Computer-Supported Collaborative Learning (2013) 8:13–39
DOI 10.1007/s11412-012-9161-0

Analyzing group coordination when solving geometry problems with dynamic geometry software

Diler Oner

Received: 12 January 2012 / Accepted: 3 December 2012 / Published online: 25 January 2013

Abstract In CSCL research, collaborative activity is conceptualized along various yet intertwined dimensions. When functioning within these multiple dimensions, participants make use of several resources, which can be social or content-related (and sometimes temporal) in nature. It is the effective coordination of these resources that appears to characterize successful collaborative activity. This study proposes a methodological approach for studying coordination of resources when solving geometry problems with dynamic geometry software. The aim is to suggest a methodological lens to capture both the content-related and social discourse within the context of geometry problem solving using dynamic geometry software. As an example, the paper also provides an analysis of a dyad's face-to-face interaction using the suggested framework.

Keywords Coordination · Geometry problem solving · Dynamic geometry software · Qualitative inquiry · Face-to-face collaboration

In CSCL research, coordination is viewed as an essential part of collaborative activity that leads to the establishment of mutual knowledge or common ground (Barron 2000). Furthermore, understanding coordination is considered necessary for building tools for collaborative work (Malone and Crowston 1990). The Oxford dictionary defines coordination as "the organization of the different elements of a complex body or activity so as to enable them to work together effectively" (Oxford University Press 2012). In CSCL, coordination, which is only noticeable in its absence (Malone and Crowston 1990), appears to be an aspect of interaction that leads to positive outcomes.

CSCL researchers explain the nature of successful collaboration through examining the coordination of different types of resources employed within interactions. Collaborators take advantage of these resources within a shared and negotiated conceptual space called the *joint problem space* (JPS) (Roschelle and Teasley 1995). A JPS is constructed "through the

D. Oner (✉)
Faculty of Education, Department of Computer Education and Educational Technology, Bogazici University, Bebek, 34342 Istanbul, Turkey
e-mail: diler.oner@boun.edu.tr

external mediational framework of shared language, situation, and activity" (ibid., pp. 70–71). It is formed as an integration of goals, descriptions of the current problem space, awareness of available problem-solving actions, and associations among these aspects. Roschelle and Teasley argued that collaborative problem solving involves two co-existing activities: solving the problem together and building a JPS. According to them, "Conversation in the context of problem solving activity is the process by which collaborators construct and maintain a JPS. Simultaneously, the JPS is the structure that enables meaningful conversation about problem solving to occur" (pp. 75–76). Thus, the notion of JPS suggests interrelated dimensions of collaborative problem solving, in which JPS functions as the ground on which content-related discourse takes place.

Therefore it is reasonable to argue that participants need to coordinate a variety of resources under these dimensions of a collaborative activity. The purpose of the current study is to propose a methodological framework to account for the content-related and social/ collaborative/ relational resources when solving geometry problems with dynamic geometry software (DGS). Content-related resources are investigated by looking at mathematical and DGS-enabled resources. Social resources are examined using the constructs of conversation analysis. In order to highlight the merit of this framework, an analysis of a dyad's face-to-face interaction when solving a construction problem with DGS is provided.

Mathematical problem solving in CSCL

Research that focuses on mathematical problem solving in CSCL shows that participants take advantage of two major kinds of resources within their collaborative activity: content-related and relational. They also highlight the role of the tools that mediate collaborative actions. It appears that the coordination among these resources is what leads to the construction of JPS. This line of research is discussed below focusing on *what* resources participants put to use and *how* they use them within the construction of their JPS.

In a micro-ethnographic case study, Cakir et al. (2009) studied the interaction among three upper-middle grade students who successfully produced and solved a pattern problem through a dual-interaction space (chat and whiteboard). The available technological resources were a shared whiteboard, text chat with graphical referencing tools to the whiteboard, and a wiki. The researchers were particularly interested in examining how participants used multiple modalities, how they made their chat postings and drawings comprehensible to each other, and how they achieved a coherence of actions that took place across different modalities.

