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A COMPREHENSIVE DEA APPROACH FOR THE RESOURCE ALLOCATION PROBLEM BASED ON SCALE ECONOMIES CLASSIFICATION*

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Abstract This paper is concerned with the resource allocation problem based on data envelopment analysis (DEA) which is generally found in practice such as in public services and in production process. In management context, the resource allocation has to achieve the effective-efficient-equality aim and tries to balance the different desires of two management layers: central manager and each sector. In mathematical programming context, to solve the resource allocation asks for introducing many optimization techniques such as multiple-objective programming and goal programming. We construct an algorithm framework by using comprehensive DEA tools including CCR, BCC models, inverse DEA model, the most compromising common weights analysis model, and extra resource allocation algorithm. Returns to scale characteristic is put major place for analyzing DMUs' scale economies and used to select DMU candidates before resource allocation. By combining extra resource allocation algorithm with scale economies target, we propose a resource allocation solution, which can achieve the effective-efficient-equality target and also provide information for future resource allocation. Many numerical examples are discussed in this paper, which also verify our work.

Key words Common weights analysis (CWA), date envelopment analysis (DEA), decision making unit (DMU), efficiency score, inverse DEA model, multiple-objective linear programming (MOLP), resource allocation problem, returns to scale (RTS).

1 Introduction

Charnes, Cooper, and Rhodes (CCR)[1] first proposed DEA method to evaluate the relative efficiency for not-for-profit organizations, such as government departments, military units, and social service entities. DEA is a valid method in both theoretical and empirical side[2−10], and it is quite available in management process, performance estimation, and behavior analysis. When DEA models are used for analyzing or solving resource allocation problems, it brings dramatic changes. For the first part, DEA can process decision making units (DMUs) with multiple inputs and multiple outputs, and it has the advantage of solving the special resource allocation with that kind of DMUs. For the second part, DEA has a tight contact with economic theory of production forms, and its models use quantitive method such as linear programming, multiplier objective programming, and parameter programming, to access the optimization process in

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economics. As a result, DEA method is adopted in solving resources allocation problem and brings a brilliant scene for both research and practical field.

The resources allocation in DEA's context includes the following categories: the resource reallocation problem[9−10,16−18], the efficient resource allocation problem[7−8,11−12], the extra resource allocation problem[19,22], the cost allocation problem[23]. Each kind of problems has its particular models' solution but the four kinds share some tools or algorithms. In this paper, we are mostly interested in the efficient allocation problem. The efficient resource allocation problem is a decision problem which is described as follows[8]. Consider a decision-making environment in which a set of units is operating under a central unit with power to control some decision parameters, such as resource of those units. The aim of the central unit is to allocate resource in such a way that the overall goals of the organization are satisfied as well as possible, or specifically, the amount of the total outputs of the units will be maximized. Golany[11] and Tamin[12] suggested models that emphasize the importance of resource reallocation as a mean of improved performance. Traditional solution for this kind of problem always focused on the integrated use of DEA and multiple-objective programming (MOLP) or goal programming[7].

Here an intuitional understanding of efficient resource allocation problem is that it is an optimization problem which should satisfy equilibrium of multiple criteria. For example, for a central manager, who controls the resources of the corporation or department, hopes to increase the whole production of the team. He/She will increase the efficiency or stimulate the staffs by allocating limited resources fairly enough to create maximization profits. Otherwise, every single staff or department will only concern with his/her own profit and hopes to get more resources than the other individuals. So, how to balance the different desires of the two management levels (the operator of each sector, and the manager of the organization) and find the efficient-effective-equality allocation is a tough task worth of challenge. It is the first major factor should be paid attention to.

Another major factor that influences the efficiency of resource allocation is the returns to scale[6]. In extra resource allocation, scale and efficiency are concerned in the solution by using CCR model. But CCR ratio provides the aggregate technical and scale efficiency[6], it could not indicate the details of returns to scale. Banker[6] indicated that returns to scale are increasing, constant, or decreasing at any pair of coordinate values (x, y) according to whether marginal product is greater than, equal to, or less than average product. So, returns to scale have a tight contact with the production ability that we should not ignored.

In this paper, we will analyze the efficient resource allocation problem to realize the efficient-effective-equality target. To satisfy the above target, we construct an algorithm framework by using comprehensive DEA tools such as CCR[1], BCC[6] models, inverse DEA model[9−10,16−17], common weights analysis model[3−4,18−19], and extra resource allocation algorithm[19]. Each model or algorithm is a sub model in our algorithm framework to solve the resource allocation problem. Here, the resource allocation problem with the plural DMUs can be treated as a selection problem[24], and our classification is according to the returns to scale of DMUs. The production frontier is also based on the BCC model which considering the variable returns to scale.

2 DEA Background

2.1 Two Basic DEA Models: CCR, BCC

DEA is a linear programming based on technique that measures the relative efficiency of comparable units, usually referred to as DMUs. Charnes, Cooper, and Rhodes introduced the CCR ratio definition[1] which generalized the single-output to single-input classical engineering-science ratio definition to multiple outputs and inputs without requiring preassigned weights, subject to the condition that the same ratio for all DMUs must be less than or equal to 1. This is done via the extremal principle incorporated in the following model:

$$
\begin{aligned}
(\mathrm{CCR_{FP}}) \quad \max \quad & h_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, \quad j = 1, 2, \cdots, n, \\
& u = (u_1, u_2, \cdots, u_s)^{\mathrm{T}} \geqslant 0, \\
& v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}} \geqslant 0,
\end{aligned}
\tag{1}
$$

where the $y_{rj}, x_{ij} > 0$ represent output and input data for DMU$_j$, an optimal $h_o^* = \max\, h_o$ will always satisfy $0 \leq h_o^* \leq 1$ with optimal solution values $u_r^*, v_i^* > 0$.

