 0$ (i. e., $\sigma(t; T, 0) = x_{\bar{n}}$) so $t_{n*} = T$ for $n \geq \bar{n}$ and $x_{n*} = 0$ for $n \leq \bar{n}$. Then

$$\begin{aligned}
&\omega_k(t, x_n) - \omega_k(t, x_{n-1}) \\
&= \omega_k(p_{n*}) - \omega_k(p_{(n-1)*}) + \int_{t_{n*}}^{t_{(n-1)*}} f_k^\mu(\tau, \sigma_k(\tau; t, x_n), u(\tau, \sigma_k(\tau; t, x_n)))\, d\tau \\
&\quad + \int_{t_{(n-1)*}}^{t} \big[f_k^\mu(\tau, \sigma_k(\tau; t, x_n), u(\tau, \sigma_k(\tau; t, x_n))) \\
&\qquad\qquad - f_k^\mu(\tau, \sigma_k(\tau; t, x_{n-1}), u(\tau, \sigma_k(\tau; t, x_{n-1})))\big]\, d\tau
\end{aligned}$$

Using the bounds on f^μ assumed in 2) of Hypotheses 2, we take absolute values and sum over n to get

$$\begin{aligned}
&\sum_{n=1}^{N} |\omega_k(t, x_n) - \omega_k(t, x_{n-1})| \\
&\leq \sum_{n=1}^{\bar{n}} |\omega_k(t_{n*}, 0) - \omega_k(t_{(n-1)*}, 0)| + \sum_{n=\bar{n}+1}^{N} |\omega_k(T, x_{n*}) - \omega_k(T, x_{(n-1)*})| \\
&\quad + \sum_{n=1}^{\bar{n}} \beta[t_{(n-1)*} - t_{n*}] + L \int_T^t \sum_{n=1}^{N} |u(\tau, \sigma_k(\tau; t, x_n)) - u(\tau, \sigma_k(\tau; t, x_{n-1}))|\, d\tau \\
&\leq \mathrm{Var}(\omega_k(\cdot, 0); [T, t]_*) \ + \ \mathrm{Var}(\omega_k(T, \cdot); [0,1]_*) \\
&\quad + \beta(t - T) + L \int_T^t \mathrm{Var}(u(\tau, \cdot); [0,1]_*)\, d\tau
\end{aligned}$$

noting that $S_\tau = (\sigma(\tau; t, x_{\bar{n}}), \cdots, \sigma(\tau; t, x_N))$ and $S_T = (0 = x_{\bar{n}*}, \cdots, x_{N*})$ each partition (part of) $[0,1]_*$ and that $(t_{\bar{n}*}, \cdots, t_{0*})$ partitions $[T, t]_*$; taking the supremum over S gives

$$\begin{aligned}
&\mathrm{Var}(\omega_k(t, \cdot); [0,1]_*) \\
&\leq \mathrm{Var}(\omega_k(\cdot, 0); [T, t]_*) + \mathrm{Var}(\omega_k(T, \cdot); [0,1]_*) \\
&\quad + \beta(t - T) + L \int_T^t \mathrm{Var}(u(\tau, \cdot); [0,1]_*)\, d\tau. \qquad (18)
\end{aligned}$$

Essentially, the same estimate holds for each k, noting only that the input data would either be at $x = 0$ for increasing σ_k or at $x = 1$ for decreasing $\sigma_{k'}$. We may then sum over k to get a similar integral estimate for $\mathrm{Var}(u(t,\cdot);[0,1]_*)$ and then apply the Gronwall inequality to bound $\mathrm{Var}(u(T+\delta,\cdot);[0,1]_*)$ directly in terms of the variations for initial data and input data.

It is important to realize here that the input data will include the effects of switching during the interval $[T, t]$ along with any exogamous input. Using the estimate recursively for

$T = 0, \delta, \cdots, m\delta, \cdots$, we have a bound on $\mathrm{Var}(u(t,\cdot);[0,1]_*)$ $(0 \le t \le T)$ in terms of T, of bounds on the variations of the initial data and of the exogamous inputs (over $[0,T]$), and a bound on the number of switches during $[0,T]$.

This proves that the solution we initially obtained in L^1 by a contraction mapping argument is, indeed, in $\boldsymbol{BV}_*$ with an estimable bound on the spatial variation $\mathrm{Var}(u(t,\cdot);[0,1]_*)$ at fixed times. Essentially, the same argument can be used to bound the temporal variation $\mathrm{Var}(u(\cdot,\bar{s});[T,T+\delta]_*)$ at a fixed location $\bar{x}$, although this is treated somewhat differently when $\bar{x}$ is a sensor location, assumed internal to $(0,1)$, or is an output boundary.