The study participants used graphical inscriptions and references, textual chat postings, and symbolic expressions across different media in a coordinated manner. Thus, both graphical and textual resources were utilized by the team members. The ways in which these were coordinated, and how this led to the construction of the group's JPS, were investigated in detail. Users brought relevant mathematical objects to other members' attention through coordinated sequences of actions performed in chat and whiteboard. Basically, group members used two referential methods to bring relevant math objects to other members' attention: marking the drawing with a different color to identify the contour of the mathematical object they had constructed (i.e., a hexagonal array) and using the referencing tool. These mechanisms directed other members' attention to the features of the shared visual field in particular ways. Group members also used deictic terms, the referential link between two spaces, and the term "like" to coordinate between the two interaction spaces.

While Cakir et al. (2009) largely focused on the coordination among the use of different interaction spaces and the content-related discourse within the JPS, Barron (2000) highlighted the relational context, focusing on the communicational processes. More specifically, she investigated the differing interactive processes between two contrasting groups (triads) of sixth graders. The study took place in a face-to-face context, and the participants worked on the first episode of *The Adventures of Jasper Woodbury*, Journey to Cedar Creek (CTGV, 1992, 1997). The interaction between the groups differed along three major dimensions: (1) the degree of mutuality in interaction, (2) the extent to which there was a joint focus of attention, and (3) the level of shared task alignment. These dimensions were described as leading towards the types of interactions that contribute or inhibit coordination. In other words, the degree of coordination among the group members was distinguished and represented along these dimensions.

Mutuality is defined as the extent to which all members could potentially make contributions and have their voices heard in interaction. Thus, it is reflected especially in the treatment of contributions by partners. High markers of mutuality included productive conflicts, transactional responses (engagement with one another's ideas), and respecting turn-taking norms; low levels of mutuality were marked by conflicts of insistence, lack of response to contributions, and violating turn-taking norms.

Joint attention is considered to be related to mutuality. The successful group sometimes worked separately, but the workbook served as the center of coordination. This group monitored the documentation of their solution by hovering over the workbook. Furthermore, attention of the partners was coordinated by showing sensitivity to each other's attention states, such as pausing in speech while others were writing or actively gaining their peers' attention before proposing an idea. While these represented high markers, low markers included treating the workbook as contested territory and individual monitoring.

Shared task alignment meant the degree of establishing a collaborative orientation to the problem rather than an independent orientation. Markers included the co-construction of solutions by the members and frequent echoing and expansion of ideas posed by others. In addition, their roles were complementary; that is, if one participant was generating ideas, the other monitored the documentation. Absence of shared task alignment was signaled by individual solution strategies and lack of responsiveness to queries.

Barron (2000, 2003) showed that rather than merely focusing on the cognitive aspects of the establishment of a JPS, one must equally pay attention to the social dimension to understand variability in collaborative accomplishment. Thus, she argued that collaboration involves an interdependent *dual-problem space* that participants must simultaneously attend to and develop: a *content* space that consists of the problem to be solved, and a *relational* space that involves the interactional challenges and opportunities. It has been further suggested that JPS integrates not only the content and participant relationships, but also includes a *temporal* aspect in longitudinal interactions, which are all shared by the collaborators (Sarmiento and Stahl 2008). Thus, collaborative activity can be conceptualized along these various intertwined dimensions.