The above fractional programming (1) can be replaced by a linear programming. The dual model is always used in realistic applications, which is expressed with a real variable θ and a nonnegative vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^{\mathrm{T}}$ of variables as follows[13]:

$$
\begin{aligned}
(\mathrm{CCR_{DLP}}) \quad \min \quad & \theta \\
\text{s.t.} \quad & X\lambda \leqslant \theta x_o, \\
& Y\lambda \geqslant y_o, \\
& \lambda \geqslant 0.
\end{aligned}
\tag{2}
$$

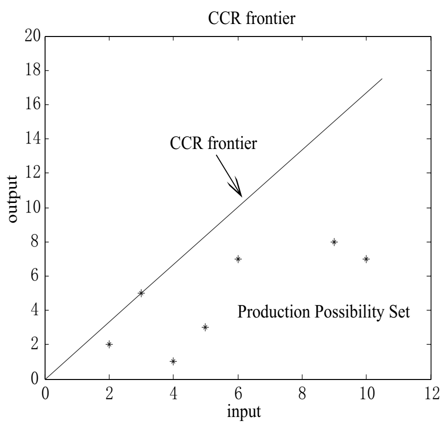

Figure 1 CCR Production Frontier

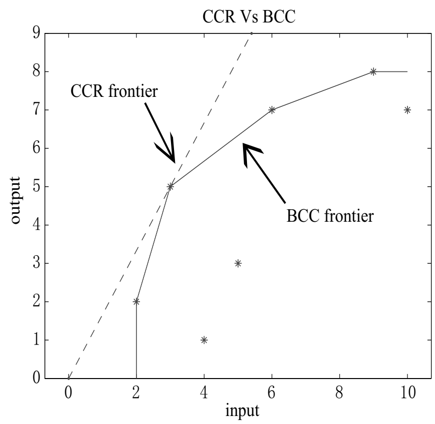

Figure 2 Frontiers of CCR model and BCC model

CCR refers to their production function as an "envelope" developed relative to observational data from all of the $j = 1, 2, \cdots, n$, DMUs, with the envelope forming an efficiency frontier relative to each DMU that is to be evaluated. Considering the DMUs with single input and single output, we regard the ray from the origin to point $(3, 5)$ as the production frontier under CCR concept. We can see from Figure 1 that the frontier touches at least one point and all points are therefore on or below this line. The region between the production frontier and the x-coordinate is called the production possibility set.

Definition 1 (Production possibility set, P) Arranging the data sets in matrices $X = (x_j)$ and $Y = (y_j)$, the production possibility set P is satisfying: $P = \{(x, y)|x \geq X\lambda,\ y \leqslant Y\lambda,\ \lambda \geq 0\}$, where λ is a semipositive vector in R^n.

The CCR model is built on the assumption of constant returns to scale (CRS) of activities which satisfies the below definition.

Definition 2 (Constant returns to scale) If an activity (x, y) in P, then the activity (tx, ty) belongs to P for any positive scalar t. We call this property the constant returns to scale assumption.

Since the very beginning of DEA studies, various extension of the CCR model have been proposed, among which the BCC (Banker-Charnes-Cooper) model is representative. The BCC model has its production possibility set and production frontiers different from CCR model. BCC model is presented as follows[6,13]:

$$
\begin{aligned}
\text{(BCC)} \quad \min \quad & \theta_B \\
\text{s.t.} \quad & X\lambda \leqslant \theta_B x_o, \\
& Y\lambda \geqslant y_o, \\
& e\lambda = 1, \\
& \lambda \geqslant 0.
\end{aligned}
\tag{3}
$$

BCC introduced variable returns to scale (VRS) which envelops data more tightly than constant returns to scale (CRS). Figure 2 gives the comparison of different production frontiers of CRS and VRS.

2.2 Returns to Scale

In economics, it is very important to analyze the economies of scale. We can define it by the formulation $y = f(x)$, where y is maximal scale of output for every input x under the assumption that technical efficiency is always achieved. Figure 3 shows the scale economies function in CCR model with single input and single output form. Curve OS' is nonlinear and indicates economies of scale, which envelopes all the n DMUs observed and also the production possibility set. Ray line OF' indicates the CCR production frontier. Points like A or C which lie within the production possibility set, are not on the CCR frontier, points like B which lies on the frontier are thought to be technically efficient.

Let (x_B, y_B) denote point B's coordinates, as can be seen in Figure 3, the slopes of the rays OS' increase with x until x_B is reached after which the slopes of the corresponding rays begin to decrease. In a similarly visualized manner, the derivative $\frac{dy}{dx}$ increases with x until the inflection point B for $f(x)$ is reached after which it begins to decrease. This means that output is changing proportionally faster than input to the left of x_B while the reverse situation occurs to the right of x_B.

Assuming that the curve OS' stands for the real scale economies function, the scale efficiency of DMU A is $SE_A = \frac{PM}{PN}$. Because the real production function is difficult to calculate, Banker et al.[6] adopted the BCC frontier to approximate the scale economies frontier, and under this assumption, the returns to scale concept was introduced. Supposed the VRS envelopment line $\overrightarrow{QABCV'}$ portrayed in Figure 3 stands for the approximate scale economies function, we can calculate some efficiency indices of DMU A as follows:

(Input) Technical Efficiency$_{\text{CRS}}$ ($\text{TE}_{I,CRS}$) $= \frac{PM}{PA}$,

(Input) Technical Efficiency$_{VRS}$ ($\text{TE}_{\text{I,VRS}}$) $= \frac{PA}{PA} = 1$,

(Input) Scale efficiency $(SE) = \frac{PM}{PA}$,
Technical and Scale Efficiency $=\frac{PM}{PA} = \mathrm{TE}_{\mathrm{I,CRS}} = \mathrm{TE}_{\mathrm{I,VRS}} * \mathrm{SE}$.

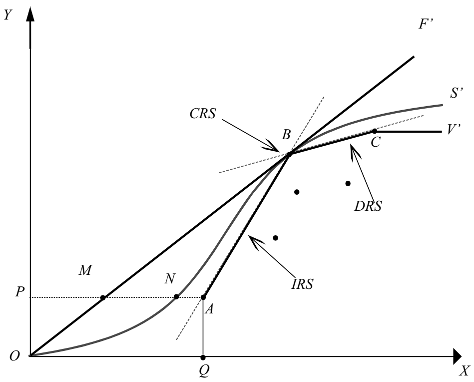

Figure 3 Returns to Scale

In classical economics, marginal product is used to measure the returns to scale; in VRS context, increasing, constant, decreasing returns to scale are used to measure the scale economies of different DMUs. As portrayed in Figure 3, the intercept value associated with the tangent line $\overrightarrow{AB}$ are presented for the increasing returns to scale type. The intercept value associated with the tangent line $\overrightarrow{BC}$ and the succeeding piecewise segment $\overrightarrow{CV'}$ on the efficient production possibility frontier are presented for the decreasing returns. Point B is constant returns to scale.

