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LINEAR QUADRATIC REGULATION FOR DISCRETE-TIME SYSTEMS WITH INPUT DELAY: SPECTRAL FACTORIZATION APPROACH*

Hongguo ZHAO · Huanshui ZHANG · Hongxia WANG · Chenghui ZHANG

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Abstract The infinite-horizon linear quadratic regulation (LQR) problem is settled for discrete-time systems with input delay. With the help of an autoregressive moving average (ARMA) innovation model, solutions to the underlying problem are obtained. The design of the optimal control law involves in resolving one polynomial equation and one spectral factorization. The latter is the major obstacle of the present problem, and the reorganized innovation approach is used to clear it up. The calculation of spectral factorization finally comes down to solving two Riccati equations with the same dimension as the original systems.

Key words Diophantine equation, infinite-horizon LQR, reorganized innovation, spectral factorization, stochastic backwards systems.

1 Introduction

The standard infinite-horizon LQR problem is commonly investigated via Dynamic Programming by using a state-space or "internal" model of the physical system[1−2]. This is a time-domain approach and yields the desired solutions in terms of an algebraic Riccati equation. The solutions to the problem also can be obtained via an alternative way, the so-called Polynomial Equation approach[3−6]. It uses transfer matrices or "external" models of the physical system, and turns out to be more akin to a frequency-domain methodology. It leads us to solve the infinite-horizon LQR problem by spectral factorization and Diophantine equation.

On the other hand, the optimal control of the systems with input/output delays has received much attention in the past decades, and a variety of methods have been developed. Among these previous works, papers [7–9] discuss the discrete-time systems while papers [10–12] discuss the continuous-time systems. For infinite-horizon LQR for discrete-time systems with input delay, one might tend to consider augmenting the systems and convert a delay problem into a delay-free problem. In this case, the optimal control law can be designed via one spectral factorization

Hongguo ZHAO · Huanshui ZHANG
School of Control Science and Engineering, Shandong University, Jinan 250061, *China.*
Email: h_g_zhao@126.com.
Hongxia WANG
Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen University Town Xili, Shenzhen 518055, *China.*
Chenghui ZHANG
School of Control Science and Engineering, Shandong University, Jinan 250061, *China.*
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and one Diophantine equation, and spectral factorization is always a key ingredient in tackling the problem. However, the spectral factorization in augmented approach heavily depends on solving higher dimension Riccati equation, whose dimension tends to increase in proportion with the length and amount of the delay involved in the underlying system. To bypass high dimension Riccati equation, paper [13] has come up with a new method, i.e., the so-called reorganized innovation approach, and has solved the state estimation problem for the systems with delayed measurement well. The reorganized innovation approach has been shown to be powerful to deal with some complicated estimation and control problems for the time-delay systems such as estimation of H_∞ fixed-lag smoothing and so on[14].

The present paper is concerned with the polynomial solutions of the LQR for the systems with delayed input, where the optimal control law will be given in view of one polynomial equation and one spectral factorization. Here, spectral factorization for time-delay systems is the key problem to be solved. We will adopt a different approach from state augmentation to calculate the spectral factor. Firstly, we introduce a stochastic backwards systems with time-delay in terms of the principle of duality between estimation and control for the time-delay systems[15]. Further, based on the stochastic backwards systems with time-delay, the ARMA innovation model can be derived. Finally, by applying the reorganized innovation approach developed in the previous works, the spectral factorization is computed with the aid of the ARMA innovation model. The spectral factor is obtained by solving two standard Riccati equations rather than an augmented algebraic Riccati equation.

The rest of the paper is organized as follows. The problem is addressed in Section 2. Section 3 presents the polynomial solutions to the present problem by using the spectral factorization approach. Based on the reorganized innovation approach and ARMA innovation model, spectral factorization is calculated in Section 4. The comparison of the computational cost between the presented approach and the conventional state augmented method for spectral factorization is given in Section 5. The conclusions are drawn in Section 6.

2 Problem Statement

We consider the linear discrete time-invariant systems with input delay

$$\boldsymbol{x}(t+1) = \Phi^{\mathrm{T}}\boldsymbol{x}(t) + \Gamma_{(0)}^{\mathrm{T}}\boldsymbol{u}_0(t) + \Gamma_{(1)}^{\mathrm{T}}\boldsymbol{u}_1(t-h), \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$ is the state, $\boldsymbol{u}_i(t) \in R^{p_i},\ i = 0, 1$, are the control input, h is the time-delay, and T stands for the transpose.

Consider the following quadratic performance index for the systems (1)

$$J = \sum_{t=0}^{\infty} \boldsymbol{u}_0^{\mathrm{T}}(t)R_0\boldsymbol{u}_0(t) + \sum_{t=h}^{\infty} \boldsymbol{u}_1^{\mathrm{T}}(t-h)R_1\boldsymbol{u}_1(t-h) + \sum_{t=0}^{\infty} \boldsymbol{x}^{\mathrm{T}}(t)Q\,\boldsymbol{x}(t), \tag{2}$$

where the matrices $R_i,\ i = 0, 1$, are positive definite and the matrix Q is non-negative definite.

The infinite-horizon LQR problem is stated as follows:

Find the input sequences $\{\boldsymbol{u}_i(t),\ i = 0, 1,\ 0 \le t < \infty\}$, which can make the resultant system asymptotically stable and minimizes the cost function J of (2) for any initial state $\boldsymbol{x}_0$.

3 Solutions to Infinite-Horizon LQR

To the best of our knowledge, it is difficult to directly deal with the problem. To go further, we introduce the following notation.

Denote

$$\boldsymbol{u}(t) = \begin{bmatrix} \boldsymbol{u}_0(t) \\ \boldsymbol{u}_1(t) \end{bmatrix}, \tag{3}$$

system (1) and quadratic performance index (2) can thus be rewritten as

$$\boldsymbol{x}(t+1) = \Phi^{\mathrm{T}}\boldsymbol{x}(t) + \overline{\Gamma}^{\mathrm{T}}(q^{-1})\boldsymbol{u}(t) \tag{4}$$

and

$$J = \sum_{t=0}^{\infty} \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t) + \sum_{t=0}^{\infty} \boldsymbol{x}^{\mathrm{T}}(t)Q\,\boldsymbol{x}(t) \tag{5}$$

respectively, where $\overline{\Gamma}^{\mathrm{T}}(q^{-1}) = \begin{bmatrix} \Gamma_{(0)}^{\mathrm{T}} & \Gamma_{(1)}^{\mathrm{T}} q^{-h} \end{bmatrix}$, q^{-1} is the backward shift operator, i.e., $q^{-1}\boldsymbol{s}(t) = \boldsymbol{s}(t-1)$, and

$$R = \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix}.$$

Let us introduce a right coprime matrix-fraction description (MFD) of the transfer matrix for the systems (4)

$$(I_n - \Phi^{\mathrm{T}}q^{-1})^{-1}\overline{\Gamma}^{\mathrm{T}}(q^{-1})q^{-1} = C^{\mathrm{T}}(q^{-1})A^{-\mathrm{T}}(q^{-1}),$$

where $C^{\mathrm{T}}(q^{-1})$ and $A^{-\mathrm{T}}(q^{-1})$ are polynomial matrices of dimensions $n \times (p_0 + p_1)$ and, respectively, $(p_0 + p_1) \times (p_0 + p_1)$.