Barron (2003) called for better theoretical understanding of collaborative learning that integrates both cognitive and relational aspects of learning. A framework suggested by Evans et al. (2011) appears to capture both the content and relational aspects of collaborative problem solving. Largely motivated by the role of gestures in different collaborative settings (computer-supported and physical), Evans et al. analyzed young children's interactions when solving a geometric puzzle. They adopted a multimodal approach in order to take into account a range of resources, including cognitive, perceptual, and physical resources that were put to use when working with mathematical ideas. Three main discourse categories were developed that represented references made to these resources: *object-, para-*, and *meta-level*. Object-level references were used to refer to an object or place in the physical world, and para-level references were used to refer to the participants', group's or speakers' opinions. Meta-level references were divided into two categories, *mathematic* and *project*, which map onto the content and relational aspects of students' interaction, respectively. When geometric or mathematic principles and properties of puzzle pieces were alluded to, the meta-level references were coded as *mathematic*, such as "*that fits*" or "*it keeps leaving that white space*." When collaborative problem-solving strategies or cooperation were evident in the talk, such as "*let's start over*" or "*my turn goes next*," the utterances were coded as *project*. The successful work of participants was attributed to "the coordination of shared meanings and coreferences to both objects and mathematical and collaborative principles" (p. 275).

Previous work focusing on mathematical problem solving in CSCL settings shows that coordinating among various types of resources is the essence of productive collaboration. Thus, understanding successful collaboration requires theoretical lenses that allow attention to be paid to the interrelated aspects of collaborative activity simultaneously. This study aims to contribute to this line of work by proposing a methodological approach to investigate the content-related and collaborative resources that are put into use in the context solving geometry problems with DGS. While Evans et al. (2011) worked with young learners (7- and 8-year-olds) on the task of a geometric tangram puzzle, the present study focuses on adult learners and the use of DGS for a geometry construction problem. As explained below, DGS provides distinct affordances that reinforce the use of theoretical and perceptual resources within geometry problem solving.

Rather than using the term "coordination of interaction" (Cakir et al. 2009), collaborative activity is framed in terms of *coordinated use of resources* (or lack thereof) as the first step in analysis. It is argued that participants rely on two major resources when working on a geometry problem within the DGS environment: (1) mathematical and tool-enabled (relevant for the content-related aspect) and (2) collaborative resources (relevant for the relational or social aspect). The basic tenet of the study is that the coordination among these resources that characterize collaborative geometry problem solving is what determines success.

The mathematical and tool-enabled resources

In order to unpack the type of mathematical resources that are available to students when solving geometry problems within a DGS environment, Laborde's (2004) distinction between two reference fields is used. One can refer to two types of reference fields to draw conclusions when solving geometrical problems: the spatio-graphical (SG) field and the theoretical (T) field. When functioning within the SG field, students refer to graphical entities and operations on graphical entities, whereas in the T field, theoretical relationships and mathematical knowledge are brought to the fore. Students may also make connections between these reference fields. In fact, expert problem solving involves the use of both reference domains and shifts between them, which is considered "an essential part of the meaning of geometry" (Laborde 2004, p.162). Thus, successful geometry problem solving can be characterized in terms of the interplay between T and SG as well as the use of both domains.

As Laborde (2004) put forward, these two aspects, T and SG, are even more interrelated in DGS environments. When geometrical objects are *constructed*, rather than *drawn*, using the theory, they embody theoretical relationships. These relationships remain invariant when the elements of the construction are altered by dragging, which is the most defining feature of DGS (Goldenberg and Cuoco 1998). This implies that DGS provides a form of feedback that is controlled by the theory underlying the construction (Laborde 2004). If, for instance, the medians of a triangle are constructed, they will concur at a point and will remain so even when the figure is altered by dragging. This will enable students to make conjectures about the medians of a triangle, which represents a shift from the SG to the T level. Furthermore, in order to construct something, one needs to use theory. If students want to construct an equilateral triangle, they need to know that all sides should be equivalent and create such a dependency among the sides. Thus, solving geometry problems with DGS is a unique context in which one can both recognize and use theoretical relationships.

Therefore, when solving geometry problems within a DGS context, students must rely on both theoretical and spatio-graphical inputs. The theoretical input comes from the mathematical knowledge that participants bring to the situation, such as theorems, definitions, and axioms. The spatio-graphical input is afforded by DGS and derives from the perceived physical properties of the figure as well as from the results of the manipulation of the figure (i.e., dragging, measuring, plotting, etc.). Students may also establish connections between these two fields.