In realistic application, CCR and BCC models are combined to determine the returns to scale of DMU[14−15]. In Figure 3, point A represents increasing returns to scale DMUs, whose output is changing proportionally faster than input. Otherwise, point B represents decreasing returns to scale DMUs, whose output grows proportionally slower than input.

2.3 Inverse DEA Method and Its Basic Model

Zhang and Cui[9] first developed a project evaluation system which used DEA as a main tool for performance forecasting and resource estimation. The problem they solved is: if the kth DMU plans to increase its outputs by Δy, how much additional inputs Δx should be allocated to the DMU to satisfy the efficiency expectation; or if the DM invests increment inputs Δx to the kth DMU, how much additional outputs Δy can the kth DMU produce? The first problem is discussed in detail in that paper, and it can be modeled to an m-dimensional parameter programming. Zhang and Cui transformed the original m-dimensional parameter problem to a one-dimensional parameter problem. Wei and Zhang[10] extended the above two problems are extended to a more general field characterized by inverse DEA problem, which extends the concept of the inverse optimization problem to the DEA context.

Inverse DEA method provides a new tool available to management scientists for performance analysis. The question is[16]: among a group of DMUs, if a DMU attempts to increase or decrease some of its input levels while maintaining its relative efficiency position among the group, what are the changes to the outputs this DMU would expect? Or the other way around, if a DMU wants to increase or decrease some of its output levels while maintaining its efficiency position, how much additional or reduced resources (inputs) are needed? Wei and Zhang[10] transformed the inverse DEA problem into a multi-objective programming problem. It is also shown that in some special cases, the inverse DEA problem can be simplified as a single-objective linear programming problem. In [17], a common algorithm is provided to solve the inverse DEA problem discussed above, the algorithm is focused on solving the following model:

$$
\begin{aligned}
(\mathrm{MOP}) \quad \max \quad & (\Delta y_{1o}, \Delta y_{2o}, \cdots, \Delta y_{so}) \\
\text{s.t.} \quad & X\lambda \leqslant \theta_o(x_o + \Delta x), \\
& Y\lambda \geqslant y_o + \Delta y, \\
& \lambda \geqslant 0,
\end{aligned}
\tag{4}
$$

where the θ_o denotes the efficiency score of DMU_o computed by CCR model.

2.4 Common Weights Analysis

Efficiency assessments incorporate a general view of the relative importance of inputs and outputs, instead of allowing different inputs and outputs weights for every DMUs. By using DEA method, we can get a category of efficient DMUs (eDMUs). But DEA cannot provide enough information to rank the eDMUs, so if one wants to further understand which DMU the best is, he/she needs another indicator to discriminate among the eDMUs, the common weights is just such an indicator.

Using DEA method to solve the common weights is first proposed by Cook et al.[3]. Andersen and Petersen[4] developed procedures for ranking only the efficient units in DEA, Liu and Peng[18] proposed common weights analysis (CWA) model to rank DMUs in the category of efficient. CWA determines an implicit datum under the assumption that the maximum efficiency is equal to 1 among the eDMUs. The CWA model is as follows:

$$
\begin{aligned}
\min \quad & \sum_{i\in E}(|\Delta_I^i| + |\Delta_O^i|) \\
\text{s.t.} \quad & \frac{\sum\limits_{j=1}^{s} u_j y_{ij} + \Delta_O^i}{\sum\limits_{j=1}^{m} v_j x_{ij} + \Delta_I^i} = 1, \quad i \in E, \\
& u_i > 0, \quad i = 1, 2, \cdots, s, \\
& v_i > 0, \quad i = 1, 2, \cdots, m, \\
& \Delta_O^i,\ \Delta_I^i, \quad \text{free}.
\end{aligned}
\tag{5}
$$

By solving the above linear programming formulation, we get a compromise common weights $cw = (v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_s)$ for efficient ranking.

Model (5) has provided a method to calculate the common weights, but it has a limitation that it is only used for efficient DMUs (eDMUs). Generally, an indicator for all the DMUs should be considered. Li and Cui[19] proposed the following θ-CWA model:

$$
\begin{aligned}
(\theta - \mathrm{CWA}) \quad \min \quad & \sum_{i=1}^{n}(|\Delta_I^i| + |\Delta_O^i|) \\
\text{s.t.} \quad & \frac{\sum\limits_{j=1}^{s} u_j y_{ij} + \Delta_O^i}{\sum\limits_{j=1}^{m} v_j x_{ij} + \Delta_I^i} = \theta_i, \quad i = 1, 2, \cdots, n, \\
& u_i > 0, \quad i = 1, 2, \cdots, s, \\
& v_i > 0, \quad i = 1, 2, \cdots, m, \\
& \Delta_O^i,\ \Delta_I^i, \quad \text{free}.
\end{aligned}
\tag{6}
$$

By solving the above linear programming formulation, we get a compromise common weights $\theta - cw = (v_1, v_2, \cdots, v_m,\ u_1, u_2, \cdots, u_s)$ for full ranking.

2.5 Extra Resource Allocation Weights

The extra resource allocation problem is described as follows[19].

suppose there are some extra resource which can be given to all or only a part of DMUs, and if we want the allocation to be most beneficial to the whole system, how the extra resource should be distributed.

This extra resource allocation problem can be generally found in practice. For example, a factory wants to distribute some premium to several outstanding staffs at the end of the year; The chief bank wants to distribute a great deal of bonus to all branch banks; The government wants to serve out food aid to different disaster areas. How should the premium, bonus or food aid be distributed to realize the fair principle and meanwhile make the whole beneficial. Since the selection of decision making units (DMUs) to receive the extra resource should depend on not only its efficiency, but also its scale, this extra resource allocation problem is complicated to be solved. Li and Cui proposed an algorithm to calculate the extra resource allocation weights by using the θ-CWA model[19]. The algorithm is shown as algorithm 1.