It can be shown[6] that the optimal control law

$$\boldsymbol{u}^*(t) = -\mathcal{F}^{-1}(q^{-1})\mathcal{G}(q^{-1})\boldsymbol{x}(t), \tag{6}$$

where $\mathcal{F}(q^{-1})$ and $\mathcal{G}(q^{-1})$ are the solutions of the following Diophantine equation

$$\mathcal{F}(q^{-1})A^{\mathrm{T}}(q^{-1}) + \mathcal{G}(q^{-1})C^{\mathrm{T}}(q^{-1}) = D^{\mathrm{T}}(q^{-1}), \tag{7}$$

where $D^{\mathrm{T}}(q^{-1})$ is a stable polynomial matrix and satisfies the following right spectral factorization

$$D(q)D^{\mathrm{T}}(q^{-1}) = C(q)QC^{\mathrm{T}}(q^{-1}) + A(q)RA^{\mathrm{T}}(q^{-1}). \tag{8}$$

In terms of the above discussion, the following theorem is now straightforward.

Theorem 3.1 *Consider the system* (1) *and quadratic performance index* (2). *The optimal LQR control* $\boldsymbol{u}_i(t),\ i = 0, 1,\ 0 \le t < \infty$ *that minimizes* (2), *is computed by*

$$\boldsymbol{u}_0^*(t) = -\,[\, I_{p_0} \quad 0 \,]\, \mathcal{F}^{-1}(q^{-1})\mathcal{G}(q^{-1})\boldsymbol{x}(t),$$

$$\boldsymbol{u}_1^*(t) = -\,[\, 0 \quad I_{p_1} \,]\, \mathcal{F}^{-1}(q^{-1})\mathcal{G}(q^{-1})\boldsymbol{x}(t).$$

Remark 3.1 The solvability and the uniqueness of the polynomial equation (7) have been well studied in the previous works, see [6,16] for details.

Remark 3.2 Although we have presented the optimal control law in Theorem 3.1, the calculation for the spectral factor $D^{\mathrm{T}}(q^{-1})$ remains to be computed. Note (8), and one possible approach to such spectral factorization is the state augmentation which, however, would result in tremendous computation.

Remark 3.3 In next section, our aim is to come up with a simple approach to above spectral factorization (8) based on ARMA innovation model. The key technique is the reorganized innovation analysis approach.

4 Spectral Factorization

4.1 ARMA Innovation Model

In this subsection, based on ARMA innovation model, the spectral factor $D^{\mathrm{T}}(q^{-1})$ in (7) can be calculated.

Next, we introduce the following backwards dual state-space model associated with systems (1) and performance index (2):

$$\boldsymbol{x}(t) = \varPhi \boldsymbol{x}(t+1) + \boldsymbol{e}(t), \tag{9}$$

$$\boldsymbol{y}_{(0)}(t) = \varGamma_{(0)}\boldsymbol{x}(t+1) + \boldsymbol{v}_{(0)}(t), \quad 0 \leq t < \infty, \tag{10}$$

$$\boldsymbol{y}_{(1)}(t) = \varGamma_{(1)}\boldsymbol{x}(t_h+1) + \boldsymbol{v}_{(1)}(t), \ \ t_h \equiv t+h, \tag{11}$$

where $\boldsymbol{x}(t) \in R^n$ and $\boldsymbol{e}(t) \in R^n$ represent the state and the system noise, $\boldsymbol{v}_{(i)}(t) \in R^{p_i}, \ i = 0, 1$ are the measurement noises, $\boldsymbol{e}(t)$ and $\boldsymbol{v}_{(i)}(t)$ are mutually uncorrelated white noises with zero means and covariance matrices $\varepsilon[\boldsymbol{e}(k)\boldsymbol{e}^{\mathrm{T}}(j)] = Q\delta_{kj}, \ \varepsilon[\boldsymbol{v}_{(i)}(k)\boldsymbol{v}_{(i)}^{\mathrm{T}}(j)] = R_i\delta_{kj}$, where δ_{kj} is Kronecker delta function, and ε denotes the mathematical expectation.

Let $\boldsymbol{y}(t)$ denote the observation of the system (9)–(11) at time t. We have

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{y}_{(0)}(t) \\ \boldsymbol{y}_{(1)}(t) \end{bmatrix}. \tag{12}$$

Exploiting (12), and the following definition can be given by

$$\boldsymbol{w}(t) \equiv \boldsymbol{y}(t) - \widehat{\boldsymbol{y}}(t|t+1), \tag{13}$$

where $\boldsymbol{w}(t)$ is called the innovation, and $\widehat{\boldsymbol{y}}(t|t+1)$ is the one-step prediction.

From (10), (11), and (13), we get

$$\boldsymbol{y}(t) = \begin{bmatrix} \varGamma_{(0)} & 0 \\ 0 & \varGamma_{(1)} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}(t+1|t+1) \\ \widehat{\boldsymbol{x}}(t_h+1|t+1) \end{bmatrix} + \boldsymbol{w}(t). \tag{14}$$