We next consider intervals $\mathcal{I}(\tau) \subset (0,1)$ for $\tau \in [T,t]$ such that for each k and each $x \in \mathcal{I}(\tau)$ one has $\sigma_k(\tau';\tau,x) \in \mathcal{I}(\tau')$ for each $\tau' \in [T,\tau]$. What we have in mind is $\mathcal{I}(\tau) = [\sigma_-(\tau;t,x_-), \sigma_+(\tau;t,x_+)]$ where $\mathcal{I}(t) = [x_-,x_+] \subset (0,1)$ and where $\sigma_\pm(\cdot)$ are the most rapidly increasing and most rapidly decreasing families of characteristic curves. We are here assuming that $t-T$ is small enough that this gives $\mathcal{I}(T) \subset (0,1)$.

Let $v_k(\tau) = \mathrm{Var}(\omega_k(\tau,\cdot);\mathcal{I}(\tau)_*)$ and $\bar{v}(\tau) = \mathrm{Var}(u(\tau,\cdot);\mathcal{I}(\tau)_*)$. Much as in the derivation above of (18) — only simpler because all of the relevant characteristics remain in $\{(\tau,s) : s \in \mathcal{I}(\tau),\ T \le \tau \le t\}$ without hitting input boundaries — we now track back a partition of $\mathcal{I}(\tau)$ along $\sigma_k(\cdot)$ for some k and use (17) to obtain

$$v_k(\tau) \le v_k(T) + \beta(\tau - T) + L\int_T^\tau \bar{v}(\tau')\,d\tau'. \tag{19}$$

Summing over k and then using the Gronwall inequality, we obtain a bound on $\bar{v}(\tau) = \mathrm{Var}(u(\tau,\cdot);\mathcal{I}(\tau))$ on $[T,t]$ in terms of $\bar{v}(T)$.

For a sensor location $\bar{x} \in (0,1)$, we now assume $t-T$ is small enough that we can take $\bar{x} \in \mathcal{I}(t)$ above with $\mathcal{I}(T) \subset (0,1)$. To estimate the sensor variation $\mathrm{Var}(u(\cdot,\bar{x}) : [T,t]_*)$, we next take

$$S = [T = \tau_0 \prec \tau_1 \prec \cdots \prec \tau_N = t].$$

Much as earlier, we use (17) to obtain

$$\begin{aligned}\omega_k(t,\tau_n) - \omega_k(t,\tau_{n-1}) = {} & \omega_k(T,x_{n*}) - \omega_k(T,x_{(n-1)*}) \\ & + \int_{\tau_{n-1}}^{\tau_n} f_k^\mu(\tau,\sigma_k(\tau;\tau_n,\bar{x}),u(\tau',\sigma_k(\tau;\tau_n,\bar{x})))\,d\tau \\ & + \int_T^{\tau_{n-1}} \big[f_k^\mu(\tau',\sigma_k(\tau;\tau_n,\bar{x}),u(\tau,\sigma_k(\tau;\tau_n,\bar{x}))) \\ & \qquad - f_k^\mu(\tau',\sigma_k(\tau;\tau_{n-1},\bar{x}),u(\tau,\sigma_k(\tau;\tau_{n-1},\bar{x})))\big]\,d\tau.\end{aligned}$$

Then, taking absolute values and summing over n, we get

$$\begin{aligned}&\sum_{n=1}^N |\omega_k(t,\tau_n) - \omega_k(t,\tau_{n-1})| \\ &\le \sum_{n=1}^N |\omega_k(T,x_{n*}) - \omega_k(T,x_{(n-1)*})| + \beta(t-T) \\ &\quad + L\int_T^t \sum_{n=1}^N |u(\tau,\sigma_k(\tau;\tau_n,\bar{x})) - u(\tau',\sigma_k(\tau;\tau_{n-1},\bar{x}))|\,d\tau.\end{aligned}$$

Noting that $[x_{n*} = \sigma_k(T;\tau_n,\bar{x})]$ and $[\sigma_k(\tau';\tau_n,\bar{x})]$ are, in reversed order, partitions of (parts of) $[\mathcal{I}(T)]_*$ and $[\mathcal{I}(\tau')]_*$, this gives

$$\mathrm{Var}(\omega_k(\cdot,\bar{x}),[T,t]_*) \le v_k(T) \ + \ \beta(t-T) \ + \ L\int_T^t \bar{v}(\tau)\,d\tau \tag{20}$$

with the observation that (19) already bounds the right hand side here in terms of $\bar{v}(T)$. We can always choose $\delta > 0$ so that $\delta < x_1/\lambda_k^\mu$ for each positive λ_k^μ and $\delta < x_N/-\lambda_k^\mu$ for each negative λ_k^μ so input up to time T cannot reach any sensor point x_n along any characteristic by $T+\delta$ and so cannot affect the observation $y(\cdot)$ on $[T,T+\delta]_*$. We can then apply Theorem 1 on each subinterval of length δ to complete the proof. ∎

While we worked with the transport equation only on a single simple segment, we note that the treatment here extends with only minor changes to the case of transport on a graph, as would be the setting for a gas pipeline network or a highway traffic system — the only essential element of that which we have not considered here is a good treatment of the nodal conditions governing the distribution of material flowing through nodes of that graph.