Algorithm 1 (Extra resource allocation algorithm):

Given n DMUs and the inputs X and outputs Y, and the $\theta - cw = (v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_s)$ computed by θ-CWA model.

Step 1 for j from 1 to n, compute $Y_j = \sum_{i=1}^{s} u_i y_{ij}$;

Step 2 compute $Y_{\sum} = \sum_{i=1}^{n} Y_i$;

Step 3 for j from 1 to n, compute $W_j = \frac{Y_j}{Y_{\sum}}$;

Output $W = (W_1, W_2, \cdots, W_n)$.

3 Inverse DEA Model Considering the Returns to Scales

3.1 Most Productive Scale Size

Given a DMU$_o$, we can determine whether its returns to scale increase, constant or decrease. The following question is, how to determine the inflexion point in the DMU$_o$'s production function curve. As can be seen in Figure 3, point B is the inflection point, which means the Most Productive Scale Size (MPSS) in the CCR production possibility set.

$$
(\mathrm{CCR}_{\mathrm{Relaxationform}}) \quad \begin{array}{ll} \min & \theta \\ \text{s.t.} & X\lambda + \widehat{s}^- = \theta x_o, \\ & Y\lambda - \widehat{s}^+ = y_o, \\ & \lambda \geqslant 0. \end{array} \tag{7}
$$

Definition 3 By using the results of the relaxation form of CCR model (7), we can estimate the MPSS of DMU$_o$ by the following formula[13]:

$$
\widehat{x}_{io}^* \Leftarrow \frac{\theta_o^* x_{io} - s_i^{-*}}{\sum\limits_{j=1}^{n} \lambda_j^*}, \quad i = 1, 2, \cdots, m \tag{8}
$$

$$
\widehat{y}_{io}^* \Leftarrow \frac{y_{ro} - s_r^{+*}}{\sum\limits_{j=1}^{n} \lambda_j^*}, \quad r = 1, 2, \cdots, s.
$$

We give a theorem as follows to explain the properties of the MPSS defined above.

Theorem 1 *The new DMU_{n+1} with inputs and outputs as $(\widehat{x}_o, \widehat{y}_o)$ calculated by Definition 3 is CCR-efficient.*

Proof Let $\lambda_{\sum} = \sum_{j=1}^{n} \lambda_j^*$, we can get that $\widehat{x}_{io} = \frac{\theta_o^* x_{io} - s_i^{-*}}{\lambda_{\sum}}$. To evaluate the DMU$_{n+1}$ through the following CCR model:

$$\begin{array}{ll} \min & \theta_{n+1} \\ \text{s.t.} & \lambda_1 x_{i1} + \cdots + \lambda_n x_{in} + \lambda_{n+1}\widehat{x}_{io} \leqslant \theta \widehat{x}_{io}, \quad i = 1, 2, \cdots, m, \\ & \lambda_1 y_{r1} + \cdots + \lambda_n y_{rn} + \lambda_{n+1}\widehat{y}_{ro} \geqslant \widehat{y}_{ro}, \quad r = 1, 2, \cdots, s. \\ & \lambda_i \geqslant 0, \quad i = 1, 2, \cdots, n+1. \end{array} \tag{9}$$

Bring (8) to (9), and let $\widehat{\lambda}_i = \sum\limits_{\lambda} \lambda_i$, $i = 1, 2, \cdots, n$, $i \neq o$, and $\widehat{\lambda}_o = \sum\limits_{\lambda} \frac{\lambda_o}{\theta_o^*} + \lambda_{n+1}$. Because there are n variables in programming (8), let $\lambda_o = 0$ and the variable $\lambda_n + 1$ instead of λ_o, we can obtain the following model:

$$\begin{array}{ll} \min & \widehat{\theta} \\ \text{s.t.} & \widehat{\lambda}_1 x_{i1} + \cdots + \widehat{\lambda}_n x_{in} + \widehat{\lambda}_o \theta_o^* x_{io} \leqslant \widehat{\theta}\theta_o^* x_{io}, \quad i = 1, 2, \cdots, m, \\ & \widehat{\lambda}_1 y_{r1} + \cdots + \widehat{\lambda}_n y_{rn} + \widehat{\lambda}_o y_{ro} \geqslant y_{ro}, \quad r = 1, 2, \cdots, s. \\ & \lambda_i \geqslant 0, \quad i = 1, 2, \cdots, n. \end{array} \tag{10}$$

Because the efficiency score of DMU$_o$ is θ_o^*, the above model has its optimal value of $\widehat{\theta} = 1$. So the DMU$_{n+1}$ is CCR-efficient. ∎

Corollary 1 *The MPSS point calculated by* (8) *corresponding to DMU_o is with constant returns-to-scale.*

3.2 Forecasting the $\Delta\theta$

Before giving our algorithm, we firstly discuss an example.

Example 1 Three DMUs with single input and single output are considered as follows.

Table 1 Example 1

DMU	A	B	C
input	5	8	10
output	5	10	12

Firstly we identify that A's returns to scale is increasing, B's constant and C's decreasing. Using the formula (8) we can also find the A's most production scale size is $(8, 10)$. If the center decision maker wants to allocate extra resource (input) to A, for example, Δx, the investor wants to know the maximal output increases. Since A's returns to scale is increasing, if its input is larger, its production efficiency score θ_{CCR} also increases. So, our main task in this section is to estimate the changes of the DMU's efficiency score $\Delta\theta_{\text{CCR}}$. This idea also comes from the elastic theory in economics.

With only taking DMU A into account, we can calculate that, the input distance between A's real scale and its most productive scale size is 3. Considering the resource is not redundant, we assume that here $0 \leqslant \Delta x \leqslant 3$, because if $\Delta x > 3$, the A's returns to scale may be deceasing kind. We can also regard 3 as a threshold value of A's scale increment. Actually, in economics, to increase input to a DMU whose returns to scale is decreasing is always not a very smart investment under a limited resource condition. So, in this paper, we will not take the decreasing returns to scale case into consideration.