Since $\widehat{\boldsymbol{x}}(t_h+1|t+1)$ is the projection of $\boldsymbol{x}(t_h+1)$ onto the linear space $\mathcal{L}\{\boldsymbol{w}(t+1), \boldsymbol{w}(t+2), \cdots\}$, by applying the projection formula in Hilbert space, $\widehat{\boldsymbol{x}}(t_h+|t+1)$ can be expressed as

$$\widehat{\boldsymbol{x}}(t_h+1|t+1) = q^h\widehat{\boldsymbol{x}}(t+1|t+1) + K(q)\boldsymbol{w}(t), \tag{15}$$

where $q^h\widehat{\boldsymbol{x}}(t+1|t+1) = \widehat{\boldsymbol{x}}(t_h+1|t_h+1)$, q is the forward shift operator, i.e., $q\boldsymbol{s}(t) = \boldsymbol{s}(t+1)$, and $K(q)$ is given by

$$K(q) = \sum_{i=1}^{h} K_i q^i. \tag{16}$$

In the above, K_i is defined as

$$K_i \equiv \varepsilon[\boldsymbol{x}(t_h+1)\boldsymbol{w}^{\mathrm{T}}(t+i)]Q_{\boldsymbol{w}}^{-1}, \quad i = 1, 2, \cdots, h, \tag{17}$$

where innovation covariance matrix $Q_{\boldsymbol{w}}$ is defined as

$$Q_{\boldsymbol{w}} \equiv \varepsilon[\boldsymbol{w}(t)\boldsymbol{w}^{\mathrm{T}}(t)]. \tag{18}$$

Substituting (15) into (14), it follows that

$$\boldsymbol{y}(t) = \overline{\Gamma}(q)\widehat{\boldsymbol{x}}(t+1|t+1) + \mathcal{K}(q)\boldsymbol{w}(t) + \boldsymbol{w}(t), \tag{19}$$

where

$$\mathcal{K}(q) = \begin{bmatrix} 0 \\ \Gamma_{(1)}K(q) \end{bmatrix} \tag{20}$$

and

$$\overline{\Gamma}(q) = \begin{bmatrix} \Gamma_{(0)} \\ \Gamma_{(1)}q^h \end{bmatrix}.$$

On the other hand, $\widehat{\boldsymbol{x}}(t+1|t+1)$ is the projection of $\boldsymbol{x}(t+1)$ onto the linear space $\mathcal{L}\{\boldsymbol{w}(t+1), \boldsymbol{w}(t+2), \cdots\}$, accordingly,

$$\widehat{\boldsymbol{x}}(t+1|t+1) = \Phi\widehat{\boldsymbol{x}}(t+2|t+2) + K_0\boldsymbol{w}(t+1), \tag{21}$$

where K_0 is defined as

$$K_0 \equiv \varepsilon[\boldsymbol{x}(t+1)\boldsymbol{w}^{\mathrm{T}}(t+1)]Q_{\boldsymbol{w}}^{-1}. \tag{22}$$

Substituting (21) into (19), we obtain the ARMA innovation model as

$$A(q)\boldsymbol{y}(t) = C(q)K_0\boldsymbol{w}(t) + A(q)\mathcal{K}(q)\boldsymbol{w}(t) + A(q)\boldsymbol{w}(t), \tag{23}$$

where $A(q)$ and $C(q)$ satisfy the following left coprime MFD of the transfer matrix for (9)–(11),

$$\overline{\Gamma}(q)(I_n - \Phi q)^{-1}q = A^{-1}(q)C(q). \tag{24}$$

In view of ARMA innovation model (23), the spectral factor $D(q)$ in (8) can be achieved by

$$D(q) = \{C(q)K_0 + A(q)\mathcal{K}(q) + A(q)\}\, Q_{\boldsymbol{w}}^{\frac{1}{2}}. \tag{25}$$

We summarize now the above discussion, and state the main results in the following theorem.

Theorem 4.1 *The spectral factor $D^{\mathrm{T}}(q^{-1})$ in* (7) *obeys that*

$$D^{\mathrm{T}}(q^{-1}) = Q_{\boldsymbol{w}}^{\frac{1}{2}}\left\{K_0^{\mathrm{T}}C^{\mathrm{T}}(q^{-1}) + \mathcal{K}^{\mathrm{T}}(q^{-1})A^{\mathrm{T}}(q^{-1}) + A^{\mathrm{T}}(q^{-1})\right\}, \tag{26}$$

where $Q_{\boldsymbol{w}}$ and K_0 are given by (18) *and* (22)*, respectively. $\mathcal{K}(q^{-1})$ can be generated by combining* (16), (20) *with* (17).

Proof By using (25), the proof is straightforward. ∎

Remark 4.1 Although we have given the solutions to spectral factor $D^{\mathrm{T}}(q^{-1})$ in Theorem 4.1, it should be noted that $D^{\mathrm{T}}(q^{-1})$ in (26) is related with unknown $K_i,\ i = 0, 1, \cdots, h$ and $Q_{\boldsymbol{w}}$. In what follows, we shall manage to compute these unknown polynomial matrices with the help of the reorganized innovation.

4.2 Computation of Q_w and K_i, $i = 0, 1, \cdots, h$

To obtain the explicit expressions of $Q_{\boldsymbol{w}}$ and K_i, $i = 0, 1, \cdots, h$, we firstly assume that the backwards systems (9)–(11) are finite-horizon, i.e., $0 \le t \le N$. Then, the observation $\boldsymbol{y}(t)$ in (12) can be rewritten as

$$\overline{\boldsymbol{y}}(t) = \begin{cases} \boldsymbol{y}_{(0)}(t), & N-h \le t \le N, \\ \begin{bmatrix} \boldsymbol{y}_{(0)}(t) \\ \boldsymbol{y}_{(1)}(t) \end{bmatrix}, & t \le N-h. \end{cases} \tag{27}$$

The linear space spanned by the measurement $\{\overline{\boldsymbol{y}}(N), \overline{\boldsymbol{y}}(N-1), \cdots, \overline{\boldsymbol{y}}(t)\}$ is denoted as $\mathcal{L}\{\overline{\boldsymbol{y}}(N), \overline{\boldsymbol{y}}(N-1), \cdots, \overline{\boldsymbol{y}}(t)\}$. As $N \to \infty$, it is easy to known that

$$\mathcal{L}\{\overline{\boldsymbol{y}}(t), \overline{\boldsymbol{y}}(t+1), \cdots\} = \mathcal{L}\{\boldsymbol{y}(t), \boldsymbol{y}(t+1), \cdots\}. \tag{28}$$

Note (27), $\boldsymbol{y}_{(1)}(t)$ is an additional measurement of the state $\boldsymbol{x}(t_h + 1)$ which is received at time instant t, i.e., a delayed measurement. Next, we are to organize the instantaneous and delayed measurements, and thus attain a delay-free measurement.