Although one would expect the content-related discourse to revolve around these resources, successful interaction cannot be characterized just along the content-related dimension. Conversational processes should also be investigated closely, as this is where the JPS is constructed and maintained (Roschelle and Teasley 1995) and where relational processes are expressed (Barron 2003).

Collaborative resources

In order to investigate the collaborative resources put into use, the constructs of conversation analysis (ten Have 1999) were employed. When doing conversation analysis, interaction among participants can be characterized in terms of analytically distinct yet interlocking organizations (ibid.). Among these, *turn-taking organization, sequence organization*, and *repair organization* were relevant to the transcript analyzed in this study.

Turn-taking organization Conversation analysts have studied the organization of turn-taking extensively. Sacks et al. (1974) identified three aspects of turn-taking: (1) turn-taking occurs, (2) one speaker tends to talk at a time, and (3) turns are taken with as little gap or overlap between them as possible. However, this does not mean that there are not any gaps or overlaps. Most overlap is an indication of turn-taking, in which participants demonstrate their close attendance to one another's talk, rather than being an interruption (Sidnell 2010).

Sequence organization Utterances in interactional talk are sequentially organized. Sequentiality points to the emergent nature of talk, such that "one thing can lead to another." ten Have (1999) clarifies the idea:

> [t]his means that any utterance in interaction is considered to have been produced for the place in the progression of the talk where it occurs (i.e. the 'slot' it fills), especially just after the preceding one, while at the same time it creates a context for its own 'next utterance.' (p. 113)

It is the sequential nature of talk that makes it difficult to divide any conversation into categories, since every utterance is a response to the previous one and sets the context for the upcoming utterances.

Repairs Along with "overlap," one other mechanism with which participants orient to the rules of turn-taking is "repair" (Hutchby and Wooffitt 1998). Incorrect word selections, mishearing, and slips of the tongue are some of the ways in which problems occur in a conversation. Repairs are ways of dealing with these troubles within the conversation. There are four types of repair composed of a matrix of possibilities, depending on who initiated and who carried out the repair: self-initiated self-repair, other-initiated self-repair, self-initiated other-repair, and other-initiated other-repair (ibid.).

The context and participants

The data that are used to apply the suggested framework come from a face-to-face setting in which students used a dynamic geometry program, the Geometer's Sketchpad (GSP) (Jackiw 2001), to solve a geometry problem. In other words, in this context collaboration is supported *around* the computer and GSP is used to mediate it (Lipponen 2002). The participants were two graduate students working toward their master's degrees in science education. They had used GSP a few times before the problem-solving session, and had learned basic menu items and how to construct geometric figures such as equilateral triangles, squares, and parallelograms.

Task

The task given to the dyad in this study was to reproduce (construct) an interactive diagram (Fig. 1). In this diagram, an equilateral triangle (EDF) is constructed inside another equilateral triangle (ABC). As point D is dragged along the side AB, the triangle EDF remains an equilateral triangle. Using Laborde's (2004) distinction, this problem can be considered a "construction problem mapping the SG and T domains" category. Although the construction problem used in the present study is not given in T—as would be the case with these types of problems—students were expected to investigate the dynamic diagram to derive its theoretical properties, and then reproduce the construction using their theoretical knowledge. In order to complete the task successfully, participants needed to realize that triangle ABC was constructed as equilateral. In addition, the line segments DA, FB, and EC were equal and dependent on each other, which theoretically implied that the triangle EDF should also be equilateral.