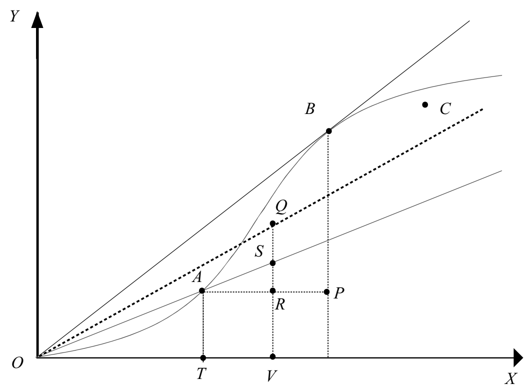

Figure 4 Example 1

We can see from the figure 4 that when increase $\Delta x = 2$ to A, we can find that the point A may reach to point Q, instead of point S which is got under the assumption that efficiency score of A is unchanged. We forecast the $\Delta\theta$ by a function $\Delta\theta = g(\Delta x)$. Because the exact function $g(\bullet)$ is hard to get, we use a linear approximate function to denote it, which is $\Delta\theta = \rho\Delta x$. As in example 1, we compute the coefficient ρ by using the reference pair $(\Delta\widetilde{x}, \Delta\widetilde{\theta})$, where $\Delta\widetilde{x}$ is the x-coordinate distance between point A and point B, which is also most productive scale size point corresponding to A, and $\Delta\widetilde{\theta} = \theta_B^* - \theta_A^*$. So we can get: $\rho = \Delta\widetilde{\theta}/\Delta\widetilde{x} = (1 - 0.8)/3 = 1/15$ and $\Delta\theta = \rho\Delta x = 2/15 \approx 0.133$. If increase 2 units of input to A, the efficiency score of A is changed to $\widehat{\theta}_A = 0.933$.

3.3 Algorithm for the New Kind of Inverse DEA Problem

Based on the MPSS concept, we give a heuristic algorithm to solve the new inverse DEA problem by using $\theta - CWA$ model. Suppose the common weights solution is $\theta - cw = (u_1, u_2, \cdots, u_s, v_1, v_2, \cdots, v_m)$, the algorithm for the improved inverse DEA problem is as follows, where the observed DMU$_o$ is with multiple inputs and multiple outputs, and the decision maker wants to allocate extra inputs Δx to DMU$_o$:

Algorithm 2: Algorithm for the improved inverse DEA problem

Step 1 Compute the efficiency score θ_o^* of DMU$_o$;

Step 2 Determine the returns to scale of DMU$_o$. W.l.o.g., we assume that the DMU$_o$'s returns to scale is increasing;

Step 3 According to definition 3, calculate the most productive scale size $(\widehat{x}_o, \widehat{y}_o)$ of DMU$_o$, where $\widehat{x}_o = (\widehat{x}_{1o}, \widehat{x}_{2o}, \cdots, \widehat{x}_{mo})$;

Step 4 Compute the ratio ρ by using the reference pair $(\Delta\widetilde{x}, \Delta\widetilde{\theta})$, where $\Delta\widetilde{\theta} = 1 - \theta_o^*$, $\Delta\widetilde{x} = \sum_{i=1}^m v_i\widehat{x}_{io} - \sum_{i=1}^m v_i x_{io}$, and the ratio is $\rho = \frac{\Delta\widetilde{\theta}}{\Delta\widetilde{x}}$;

Step 5 Compute the real $\Delta\overline{x} = \sum_{i=1}^n v_i\Delta x_i$, we assume that $\Delta\overline{x} \le \Delta\widetilde{x}$. So $\Delta\theta = \rho\Delta\overline{x}$;

Step 6 The $\widehat{\theta}_o = \theta_o^* + \Delta\theta$. Bring the new parameters to model (4), to obtain the Δy.

4 Approach to the Resource Allocation

4.1 Scheme for the Resource Allocation

In chapter 3, we have discussed the algorithm for resource allocation focused on a single

DMU. When the number of DMUs considered is more than one, the problem turns out to be more complicated. Generally, the resource allocation problem with the plural sections (DMUs) should be treated firstly as a selection problem, and then as an investment problem. In the process, scholars always paid much attention to the optimization and efficiency factors but the detail of the selection technique. In this paper, we will emphasize the importance of the selection technique and also combine the optimization tricks into our solution.

Firstly we classify all the DMUs and choose some special candidates to allocate resources. There are six categories in DEA's literature as shown in table 2. Between the six categories, only efficient DMUs with constant returns to scale is of equilibrium meaning, which could be regarded as a best condition both on technical skill and also on scale economies.

Table 2 Category Priority of Candidates

	Technical efficient	Technical inefficient
Increasing returns to scale	1	3
Constant returns to scale	2	4
Decreasing returns to scale	*	*

In Table 2, the number stands for the priority order of the importance of the DMU type in resource allocation. For instance, efficient DMUs with increasing RTS are in the first order in resource allocation, which means that these DMUs take priority of getting resources. We regard this type of DMUs as with priority because these DMUs gain the equilibrium in technical skill, but not reach its maximal economical scale. So these DMUs are considered the most potential candidates. But if there are no efficient DMUs with increasing returns to scale, then efficient DMUs with constant returns to scale may have the priority of getting resource. Because efficient DMUs has the higher technique level than inefficient DMUs, and constant returns to scale is better in economics scales than decreasing returns to scale. Here, the first criteria according to which we determine the candidates is the technical efficiency and the second criteria is economical scale's potential. But for decreasing returns to scale DMUs, because their scale elastics are the lowest, it is not advised to increase their inputs. So, according to Table 3, we can divide all the DMUs into 5 layers and decide the set of DMU candidates who will get the additional resources.

The next step is to determine the allocation weights. This is a very key step because the allocation weights should be equitable and fair enough. As the goal of the resource allocation problem is to realize the maximal of the total outputs, and when it is referred to multiple inputs and multiple outputs, the problem becomes a multiple criteria one. It is naturally to solve the problem from two kinds of ways: 1) directly solve a multiple-objective programming, which is hard to achieve a unique solution; 2) decompose the problem and construct a scheme by using many sub models. The above two ways both have their advantages and disadvantages. For the former way, the multiple-objective character is closer to the real complicated environment, but just because too much criteria are considered, it is hard to handle all of the factors. For the later way, it usually analyzes one unit at a time and integrated all the solutions by using a general multiplier.