1) Reorganized Innovation

In view of (10)–(11), and (27), it is clear that the linear space $\mathcal{L}\{\overline{\boldsymbol{y}}(N), \overline{\boldsymbol{y}}(N-1), \cdots, \overline{\boldsymbol{y}}(t)\}$ can be reorganized equivalently as

$$\mathcal{L}\{\{\boldsymbol{y}_2(s)\}_{s=N}^{t_h}, \boldsymbol{y}_1(t_h - 1), \cdots, \boldsymbol{y}_1(t)\}.$$

$\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ satisfy

$$\boldsymbol{y}_1(t) = \Gamma_1 \boldsymbol{x}(t+1) + \boldsymbol{v}_1(t), \tag{29}$$

$$\boldsymbol{y}_2(t) = \Gamma_2 \boldsymbol{x}(t+1) + \boldsymbol{v}_2(t), \tag{30}$$

where

$$\Gamma_1 = \Gamma_{(0)}, \quad \Gamma_2 = \begin{bmatrix} \Gamma_{(0)} \\ \Gamma_{(1)} \end{bmatrix}, \quad \boldsymbol{y}_1(t) = \boldsymbol{y}_{(0)}(t), \quad \boldsymbol{y}_2(t) = \begin{bmatrix} \boldsymbol{y}_{(0)}(t) \\ \boldsymbol{y}_{(1)}(t-h) \end{bmatrix},$$

and

$$\boldsymbol{v}_1(t) = \boldsymbol{v}_{(0)}(t), \quad \boldsymbol{v}_2(t) = \begin{bmatrix} \boldsymbol{v}_{(0)}(t) \\ \boldsymbol{v}_{(1)}(t-h) \end{bmatrix}$$

with zero means and covariance matrices

$$Q_{\boldsymbol{v}_1} = R_0, \quad Q_{\boldsymbol{v}_2} = \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix},$$

respectively.

Obviously, the new measurements $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are no longer with time-delay. With the new measurements, we can introduce the following sequence

$$\boldsymbol{w}(t, 1) = \boldsymbol{y}_1(t) - \widehat{\boldsymbol{y}}_1(t, 1), \tag{31}$$

$$\boldsymbol{w}(t, 2) = \boldsymbol{y}_2(t) - \widehat{\boldsymbol{y}}_2(t, 2), \tag{32}$$

where $\widehat{\boldsymbol{y}}_1(t, 1)$ is the projection of $\boldsymbol{y}_1(t)$ onto the linear space $\mathcal{L}\left\{\{\boldsymbol{y}_2(i)\}_{i=N}^{t_h}, \{\boldsymbol{y}_1(i)\}_{i=t_h-1}^{t+1}\right\}$, and $\widehat{\boldsymbol{y}}_2(t, 2)$ is the projection of $\boldsymbol{y}_2(t)$ onto the linear space $\mathcal{L}\left\{\{\boldsymbol{y}_2(i)\}_{i=N}^{t+1}\right\}$.

In view of (29)–(32),

$$\boldsymbol{w}(t,1) = \Gamma_1\widetilde{\boldsymbol{x}}(t+1|t+1,1) + \boldsymbol{v}_1(t),$$
$$\boldsymbol{w}(t,2) = \Gamma_2\widetilde{\boldsymbol{x}}(t+1|t+1,2) + \boldsymbol{v}_2(t),$$

where

$$\widetilde{\boldsymbol{x}}(t+1|t+1,1) = \boldsymbol{x}(t+1) - \widehat{\boldsymbol{x}}(t+1|t+1,1),$$
$$\widetilde{\boldsymbol{x}}(t+1|t+1,2) = \boldsymbol{x}(t+1) - \widehat{\boldsymbol{x}}(t+1|t+1,2),$$

in the above, $\widehat{\boldsymbol{x}}(t{+}1|t{+}1,1)$ is the projection of $\boldsymbol{x}(t{+}1)$ onto linear space $\mathcal{L}\left\{\{\boldsymbol{y}_2(i)\}_{i=N}^{t_h}, \boldsymbol{y}_1(t_h-1), \cdots, \boldsymbol{y}_1(t{+}1)\right\}$, and $\widehat{\boldsymbol{x}}(t{+}1|t{+}1,2)$ is the projection of $\boldsymbol{x}(t{+}1)$ onto the linear space $\mathcal{L}\left\{\{\boldsymbol{y}_2(i)\}_{i=N}^{t+1}\right\}$.

Based on the above discussion, we give the following lemma.

Lemma 4.1 $\{\boldsymbol{w}(N,2),\cdots,\boldsymbol{w}(t_h,2),\boldsymbol{w}(t_h-1,1),\cdots,\boldsymbol{w}(t,1)\}$ *is uncorrelated white noise sequence and spans the same space as* $\mathcal{L}\{\overline{\boldsymbol{y}}(N),\overline{\boldsymbol{y}}(N-1),\cdots,\overline{\boldsymbol{y}}(t)\}$.

Proof The proof is similar to the Lemma 2.1 in [13]. ∎

Remark 4.2 From Lemma 4.1 and (28), as $N\to\infty$, the linear space $\mathcal{L}\{\boldsymbol{y}(t),\boldsymbol{y}(t+1),\cdots\}$ is equivalent to $\mathcal{L}\{\boldsymbol{w}(t,1),\cdots,\boldsymbol{w}(t_h-1,1),\boldsymbol{w}(t_h,2),\cdots\}$. As usual, $\{\boldsymbol{w}(t,1),\cdots,\boldsymbol{w}(t_h-1,1),\boldsymbol{w}(t_h,2),\boldsymbol{w}(t_h+1,2),\cdots\}$ is called the reorganized innovation sequence.

By use of Lemma 4.1, as $t=0$ and $N\to+\infty$, state equation (9) together with reorganized measurement (30) can bring forth the following steady-state Riccati equation

$$P_2 = \Phi P_2\Phi^{\mathrm{T}} + Q - \Phi P_2\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\Gamma_2P_2\Phi^{\mathrm{T}}, \tag{33}$$

where P_2 is the steady-state state estimation error covariance matrices, and $Q_{\boldsymbol{w}_2}$ denotes the steady-state covariance matrices of $\boldsymbol{w}(\cdot,2)$, and satisfies

$$Q_{\boldsymbol{w}_2} = \Gamma_2P_2\Gamma_2^{\mathrm{T}} + Q_{\boldsymbol{v}_2}. \tag{34}$$

Similarly, as $t=0$ and $N\to+\infty$, state equation (9) and reorganized measurement (29) can yield the following steady-state Riccati equation

$$P_1(i+1) = \Phi P_1(i)\Phi^{\mathrm{T}} + Q - \Phi P_1(i)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(i,1)\Gamma_1P_1(i)\Phi^{\mathrm{T}}, \quad i\ge 0, \tag{35}$$
$$P_1(0) = P_2,$$

where $P_1(i)$ is steady-state state estimation error covariance matrices, and $Q_{\boldsymbol{w}}(i,1)$ denotes the steady-state covariance matrices of $\boldsymbol{w}(i,1)$, and admits

$$Q_{\boldsymbol{w}}(i,1) = \Gamma_1P_1(i)\Gamma_1^{\mathrm{T}} + Q_{\boldsymbol{v}_1}, \quad i>0. \tag{36}$$