Data collection and analysis

The interaction between the students was captured through two video cameras. Their interaction with the DGS was captured using a screen capture program that saved the log of their session on the GSP screen. The data analyzed is a 10 min-long dialogue in which the participants were able to successfully construct the given interactive figure (Fig. 1). The researcher (the author) held a moderator role during this interaction providing explanations about the task, and help regarding the use of GSP functions if needed. The data were transcribed using the notations of conversation analysis (ten Have 1999), paying attention to what was said as well as what was done during the talk. Some screen shots from the problem solving session were also provided as figures. The transcript was translated into English by the author.

The data analysis involved two simultaneous engagements: paying attention to both (a) mathematical and tool (GSP)-enabled resources, and (b) collaborative resources.

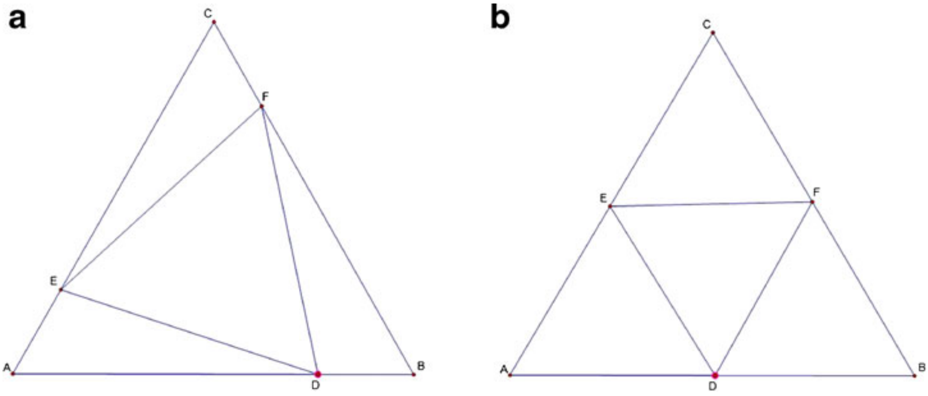

Fig. 1 The two states of the interactive diagram (as Point D is dragged along AB)

The participants' mathematical and GSP-enabled resources were investigated paying attention to the reference fields employed by them. Based on the work of Arzarello et al. (2002), Laborde (2004), and Olivero and Robutti (2007), the following indicators were determined to operationalize the reference fields to which students might refer, and their moves between them:

(1) The indicators of referring to the SG field are: (a) referring to SG aspects of given figures (Laborde 2004); (b) exploring the situation rather randomly by taking measurements of some elements to identify relationships (wandering measurement) (Olivero and Robutti 2007); and (c) using measurements to validate a perception or an intuition about the figure (perceptual measuring) (ibid.).
(2) The indicator of referring to the T field is making a reference to theorems, rules and their implications (Laborde 2004).
(3) The indicators of the move from SG to T are: (a) interpretation of SG in geometrical terms (such as, "this is an equilateral triangle") due to an indicated or observed property in the diagram (Laborde 2004); (b) giving a geometrical reason for something observed in the behavior of the diagram—more specifically, when students use theory to explain the figure (ibid.); (c) using empirical data (due to wandering or perceptual measurement) to make conjectures (Olivero and Robutti 2007).
(4) The indicators of the move from T to SG are: (a) Prediction based on geometrical knowledge or what should happen in the SG domain (Laborde 2004); (b) experimentation on the diagram based on geometrical knowledge (ibid.); (c) using measurements to verify a formulated conjecture (validation measuring) or to verify a theorem that is proved (proof measuring) (Olivero and Robutti 2007); (d) using the drag mode to test a conjecture (Arzarello et al. 2002).

In order to explicate the collaborative resources participants were relying on, the transcript was analyzed in terms of turn-taking organization, sequence organization and repair organization (ten Have 1999). In terms of turn taking, gaps or overlaps in the talk (if any) and who was addressing whom was investigated. In terms of sequence organization, patterns of subsequent actions were analyzed looking at who initiated utterances, who responded, and how. Finally, for repairs, incorrect word selections, mishearing, and slips of the tongue, and who initiated and who performed the repair were determined.