We prefer the second way. By using the $\theta - CWA$ model, a general multiplier can be got. By directly using the multiplier into extra resource allocation algorithm (Algorithm 1), we get a set of allocation weights. Here, some technique should be adopted when using extra resource allocation algorithm, because the target economics scale is different from the output scale considered in extra resource allocation. The same as the concept most productive scale size (MPSS) has been defined by formula (8), we set the target economics by using the average outputs' scale of efficient DMUs with constant returns to scale, and set the extra resource allocation according to the distance between the DMU' scale and the target scale. Here the output scale of efficient DMUs with constant returns to scale is referred to be the target or ideal economical scale. As in Example 1, DMU A's MPSS is the same as DMU B's scale, and DMU B is an efficient DMU with constant returns to scale. It is not a coincidence, because the target scale should be related to the reference DMU's scale, and also an ideal status for all the DMUs considered. As a result, no other DMU is more ideal than efficient DMUs not only at technical skill but also at economical scale side. The computing detail of this part can be seen in Example 2–4.

After calculating the Δx for each DMU considered, inverse DEA model is used to compute the corresponding Δy. Then we can get a set of new DMUs and compute their efficiency scores, respectively. By comparing the results, we can give the analysis of production frontier and economics scale.

The schedule of our solution to resource allocation problem is shown in Figure 5.

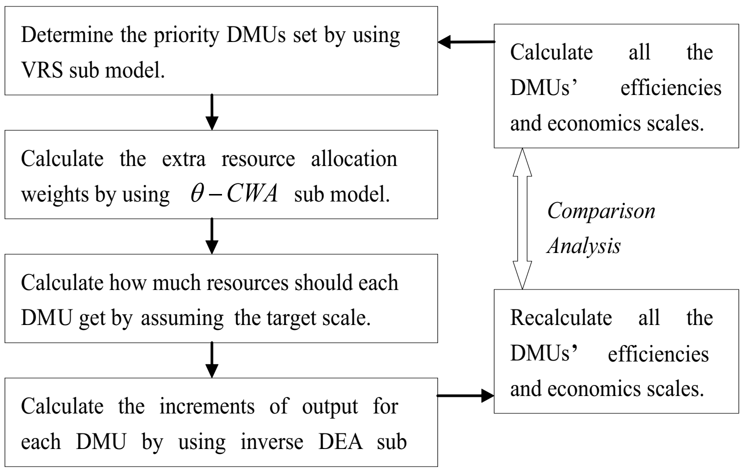

Figure 5 Scheme

Although the framework or algorithm scheme portrayed in Figure 5 seems like a static technique for resource allocation problem, it can be flexibly used in dynamic process. We regard the resource allocation as effective and efficient because it can stimulate the efficiency of the organization, and the improvements brought by the current resource allocation should have the ability to bring new benefit in future. So, we focus on current resource allocation, without consideration of the further performance and management forecasting.

We investigated a new inverse DEA method in Subsection 3.3. Algorithm 2 concerns resource allocation to inefficient DMUs with increasing returns, whose priority order is low shown in Table 2. In most cases, this kind of DMUs may not be considered according to Figure 5. That is because we mainly concern with the case that the allocated resource is not abundant, and less than the scale economies target. When the resource is as abundant as more than the scale threshold value, the inefficient DMUs with increasing returns should be firstly emphasized, and then the Algorithm 2 is available and valid.

4.2 An Extreme Case of Resource Allocation

Since the aim of the resource allocation problem is to realize the maximal of the outputs increments, an intuitive idea is to allocate all the additional resources to the most productive DMU. In many competitions or selections, only a few excellent candidates may have the chance to win out; in economics, different entities have different production levels, if one has the greatest marginal product, it could be regarded as the most promising candidate. In these cases, it is wise for the decision maker to put major importance to the excellent candidates. If the resources are limited, the decision maker will evaluate and forecast which one will produce maximal outputs with the limited resource. This target is arrived by using inverse DEA method.

Take the selection process of the athletes as an example: when a big game is coming, only one or two athletes can get the access to the game. In this case, the coach who masters limited resource will certainly give all the resource to the most promising athletes, because they have the biggest possibilities to win the prize. Otherwise, if the coach allocate the limited resource to some athletes with low present records, the athletes are not likely to create higher records than the athletes with high performance records.

This part is an extreme case of resource allocation, and our principle is to allocate all the resources to the best candidate DMU. This aim can be realized by the following algorithm.

Algorithm 3: Algorithm for the extreme case of resource allocation

Step 1 Given a set of DMUs and its inputs and outputs indices, compute each DMU's efficiency score θ by using CCR model. Pick out the efficient DMUs into the set of eDMUs.

Step 2 Assuming allocate all the Δx to one DMU, use inverse DEA model to compute Δy_i for every efficient $DMU_i \in eDMUs$.

Step 3 By using CWA model as a multiplier for vector $(y_1, y_2, \cdots, y_s)$, change the s-dimensional vector to a real number. Choose the biggest Δy and its corresponding DMU is the candidate.

5 Examples

In this chapter, we will provide many examples and provides details of algorithms discussed above, to illustrate our approach to the resource allocation problem.

5.1 A Simple Example

Considering the following example, each DMU investigated is with single input and single output.

Example 2 Seven DMUs are investigated. If the decision maker has some additional resource as $\Delta x = 0.5$, how to allocate Δx? Numerical case displays in Table 3.

Table 3 Example 2's Data

	A	B	C	D	E	F	G
input	2	3	6	9	5	4	10
output	2	5	7	8	3	1	7

DMU A is the only efficient unit with increasing returns to scale. Allocate $\Delta x = 0.5$ to DMU A, and use inverse DEA method to compute Δy_B. DMU A is changed from $A = (2, 2)$ to $\widehat{A} = (2.5, 4.16667)$. Compute $\theta_{\widehat{A},\mathrm{CRS}}$ under the new production possibility set $\{\widehat{A}, B, C, D, E, F, G\}$. We can analyze the comparison results before and after the resource allocation in Table 4.

We mentioned in the introduction part that our aim of resource allocation is to make an efficient-effective-equality allocation that can balance the different desires of two management layers. In Example 2, we can make the following comparison analysis.