Further, for the sake of convenience to discuss, we now define

$$\mathcal{M}_2(t-j,t) \equiv \varepsilon[\boldsymbol{x}(t-j)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t|t,2)],$$
$$\mathcal{M}_1(t-j,t-i) \equiv \varepsilon[\boldsymbol{x}(t-j)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t-i|t-i,1)], \quad i>0,$$

with

$$\widetilde{\boldsymbol{x}}(t|t,2) = \boldsymbol{x}(t) - \widehat{\boldsymbol{x}}(t|t,2),$$
$$\widetilde{\boldsymbol{x}}(t-i|t-i,1) = \boldsymbol{x}(t-i) - \widehat{\boldsymbol{x}}(t-i|t-i,1),$$

where $\widehat{\boldsymbol{x}}(t|t,2)$ is the projection of $\boldsymbol{x}(t)$ onto the linear space $\mathcal{L}\left\{\{\boldsymbol{w}(s,2)\}_{s=N}^{\mathrm{T}}\right\}$, and $\widehat{\boldsymbol{x}}(t-i|t-i,1)$ the projection of $\boldsymbol{x}(t-i)$ onto the linear space $\mathcal{L}\left\{\{\boldsymbol{w}(s,2)\}_{s=N}^{\mathrm{T}},\{\boldsymbol{w}(s,1)\}_{s=t-1}^{t-i}\right\}$.

As $t=0$ and $N\to+\infty$, $\mathcal{M}_2(t-j,t)$ and $\mathcal{M}_1(t-j,t-i)$ will be independent of the time t, which is rewritten as $\mathcal{M}_2(j,0)$ and $\mathcal{M}_1(j,i)$, and can be achieved by

$$\mathcal{M}_2(j,0)=\begin{cases}P_2[\mathcal{A}_2^{\mathrm{T}}]^{-j}, & j\le 0,\\ \Phi^j P_2, & j>0,\end{cases} \tag{37}$$

$$\mathcal{M}_1(j,i)=\begin{cases}P_1(j)\mathcal{A}_1^{\mathrm{T}}(j)\cdots\mathcal{A}_1^{\mathrm{T}}(i-1), & i\ge j,\\ \Phi^{j-i}P_1(i), & i<j,\end{cases} \tag{38}$$

where

$$\mathcal{A}_2=\Phi-\Phi P_2\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\Gamma_2,$$

$$\mathcal{A}_1(i)=\Phi-\Phi P_1(i)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(i,1)\Gamma_1,\quad i>0,$$

and P_2 and $P_1(i)$ are calculated by (33) and (35). $Q_{\boldsymbol{w}}(i,1)$ and $Q_{\boldsymbol{w}_2}$ are calculated via (36) and (34).

2) Solutions to $Q_{\boldsymbol{w}}$ and $K_i,\ i=0,1,\cdots,h$

In order to stress that the unknown matrices $Q_{\boldsymbol{w}}$ and $K_i,\ \ i=0,1,\cdots,h$, play important roles in designing the optimal control law, we give following theorems based on the reorganized innovation approach. The calculation does not require the augmented systems.

Theorem 4.2 *The steady-state innovation covariance matrix $Q_{\boldsymbol{w}}$ complies with*

$$Q_{\boldsymbol{w}}=\begin{bmatrix}\Gamma_{(0)}\mathcal{M}_1(h-1,h-1)\Gamma_{(0)}^{\mathrm{T}}+R_0 & \Gamma_{(0)}[\mathcal{M}_1(-1,h-1)]^{\mathrm{T}}\Gamma_{(1)}^{\mathrm{T}}\\ \Gamma_{(1)}\mathcal{M}_1(-1,h-1)\Gamma_{(0)}^{\mathrm{T}} & \Gamma_{(1)}\mathcal{P}\Gamma_{(1)}^{\mathrm{T}}+R_1\end{bmatrix}, \tag{39}$$

where $\mathcal{M}_1(\cdot,\cdot)$ is calculated by (38), and

$$\begin{aligned}\mathcal{P}=\;&P_2-\mathcal{M}_2(0,0)\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\Gamma_2[\mathcal{M}_2(0,0)]^{\mathrm{T}}\\&-\sum_{i=1}^{h-1}\mathcal{M}_1(-1,i-1)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(i,1)\Gamma_1[\mathcal{M}_1(-1,i-1)]^{\mathrm{T}},\end{aligned}$$

while $Q_{\boldsymbol{w}}(i,1),\ i=1,2,\cdots,h$ and P_2 are given by (36) and (33), respectively.

Proof See Appendix A. ∎

What comes on next is, based on the reorganized innovation approach, to give the explicit formula of $K_i,\ i=0,1,\cdots,h$, in the following theorem.

Theorem 4.3 *K_0 obeys that*

$$K_0=[\,\mathcal{M}_1(h-1,h-2)\quad \mathcal{S}\,]\begin{bmatrix}\Gamma_{(0)} & 0\\ 0 & \Gamma_{(1)}\end{bmatrix}^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}, \tag{40}$$

where

$$\begin{aligned}\mathcal{S}=P_2&-\sum_{j=-1}^{0}\mathcal{M}_2(h-j,0)\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\Gamma_2[\mathcal{M}_2(-j-1,0)]^{\mathrm{T}}\\&-\sum_{j=1}^{h-2}\mathcal{M}_1(h-1,j-1)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(j,1)\Gamma_1[\mathcal{M}_1(-2,j-1)]^{\mathrm{T}}.\end{aligned}$$