Table 1 Summary of SG and T levels and moves between them in the first phase

Sequence	Reference fields	Explanation
1	$(\text{SG})_{\#1}$	Use of measurement to validate the perception that “Are the line segments equal?” (1st phase, Line 5).
2	$(\text{SG to T})_{\#1}$	Making the conjecture that the triangle is equilateral—interpreting the SG in geometrical terms. (1st phase, Line 29).
3	$(\text{SG})_{\#2}$	Taking the measurements as exact and point D as located exactly as a midpoint (1st phase, Lines 31–33). (Implicit)
4	$(\text{T to SG})_{\#1}$	Using measurement to verify a conjecture about the figure—that the outer triangle (ABC) was equilateral (1st phase, Lines 35–37).

Findings

The problem-solving session of the two participants, who will be called Ayla and Mete here, involved five phases: (1) investigating the triangle ABC and line segments DA, FB, and EC; (2) investigating the triangle EDF and other smaller triangles; (3) suggesting solution strategies; (4) constructing the outer equilateral triangle (A′B′C′); and (5) constructing the inner equilateral triangle (E′D′F′). Each phase was examined by considering the reference fields from which the participants drew. Also, excerpts from each phase were selected focusing on the most notable aspects, and were analyzed in detail by considering the turn-taking, sequence, and repair organizations of the conversation (ten Have 1999).

In the first two phases of their problem solving session, the participants investigated the given figure, trying to understand its properties. In phase 1, they realized that the line segments DA, FB, and EC were equivalent, and that the outer triangle, triangle ABC, was equilateral, by using the measure function. In phase 2, they explored the properties of inner triangles and recognized that the inner triangle, triangle EDF, was also equilateral. During these investigations, they made use of both reference fields (T and SG), and moved between them frequently (Tables 1 and 2).

Phase 1: Investigating the triangle ABC and line segments DA, FB, and EC

The problem solving session of the dyad started with the following conversation (Excerpt 1).

Excerpt 1

Line	Time	Speaker	What is said	What is done
1	0:0:42;15	**Researcher:**	we will replicate this figure	
2		**Mete:**	(2.0) hah goodness gracious	Mete smiles.
3		**Mete:**	hmm	
4		**Researcher:**	you could-	
5	0:0:50;17	**Ayla:**	[are t:these <u>equal</u> (.) these distances	Ayla points at line segments DA, FB, and EC on the screen.
6		**Researcher:**	have a look	
7		**Ayla:**	<u>let’s measure</u>	They select the line segment DA.

8		**Researcher:**	um the figure has a special property let me tell you while you are measuring when you move D the figure also moves according to a rule	They measure line segment DA. They select line segment EC and measure it.
9	0:01:09;16	**Mete:**	°equal°	
10		**Ayla:**	°uh huh°	Ayla nodes.
11		**Ayla:**	((inaudible whispering voice))	Finally, they select the line segment FB and measure it .
12		**Mete:**	((inaudible whispering voice))	
13		**Ayla:**	(2.5) yes=	The lengths of all three line segments are shown on the screen as 1.96 cm.
14	0:01:21;14	**Mete:**	=we move D	Mete drags D along AB.
15		**Ayla:**	(2.0) ° the others are also moving °=	
16		**Mete:**	=oh very good ve:ery good	
17	0:01:27;24	**Mete:**	(2.1) hmm right there	Mete drags D somewhere in the middle of AB.
18		**Ayla:**	(1.5) hmm=	
19		**Mete:**	=hmm	
20		**Researcher:**	hmm what's there=	
21		**Ayla:**	=hmm hihi	Ayla giggles.
22		**Mete:**	[nice figure hihihi	
23	0:01:36;16	**Mete:**	now I got suspicious of the thing	
24		**Mete:**	here hm-	Mete moves the mouse over point E.
25		**Mete:**	this is an amazing triangle	
26		**Ayla:**	ehehe	Ayla laughs.
27		**Mete:**	[we did not measure EF did we	Mete selects line segment EF.
28		**Ayla:**	[it is a magnificent triangle	
29	0:01:42;27	**Ayla:**	it is an equilateral triangle=	
30		**Mete:**	=probably it seems so	
31	0:01:44;28	**Mete:**	let's measure this	Mete wants to measure the line segment EF.
32		**Ayla:**	(1.2) shall we measure the angles=	The length of EF is shown on the screen, which is different from the lengths DA, EC or FB.
33		**Mete:**	[nope I mean it does not look so	
34	0:01:46;26	**Ayla:**	why (1.2) °shall we measure the angles ((inaudible))°	
35				They select point E, point C, and then F in order and measure the angle ECF and its measure is displayed as 60° on the screen.
36				They measure angles FBD and EAD— the other interior angles of the triangle ABC.
37	0:02:18;20	**Mete:**	(27.1) equilateral triangle (.) the one outside	