Table 4 Example 2's result

DMU	Before Resource Allocation			After Resource Allocation		
	Score	RTS	RTS of Projected DMU	Score	RTS	RTS of Projected DMU
A	1	increasing		1	constant	
B	1	constant		1	constant	
C	1	decreasing		1	decreasing	
D	1	decreasing		1	decreasing	
E	0.4666667		increasing	0.5		constant
F	0.5		increasing	0.625		constant
G	0.6		decreasing	0.6		decreasing

i) Effectiveness. For DMUs who get resources, such as DMU A in Example 2, its maximal profit is realized. And as can be seen from Figure 6, DMU A reaches CRS frontier after resource allocation and its returns to scale becomes constant which means equilibrium is arrived. So, the resource is effectively allocated.

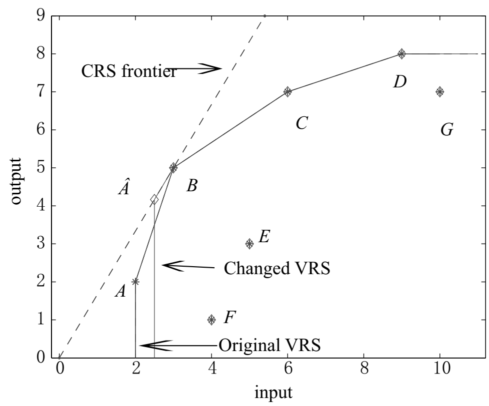

Figure 6 Frontier improved in example 2

ii) Efficiency. We can draw the VRS-frontiers before and after the resource allocation shown in Figure 6, it is obvious that the whole VRS production frontier is improved. For the central manager who hopes to increase the whole efficiency of the organization, it is a piece of delightful news if the whole VRS production frontier is improved. And we also regard that the whole scale economies efficiency is also well improved after allocation. This characteristic is especially obvious in Example 4 whose returns to scale changes into constant from decreasing after allocation.

iii) Equality. For the DMUs who don't get allocated any resource, are also regarded to benefit from the allocation process. We can see from Table 4 that find that DMU E, F, G's efficiency scores are increased after allocating additional resources to DMU A. Moreover, in Examples 3 and 4, some inefficient DMUs change into efficient with increasing returns after allocation. Although these DMUs don't get any additional resources, they may get a bigger possibility to become candidates in the next allocation. So, the allocation takes an equality consideration of all the DMUs with great efficiency of potential to be efficient.

5.2 Increasing Efficient Category

Example 3 25 A real-life case, which consists 25 supermarkets that are suited in finland and belong to the same chain. The original problem is simplified by taking into consideration only two outputs variables (Sales and Size) and two inputs (Man-hours and Size)[8]. The data set is given in Table 5. Suppose there are additional inputs resources $\Delta x = (20, 1)$ could be allocated to any DMUs considered, how to allocate these resources?

Table 5 Example 3's Data

Supermarket	Man-hours 10^3 h	Size 10^3 m^2	Sales 10^6 FIM	Profit 10^6 FIM
i	x_{1i}	x_{2i}	y_{1i}	y_{2i}
A	79.1	4.99	115.3	1.71
B	60.1	3.3	75.2	1.81
C	126.7	8.12	225.5	10.39
D	153.9	6.7	185.6	10.42
E	65.7	4.74	84.5	2.36
F	76.8	4.08	103.3	4.35
G	50.2	2.53	78.8	0.16
H	44.8	2.47	59.3	1.3
I	48.1	2.32	65.7	1.49
J	89.7	4.91	163.2	6.26
K	56.9	2.24	70.7	2.8
L	112.6	5.42	142.6	2.75
M	106.9	6.28	127.8	2.7
N	54.9	3.14	62.4	1.42
O	48.8	4.43	55.2	1.38
P	59.2	3.98	95.9	0.74
Q	74.5	5.32	121.6	3.06
R	94.6	3.69	107	2.98
S	47	3	65.4	0.62
T	54.6	3.87	71	0.01
U	90.1	3.31	81.2	5.12
V	95.2	4.25	128.3	3.89
W	80.1	3.79	135	4.73
X	68.7	2.99	98.9	1.86
Y	62.3	3.1	66.7	7.41

We assume that the production possibility set P is based on the VRS assumption and pick out 4 DMUs: G, H, I, K, as candidates because these four DMUs are efficient DMUs with increasing returns to scale. By using θ-CWA model, we can get the most compromising common weights as $\theta - cw = (5, 41.25, 4.11, 18.46)$ and calculate the aggregate output of each efficient DMU. Results are shown in Table 6.

As C's aggregate output is greater than DMU D's whose returns to scale is decreasing, only DMU J, W, Y are taken into consideration in setting the target economics scale. Take the average of the three aggregate outputs, we can empirically get the target scale as 613.13. Then we can calculate the distance output of the four candidates DMUs respectively and get the extra resource allocation weights. According to the weights we allocate the input increments to the four DMUs and update the new DMUs' inputs and outputs indices. Results are shown in Table 7.

We can see from Table 8 that after resource allocation: 1) DMU G, H, I, K become constant

Table 6 Example 3's Data

DMU	Score	RTS	Aggregate output	Distance output	extra resource allocation weights
G	1	increasing	326.8	286.3	0.235
H	1	increasing	267.7	345.4	0.284
I	1	increasing	297.5	315.6	0.259
K	1	increasing	342.3	270.9	0.222
D	1	decreasing	955.2	×	0
C	1	constant	1118.6	×	0
J	1	constant	786.3	√	0
W	1	constant	642.2	√	0
Y	1	constant	410.9	√	0

Table 7 Updated data for example 3

Supermarket	Man-hours 10^3 h	Size 10^3 m^2	Sales 10^6 FIM	Profit 10^6 FIM
$\widehat{G}$	50.90	2.77	95.55	1.82
$\widehat{H}$	50.47	2.71	90.79	3.44
$\widehat{I}$	53.28	2.56	90.41	3.2
$\widehat{K}$	61.35	2.48	88.16	3.09

returns to scale; 2) DMU S becomes efficient from inefficient; 3) from the numerical comparison, we can gain the conclusion that the whole frontier is improved.

5.3 Constant Efficient Category

Example 4 7 hospitals are investigated, if the decision maker has some additional resource as $\Delta x = (10, 30)$, how to allocate Δx? Numerical case displays in Table 9[13].