$K_i,\ i=1,2,\cdots,h$, satisfy that

$$K_i = [\,\mathcal{M}_1(-1,h-i-1) \quad \mathcal{N}(i)\,]\begin{bmatrix} \mathit{\Gamma}_{(0)} & 0 \\ 0 & \mathit{\Gamma}_{(1)} \end{bmatrix}^{\mathrm{T}} Q_{\boldsymbol{w}}^{-1}, \tag{41}$$

where

$$\begin{aligned}\mathcal{N}(i) = \mathcal{M}_2(i,0) &- \sum_{j=-i}^{0}\mathcal{M}_2(-j,0)\mathit{\Gamma}_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\mathit{\Gamma}_2[\mathcal{M}_2(-j-i,0)]^{\mathrm{T}} \\ &- \sum_{j=1}^{h-i-1}\mathcal{M}_1(-1,j-1)\mathit{\Gamma}_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(j,1)\mathit{\Gamma}_1[\mathcal{M}_1(-i-1,j-1)]^{\mathrm{T}}\end{aligned}$$

with $\mathcal{M}_1(\cdot,\cdot),\mathcal{M}_2(\cdot,\cdot),Q_{\boldsymbol{w}}(\cdot,1)$, and $Q_{\boldsymbol{w}_2}$ are computed as in (38), (37), (36), *and* (34), *respectively.*

Proof See Appendix B. ∎

Remark 4.3 Theorem 4.2 and Theorem 4.3 have given $Q_{\boldsymbol{w}}$ and $K_i,\ i=0,1,\cdots,h$, based on projection theory and time-domain reorganized innovation approach. Thus, by applying Theorem 4.2 and Theorem 4.3, spectral factor $D^{\mathrm{T}}(q^{-1})$ can easily be computed by Theorem 4.1. It should be pointed out that the reorganized innovation is completely different from the innovation in conventional Kalman filtering formulation, which is defined in (13).

5 Comparison of Computational Cost

Since the spectral factorization is the key problem solved in this paper, the section is devoted to compare the computational cost of the spectral factorization via the presented approach and the state augmented method.

Now, we introduce an augmented state

$$\boldsymbol{x}_a^{\mathrm{T}}(t+1) = [\,\boldsymbol{x}^{\mathrm{T}}(t+1) \quad \boldsymbol{x}^{\mathrm{T}}(t+2) \quad \cdots \quad \boldsymbol{x}^{\mathrm{T}}(t+h+1)\,]\,. \tag{42}$$

The backwards state-space model (9)–(11) can be rewritten as an augmented systems

$$\boldsymbol{x}_a(t) = \mathit{\Phi}_a\boldsymbol{x}_a(t+1) + \boldsymbol{e}_a(t), \tag{43}$$

$$\boldsymbol{y}(t) = \mathit{\Gamma}_a\boldsymbol{x}_a(t+1) + \boldsymbol{v}(t), \tag{44}$$

where

$$\mathit{\Phi}_a = \begin{bmatrix} \mathit{\Phi} & & & & \\ I_n & 0 & & & \\ & I_n & \ddots & & \\ & & \ddots & \ddots & \\ & & & I_n & 0 \end{bmatrix}, \quad \mathit{\Gamma}_a = \begin{bmatrix} \mathit{\Gamma}_{(0)} & \cdots & 0 \\ 0 & \cdots & \mathit{\Gamma}_{(1)} \end{bmatrix},$$

and

$$\boldsymbol{v}^{\mathrm{T}}(t) = \left[\,\boldsymbol{v}_{(0)}^{\mathrm{T}}(t) \quad \boldsymbol{v}_{(1)}^{\mathrm{T}}(t)\,\right], \quad \boldsymbol{e}_a^{\mathrm{T}}(t) = [\,\boldsymbol{e}^{\mathrm{T}}(t) \quad 0 \quad \cdots \quad 0\,]$$

are mutually uncorrelated white noises with zero means and covariance matrices $Q_{\boldsymbol{v}} = R$,

$$Q_{\boldsymbol{e}_a} = \begin{bmatrix} Q & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix},$$

respectively.

In light of (43)–(44), spectral factor $D(q)$ is directly calculated by

$$D(q) = [C_a(q)K_a + A_a(q)]Q_{\boldsymbol{w}_a}^{\frac{1}{2}}, \tag{45}$$

where $K_a = \Phi_a P_a \Gamma_a^{\mathrm{T}} Q_{\boldsymbol{w}_a}^{-1}$ and $Q_{\boldsymbol{w}_a} = Q_{\boldsymbol{v}} + \Gamma_a P_a \Gamma_a^{\mathrm{T}}$ are the Kalman gain and innovations variance, respectively, and P_a satisfies the algebraic Riccati equation

$$P_a = \Phi_a P_a \Phi_a^{\mathrm{T}} + Q_{\boldsymbol{e}_a} - K_a Q_{\boldsymbol{w}_a}^{-1} K_a^{\mathrm{T}}.$$

In (45), $C_a(q)$ and $A_a(q)$ are produced via a left MFD of the transfer matrix for the augmented systems (43) and (44) as follows

$$\Gamma_a(I_{n+hn} - \Phi_a q)^{-1} q = A_a^{-1}(q) C_a(q),$$

where $A_a(q)$ and $C_a(q)$ are $(p_0 + p_1) \times (p_0 + p_1)$ and $(p_0 + p_1) \times (n + hn)$ polynomial matrices, respectively.

According to (45), $D^{\mathrm{T}}(q^{-1})$ is easily computed as

$$D^{\mathrm{T}}(q^{-1}) = Q_{\boldsymbol{w}_a}^{\frac{1}{2}}[K_a^{\mathrm{T}} C_a^{\mathrm{T}}(q^{-1}) + A_a^{\mathrm{T}}(q^{-1})]. \tag{46}$$

As well-known, the computational cost is measured by the amount of multiplication and division[13]. Observe (46) and (26), it is not difficult to find that, the former involves much higher dimension matrices operation, and thus yields much more multiplication and division, intuitive higher computational count than the latter. If it is not enough, the below numerical example can supply more convincing argument.

We consider the systems (9)–(11) with $n = p_0 = p_1 = 1$. The results of computational cost for the state augmentation and the presented approach are shown in Figure 1.

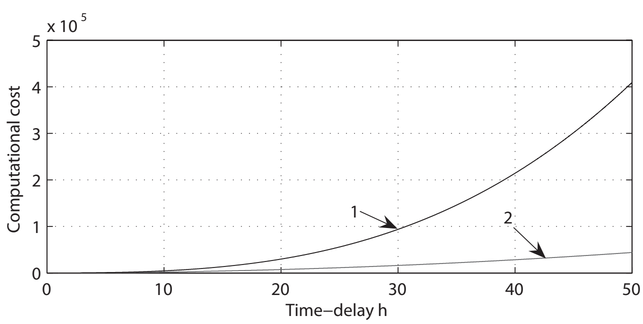

Figure 1 1 state augmentation; 2 the presented approach

Obviously, when the delay h is sufficiently large, it is easy to know that the computational cost for the presented approach is less than that for the state augmentation. Moreover, the larger the delay h is, the larger the difference of the computational cost between the state augmentation and the presented approach is, which implies the presented approach in this paper is more effective.