Table 2 Descriptions of SG and T levels and moves between them in the second phase

Sequence	Reference fields	Explanation
1	$(\text{SG to T})_{\#2}$	Making the conjecture that EDF was also equilateral (2nd phase, Line 44).
2	$(\text{T to SG})_{\#2}$	Use of measurements (and geometrical knowledge) to verify the conjecture that EDF was equilateral (2nd phase, Lines 45–51).
3	$(\text{T})_{\#1}$	Referring to geometrical knowledge about the sum of interior angles of a triangle to derive the measure of the third angle (2nd phase, Line 51).
4	$(\text{SG to T})_{\#3}$	Making the conjecture that EAD, CEF, and FDB were also equilateral (2nd phase, Line 52).
5	$(\text{T})_{\#2}$	Referring to geometrical knowledge about alternate interior angles being congruent when formed by two parallel lines (2nd phase, Lines 62–67). (Implicit)
6	$(\text{SG})_{\#3}$	Ignoring the fact that D was a draggable point, and points E and F were dependent on it, choosing angles to measure that form alternate interior angles (2nd phase, Line 62). (Implicit)
7	$(\text{T to SG})_{\#3}$	Attempting to verify that EAD, CEF, and FDB were equilateral (2nd phase, Line 62).
8	$(\text{T to SG})_{\#4}$	Prediction on SG level based on geometrical knowledge: "Sixty we could make it though / when this is exactly parallel to this" (2nd phase, Lines 70–71).

When this transcript is examined in light of turn-taking organization, one can realize that the participants talked to each other without leaving many gaps between their turns. (The equal signs in the transcript indicate that there are no intervals between successive turns). In addition, their talk sometimes overlapped. However these overlaps did not necessarily indicate a close orientation to each other's talk. Although Ayla was mostly aligning with Mete, Mete was not always addressing her. More specifically, in line 21 while Ayla giggled, Mete seemed to respond to the researcher in the following line (Excerpt 1, Line 22). In line 27, Mete was mostly occupied with measuring the line segment EF as Ayla kept on aligning with him in line 28. In addition, both participants were addressing the researcher as the third interlocutor. Looking at the transcript in terms of sequence organization will better help to unpack the interaction structure.

As soon as the problem statement was explained to them, Mete seemed puzzled (Excerpt 1, Line 2). Yet it was Ayla who wondered whether lengths of the line segments DA, FB, and EC were equal. After the researcher told them to "have a look," Ayla suggested measuring and Mete accepted the offer. While measuring both participants had a whispering voice, which was inaudible, yet they seemed to be helping each other with taking the measurement. Ayla responded to her question with a "yes" (Excerpt 1, Line 13) when the lengths of all three line segments were shown on the screen as 1.96 cm. In terms of the reference fields, the participants were operating at the SG level (#1, Table 1), as they used *perceptual measuring* (Olivero and Robutti 2007) to validate a perception ("are the line segments equal?").