4 Convergence in $\boldsymbol{BV}_*$

Let us review briefly the formulation of $\boldsymbol{BV}_*$. We consider, first, the switching signal $\mu(\cdot)$. This has a well-defined sequential order with transitions $\mu_k \curvearrowright \mu_{k+1}$ and is associated with the passage of (normal) time so each such modal transition occurs at a specified switching time t_k. For admissilbility of such a switching signal we require

1) order is preserved: $k \ge k'$ implies $t_k \ge t_{k+1}$ (note that we do not require that the $\{t_k\}$ be distinct on our slow scale);

2) there are only finitely many such transitions within any finite interval.

The interpretation here is that a set of switching actions taking place "at the same time" really are occurring in sequence on the fast scale, which is left largely unmodeled, so we may have $t_k = t_{k+1}$ but still $t_k \prec t_{k+1}$. As noted, one notational device for recognizing this is the use of the augmented time interval $\mathcal{T}_* \subset \mathbb{R}\times\mathbb{Z}$ as in (1). The variation $\mathrm{Var}(\mu(\cdot);\mathcal{I})$ is here defined as the number of modal transitions within $\mathcal{I}$.

For eventual purposes of considering both 'well-posedness' and optimal switching, we wish a topology for these sequences and, in the presence of a bound as in 2) above on the number of switchings, take $\mu^\nu \to \bar{\mu}(\cdot)$ to mean that, for each k, one has both $t_k^\nu \to \bar{t}_k$ and $\mu_k^\nu = \bar{\mu}_k$ for large ν. Note that the number of distinct (normal time) switching times cannot increase in the limit. Bounding the number of switchings and the (normal time) length of the interval will ensure compactness for this topology.

Next, consider the construction of such a switching signal dynamically by feedback. We are here assuming that at each moment t (of the effective time — which is also being created dynamically) we have a sensor output value $y(t) \in \mathcal{Y}$, obtained by (partial) observation of the state and perhaps of some external inputs and that (5) uses this to construct $\mu(t)$.

What is needed for an effective theory is that our definition of the appropriate space $\boldsymbol{BV}_*$ should have the properties:

(a) If we have $y(\cdot) \in \boldsymbol{BV}_*$, then any resulting switching signal $\mu(\cdot)$ produced through the rules (5) should be admissible as above — this is Theorem 1.

(b) If $\mu(\cdot)$ is admissible on $[0,T]$, then the sensor output $y(\cdot)$ produced by the dynamics and observation $\mathcal{P}$ being considered will be in $\boldsymbol{BV}_*([0,T]_*)$ — this the principal point of the argument for Theorem 2.

It is significantly more difficult to give a good general description of the corresponding $\boldsymbol{BV}_*$ for $\mathcal{Y}$-valued functions. For our application it turned out to be entirely satisfactory to define this, both in time and in space, essentially as above for $\mu(\cdot)$, but we note some potential difficulties when we would wish to consider limits and the desirable property:

(c) Each of the reciprocal maps $y(\cdot) \mapsto \mu(\cdot)$ and $\mu(\cdot) \mapsto y(\cdot)$ of (a), (b) will be continuous, using our topology for $\boldsymbol{BV}_*$.

Note that we have not considered this last property in any detail here, but the essential point is that our treatment of the fast dynamics should be rate-independent when normal time interswitching subintervals $[a,b]^\nu$ collapse in the limit $(a^\nu, b^\nu \to \bar{a})$ — the interesting situation is the possibility that this may happen with $\mathrm{Var}(y^\nu, [a,b]^\nu) \not\to 0$ so some slow scale evolution is becoming nontrivial fast scale behavior: we may think of the 'value' $y(\bar{a})$ as some actual fast scale function but, since we leave this unmodeled, we may think of it as an equivalence class of these modulo fast scale order-preserving reparameterizations. What is then needed is inclusion in the collapsed form of just enough information that the output of the map should depend only on this. One might consider for this the more detailed description in [12] for the context of piecewise continuous functions with similar augmentation of the intervals.

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