6 Conclusions

In this paper, the infinite-horizon LQR problem for discrete-time systems with input delay has been investigated. The optimal controller is designed via one spectral factorization and one polynomial equation, where the key technique for deriving spectral factorization is the time-domain reorganized innovation approach. In contrast to the state augmentation, the presented approach is much simpler for derivation and calculation, especially when the time-delay is larger.

Appendix A: Proof of the Theorem 4.2

From (13), we have

$$\boldsymbol{w}(t) = \begin{bmatrix} \Gamma_{(0)} & 0 \\ 0 & \Gamma_{(1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t+1) - \widehat{\boldsymbol{x}}(t+1|t+1) \\ \boldsymbol{x}(t_h+1) - \widehat{\boldsymbol{x}}(t_h+1|t+1) \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{(0)}(t) & 0 \\ 0 & \boldsymbol{v}_{(1)}(t) \end{bmatrix}, \tag{A.1}$$

where $\widehat{\boldsymbol{x}}(t+1|t+1)$ and $\widehat{\boldsymbol{x}}(t_h+1|t+1)$ are the projection of $\boldsymbol{x}(t+1)$ and $\boldsymbol{x}(t_h+1)$ onto the linear space $\mathcal{L}\{\boldsymbol{w}(t+1), \boldsymbol{w}(t+2), \cdots,\}$, which is equivalent with linear space

$$\mathcal{L}\{\boldsymbol{w}(t+1,1), \cdots, \boldsymbol{w}(t_h-1,1), \boldsymbol{w}(t_h,2), \cdots\}.$$

Then $\widehat{\boldsymbol{x}}(t_h+1|t+1)$ becomes the projection of $\boldsymbol{x}(t_h+1)$ onto the linear space

$$\mathcal{L}\{\boldsymbol{w}(t+1,1), \cdots, \boldsymbol{w}(t_h-1,1), \boldsymbol{w}(t_h,2), \cdots\}.$$

Therefore, using projection formula, $\widehat{\boldsymbol{x}}(t_h+1|t+1)$ can be formulated as

$$\begin{aligned} \widehat{\boldsymbol{x}}(t_h+1|t+1) &= \mathrm{Proj}\{\boldsymbol{x}(t_h+1)|\boldsymbol{w}(t+1,1), \cdots, \boldsymbol{w}(t_h-1,1), \boldsymbol{w}(t_h,2), \boldsymbol{w}(t_h+1,2)\cdots,\} \\ &= \widehat{\boldsymbol{x}}(t_h+1|t_h+1,2) + \varepsilon[\boldsymbol{x}(t_h+1)\boldsymbol{w}^{\mathrm{T}}(t_h,2)]Q_{\boldsymbol{w}_2}^{-1}\boldsymbol{w}(t_h,2) \\ &\quad + \sum_{i=1}^{h-1} \varepsilon[\boldsymbol{x}(t_h+1)\boldsymbol{w}^{\mathrm{T}}(t_h-i,1)]Q_{\boldsymbol{w}}^{-1}(i,1)\boldsymbol{w}(t_h-i,1) \\ &= \widehat{\boldsymbol{x}}(t_h+1|t_h+1,2) + \mathcal{M}_2(0,0)\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\boldsymbol{w}(t_h,2) \\ &\quad + \sum_{i=1}^{h-1} \mathcal{M}_1(-1,i-1)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(i,1)\boldsymbol{w}(t_h-i,1). \end{aligned} \tag{A.2}$$

Note that

$$\hat{\boldsymbol{x}}(t+1|t+1) = \widehat{\boldsymbol{x}}(t+1|t+1,1). \tag{A.3}$$

By substituting (A.2) and (A.3) into (A.1), the innovation $\mathbf{w}(t)$ allows us to be rewritten as

$$\boldsymbol{w}(t) = \begin{bmatrix} \Gamma_{(0)} & 0 \\ 0 & \Gamma_{(1)} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{x}}(t+1|t+1,1) \\ \varsigma(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{(0)}(t) & 0 \\ 0 & \boldsymbol{v}_{(1)}(t) \end{bmatrix}, \tag{A.4}$$

where

$$\widetilde{\boldsymbol{x}}(t+1|t+1,1)=\boldsymbol{x}(t+1)-\widehat{\boldsymbol{x}}(t+1|t+1,1)$$

and

$$\begin{aligned}\varsigma(t)&=\boldsymbol{x}(t_h+1)-\widehat{\boldsymbol{x}}(t_h+1|t+1)\\&=\widetilde{\boldsymbol{x}}(t_h+1|t_h+1,2)-\mathcal{M}_2(0,0)\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\boldsymbol{w}(t_h,2)\\&\quad-\sum_{i=1}^{h-1}\mathcal{M}_1(-1,i-1)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(i,1)\boldsymbol{w}(t_h-i,1).\end{aligned}$$

Consequently, the innovation covariance matrix $Q_{\boldsymbol{w}}$ is given by

$$Q_{\boldsymbol{w}}=\begin{bmatrix}\Gamma_{(0)}&0\\0&\Gamma_{(1)}\end{bmatrix}\Theta\begin{bmatrix}\Gamma_{(0)}&0\\0&\Gamma_{(1)}\end{bmatrix}^{\mathrm{T}}+\begin{bmatrix}R_0&0\\0&R_1\end{bmatrix},\tag{A.5}$$

where

$$\Theta=\begin{bmatrix}\varepsilon[\widetilde{\boldsymbol{x}}(t+1|t+1,1)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+1|t+1,1)]&\varepsilon[\widetilde{\boldsymbol{x}}(t+1|t+1,1)\varsigma^{\mathrm{T}}(t)]\\ \varepsilon[\varsigma(t)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+1|t+1,1)]&\varepsilon[\varsigma(t)\varsigma^{\mathrm{T}}(t)]\end{bmatrix}.$$

In view of (38), the following equation can be easily obtained:

$$\varepsilon[\widetilde{\boldsymbol{x}}(t+1|t+1,1)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+1|t+1,1)]=\mathcal{M}_1(h-1,h-1).\tag{A.6}$$

Also, by considering the fact that $\varsigma(t)$ is uncorrelated with $\boldsymbol{w}(t_h-i,1),i=1,2,\cdots,h-1$, it follows that

$$\begin{aligned}\varepsilon[\varsigma(t)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+1|t+1,1)]&=\varepsilon[\widetilde{\boldsymbol{x}}(t_h+1|t_h+1,2)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+1|t+1,1)]\\&=\mathcal{M}_1(-1,h-1).\end{aligned}\tag{A.7}$$