Doctor and nurse are regarded as inputs indices, outpatient and inpatient are regarded as outputs indices. By using VRS model we can calculate each hospital's efficiency score, and determine the set of candidates from efficient DMUs as $\{A, B, D\}$ whose returns to scale is constant. By using θ-CWA model we can get the common weights as $cw = (1.0561, 1.0948, 1.002, 1.0042)$. Then we calculate the aggregate output of each efficient DMU and choose the minimal output from the DMUs whose returns to scale are decreasing. So, we regard 308.8 as target economical scale.

By using Algorithm 1, we can determine the extra resource allocation weights and update DMU A, B, D. The results is shown in Table 10.

We can see from Table 10 that after resource allocation: 1) DMU A, B, D become constant returns to scale; 2) DMU E becomes efficient from inefficient; 3) DMU G's returns to scale becomes constant from decreasing; 4) from the numerical comparison, we can gain the conclusion that the whole frontier is improved.

5.4 An Extreme Case

Example 5 Considering the Example 4, the decision maker want to allocate $\Delta x = (30, 10)$ to only one hospital, how to allocate the additional resources.

The solution of this problem is got through 3 steps as follows:

Step 1 Determine the set of eDMUs $= \{A, B, D\}$;

Table 8 Example 3's result

DMU	Before Resource Allocation			After Resource Allocation		
	Score	RTS	RTS of Projected DMU	Score	RTS	RTS of Projected DMU
A	0.85		increasing	0.81		constant
B	0.85		increasing	0.83		increasing
C	1	constant		1	constant	
D	1	decreasing		1	decreasing	
E	0.84		increasing	0.76		increasing
F	0.86		increasing	0.78		constant
G	1	increasing		1	constant	
H	1	increasing		1	constant	
I	1	increasing		1	constant	
J	1	constant		1	constant	
K	1	increasing		1	constant	
L	0.75		constant	0.76		constant
M	0.68		increasing	0.66		constant
N	0.84		increasing	0.9		increasing
O	0.92		increasing	0.98		increasing
P	0.98		increasing	0.9		constant
Q	0.94		increasing	0.9		constant
R	0.84		increasing	0.81		constant
S	0.99		increasing	1	increasing	
T	0.88		increasing	0.87		increasing
U	0.88		increasing	0.86		constant
V	0.85		increasing	0.85		constant
W	1	constant		1	constant	
X	0.98		increasing	0.93		constant
Y	1	constant		1	constant	

Table 9 Example 4's Data

hospital	doctor	nurse	outpatient	inpatient
A	20	151	100	90
B	19	131	150	50
C	25	160	160	55
D	27	168	180	72
E	22	158	94	66
F	55	255	230	90
G	33	235	220	88

Step 2 When increase $\Delta x = (10, 30)$ to DMU A, B, D, respectively, using inverse DEA model we can calculate Δy as $\Delta y_A = (63.56, 0)$, $\Delta y_B = (22.5, 19)$, $\Delta y_D = (32.14, 12.86)$;

Step 3 Using CWA model we can calculate the most compromising common weights $(u_1, u_2) = (1.002, 1.0042)$, so we can compute the weighted $\Delta\widetilde{y}$ for DMU A, B, D, the answer is $\Delta\widetilde{y}_A = 63.69$, $\Delta\widetilde{y}_B = 41.62$, and $\Delta\widetilde{y_D} = 45.12$.

Because of $\Delta\widetilde{y}_D \leq \Delta\widetilde{y}_B \leq \Delta\widetilde{y}_A$, the DMU$_A$ is the most promising DMU, and we suggest allocate all the additional resource to DMU$_A$.

Example 5 gives us a clear image about the method proposed by us. Our principle is to allocate all additional resource to only one unit which is the most promising one. Traditional

Table 10 Example 4's result

DMU	Before Resource Allocation			After Resource Allocation		
	Score	RTS	RTS of Projected DMU	Score	RTS	RTS of Projected DMU
A	1	constant		1	constant	
B	1	constant		1	constant	
C	0.90		constant	0.92		increasing
D	1	constant		1	constant	
E	0.88		constant	1	increasing	
F	1	decreasing		1	decreasing	
G	1	decreasing		1	constant	

methods for resource allocation problem suggest allocate resource to many DMUs under some combination of Δx. Maybe the combination can realize the optimal proportional equilibrium, which means, take Example 5 as regarded, allocating $\Delta x_A = (\Delta x_{A1}, \Delta x_{A2})$ to DMU$_A$, and $\Delta x_D = (\Delta x_{D1}, \Delta x_{D2})$ to DMU$_D$, where $\Delta x = \Delta x_A + \Delta x_D$ may be better than allocating all the Δx to DMU$_A$. The question is, we can not prove which combination is best unless we have calculated all the possibilities. So, we avoid the multiple-objective programming way and choose the decomposed method. Actually the inverse DEA model is similar to a multiple-objective programming, which takes into account of many criteria in the resource allocation problem.

6 Conclusion

The resource allocation problem is currently under active research in the DEA literature[13,20]. In this paper we investigate an efficient-effective-equality resource allocation framework, and we introduce various returns to scale model, inverse DEA model, common weight analysis model, and extra resource allocation algorithm for supporting our method. Various returns to scale results help the selection process be more efficient, and the scale economies become one of the major factors to evaluate the resource allocation. Another metric of our method is that it can easily takes into considerations the decision makers' preferences. We use CWA or θ-CWA models to obtain the most compromising common weights, which is useful when there are no additional preferences from the decision makers, otherwise the outputs multiplier can be given considering the preferences. As in Example 5, if the decision maker is more care about the "outpatient" index, he/she can higher the weight corresponding to the output index, in which case DMU$_A$ may be chosen instead of DMU$_D$. Extra resource allocation algorithm is adopted because it considers not only the efficiency but also the scales of DMU, which is consistent with the our aim of resource allocation.

The efficient-effective-equality resource allocation we propose can provide many further suggestions for managers to evaluate performance. It is not a statistical process; it can be operated as a dynamical process. Our further research on this topic will be focused on the real-time resource allocation method and its production forecasting. In that research, the expanding of the scale economies is also an interesting topic, and every former result can be used in the next resource allocation to forecast the future performance and production ability.

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