Further, taking into account (A.5), we have

$$\begin{aligned}\varepsilon[\varsigma(t)\varsigma^{\mathrm{T}}(t)]&=P_2-\mathcal{M}_2(0,0)\Gamma_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\Gamma_2[\mathcal{M}_2(0,0)]^{\mathrm{T}}\\&\quad-\sum_{i=1}^{h-1}\mathcal{M}_1(-1,i-1)\Gamma_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(i,1)\Gamma_1[\mathcal{M}_1(-1,i-1)]^{\mathrm{T}}.\end{aligned}\tag{A.8}$$

Substituting (A.6)–(A.8) into (A.5), (39) can be obtained. ▮

Appendix B: Proof of the Theorem 4.3

Note that (A.4), we obtain

$$\boldsymbol{w}(t+i)=\begin{bmatrix}\Gamma_{(0)}&0\\0&\Gamma_{(1)}\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{x}}(t+i+1|t+i+1,1)\\\varsigma(t+i)\end{bmatrix}+\begin{bmatrix}\boldsymbol{v}_{(0)}(t+i)&0\\0&\boldsymbol{v}_{(1)}(t+i)\end{bmatrix},\tag{B.1}$$

where

$$\widetilde{\boldsymbol{x}}(t+i+1|t+i+1,1) = \boldsymbol{x}(t+i+1) - \widehat{\boldsymbol{x}}(t+i+1|t+i+1,1)$$

and

$$\varsigma(t+i) = \boldsymbol{x}(t_h+i+1) - \widehat{\boldsymbol{x}}(t_h+i+1|t+i+1).$$

Applying the projection formula and the reorganization innovation, we get

$$\begin{aligned}
\varsigma(t+i) &= \boldsymbol{x}(t_h+i+1) - \mathrm{Proj}\{\boldsymbol{x}(t_h+i+1)|\boldsymbol{w}(t+i+1,1),\cdots,\boldsymbol{w}(t_h-1,1),\\
&\quad \boldsymbol{w}(t_h,2),\cdots,\boldsymbol{w}(t_h+i,2),\boldsymbol{w}(t_h+i+1,2),\cdots\}\\
&= \widetilde{\boldsymbol{x}}(t_h+i+1|t_h+i+1,2) - \sum_{j=-i}^{0}\varepsilon[\boldsymbol{x}(t_h+i+1)\boldsymbol{w}^{\mathrm{T}}(t_h-j,2)]Q_{\boldsymbol{w}_2}^{-1}\boldsymbol{w}(t_h-j,2)\\
&\quad - \sum_{j=1}^{h-i-1}\varepsilon[\boldsymbol{x}(t_h+i+1)\boldsymbol{w}^{\mathrm{T}}(t_h-j,1)]Q_{\boldsymbol{w}}^{-1}(j,1)\boldsymbol{w}(t_h-j,1)\\
&= \widetilde{\boldsymbol{x}}(t_h+i+1|t_h+i+1,2) - \sum_{j=-i}^{0}\mathcal{M}_2(-j-i,0)\mathit{\Gamma}_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\boldsymbol{w}(t_h-j,2)\\
&\quad - \sum_{j=1}^{h-i-1}\mathcal{M}_1(-i-1,j-1)\mathit{\Gamma}_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(j,1)\boldsymbol{w}(t_h-j,1). \qquad \text{(B.2)}
\end{aligned}$$

Substituting (B.1) into (17), and considering the fact that $\boldsymbol{x}(t_h+1)$ is uncorrelated $\boldsymbol{v}_{(0)}(t+i),\boldsymbol{v}_{(1)}(t+i),\ i=0,1,\cdots,h$, it follows that

$$K_i = \begin{bmatrix}\varepsilon[\boldsymbol{x}(t_h+1)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+i+1|t+i+1,1)] & \varepsilon[\boldsymbol{x}(t_h+1)\varsigma^{\mathrm{T}}(t+i)]\end{bmatrix}\begin{bmatrix}\mathit{\Gamma}_{(0)} & 0\\ 0 & \mathit{\Gamma}_{(1)}\end{bmatrix}^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}. \qquad \text{(B.3)}$$

In view of (38), we have that

$$\varepsilon[\boldsymbol{x}(t_h+1)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t+i+1|t+i+1,1)] = \mathcal{M}_1(-1,h-i-1). \qquad \text{(B.4)}$$

Substituting (B.2) into $\varepsilon[\boldsymbol{x}(t_h+1)\varsigma^{\mathrm{T}}(t+i)]$, it yields that

$$\begin{aligned}
\varepsilon[\boldsymbol{x}(t_h+1)\varsigma^{\mathrm{T}}(t+i)] &= \varepsilon[\boldsymbol{x}(t_h+1)\widetilde{\boldsymbol{x}}^{\mathrm{T}}(t_h+i+1|t_h+i+1,2)]\\
&\quad - \sum_{j=-i}^{0}\varepsilon[\boldsymbol{x}(t_h+1)\boldsymbol{w}^{\mathrm{T}}(t_h-j,2)]Q_{\boldsymbol{w}_2}^{-1}\mathit{\Gamma}_2[\mathcal{M}_2(-j-i,0)]^{\mathrm{T}}\\
&\quad - \sum_{j=1}^{h-i-1}\varepsilon[\boldsymbol{x}(t_h+1)\boldsymbol{w}^{\mathrm{T}}(t_h-j,1)]Q_{\boldsymbol{w}}^{-1}(j,1)\mathit{\Gamma}_1[\mathcal{M}_1(-i-1,j-1)]^{\mathrm{T}}\\
&= \mathcal{M}_2(i,0) - \sum_{j=-i}^{0}\mathcal{M}_2(-j,0)\mathit{\Gamma}_2^{\mathrm{T}}Q_{\boldsymbol{w}_2}^{-1}\mathit{\Gamma}_2[\mathcal{M}_2(-j-i,0)]^{\mathrm{T}}\\
&\quad - \sum_{j=1}^{h-i-1}\mathcal{M}_1(-1,j-1)\mathit{\Gamma}_1^{\mathrm{T}}Q_{\boldsymbol{w}}^{-1}(j,1)\mathit{\Gamma}_1[\mathcal{M}_1(-i-1,j-1)]^{\mathrm{T}}. \qquad \text{(B.5)}
\end{aligned}$$

Substituting (B.4) and (B.5) into (B.3), we can prove (41). By applying the similar approach, K_0 can be obtained as (40). ∎

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