In line 14, Mete narrated his action with the GSP as he dragged point D along the side AB. His utterance functioned as an "evidence of understanding" of what the researcher said, thus could be considered a *contribution* (Clark and Schaefer 1989). And Ayla's next utterance, "the others are also moving" (Excerpt 1, Line 15), functioned in two different ways. She completed Mete's sentence, which she must have perceived as not finished. Meanwhile, she was also responding to the researcher's *presentation*. With this line, she managed to align both with the researcher and with Mete.

Mete, on the other hand, was preoccupied with something else. He moved point D somewhere in the middle of line segment AB and said "hmm right there" (Excerpt 1, Line 17). He said he "got suspicious of the thing" (Excerpt 1, Line 23). Later he wanted to measure the line segment EF (Excerpt 1, Line 27). But here he abruptly left this line of thought and did not articulate any reason. Rather he said: "this is an amazing triangle" (Excerpt 1, Line 25). Ayla again used utterances that showed alignment with him in two different ways. First was when Mete seemed puzzled and uttered "hmm" (Excerpt 1, Lines 17 and 19). Ayla responded to his puzzlement verbatim. The next was when she uttered the parallel expression "it is a magnificent triangle," which matched Mete's "this is an amazing triangle" (Excerpt 1, Line 25). Next Ayla made the conjecture that the triangle was an equilateral one (Excerpt 1, Line 29). As she interpreted the SG in geometrical terms, this represented a move from SG to T (#1, Table 1). However, it was not clear which triangle she was referring to as both triangles looked equilateral at that particular time—point D was moved somewhere in the middle of line segment AB (Fig. 2c). Mete seemed to agree: "probably it seems so" (Excerpt 1, Line 30). Yet he continued to pursue his unarticulated perception about the line segment EF by measuring it. Thus, while seeming that he was carrying the conversation with Ayla, he was mostly carrying the conversation with himself. At this point Ayla noticed the problem of foci and suggested to measure the angles, again addressing Mete (Excerpt 1, Line 32). Seeing that EF was different from the lengths of other line segments (i.e., DA, FB, and EC), Mete said "nope I mean it does not look so," which seemed as an utterance rather taking place in his *personal channel* (Sfard 2001). Mete's utterance also suggested that he thought that all four (smaller) triangles must be congruent when D is located in the middle of AB. As he took measurements as exact and point D as located exactly as a midpoint, he stayed at the SG level (#2, Table 1). Then Ayla repeated her initial suggestion yet this time with a lower voice "Why? Shall we measure the angles?" (Excerpt 1, Line 34). This suggested a trouble in the communication, as Mete's action was probably not meaningful to Ayla at this point. Based on his earlier statement, she must have been expecting the relevant action, which was to show that the triangle was equilateral.

Mete next decided to attend to Ayla's suggestion as he measured the interior angles of the triangle ABC. Finding that they were all 60°, the pair concluded that the outer triangle was equilateral. As they used measurements to verify a conjecture about the figure, they moved from T to SG level (#1, Table 1). Thus, it can be inferred from here that, as Ayla seemed satisfied, her conjecture in line 29 was about the outer triangle, ABC, although it was not clearly articulated earlier. Mete must also have understood this as he measured the correct angles.

Phase 2: Investigating the triangle EDF and other small triangles

The second phase, in which the participants investigated the triangle EDF and other smaller triangles, showed similarities with the phase 1. Both phases were marked by some of Mete's individual actions, which differed noticeably from those in other phases. Although Ayla kept on addressing Mete during the conversation, Mete was again partly interacting with his geometrical knowledge and the software, and his actions were not all clear to Ayla. In this phase, Mete's interest in the line segment EF became clearer, which was evident in the following conversation (Excerpt 2).

Excerpt 2

Line	Time	Speaker	What is said	What is done
43	0:02:23;18	**Ayla:**	ECF, ED((inaudible)) <u>inside</u> (.) <u>the ones inside are too-</u>	
44		**Mete:**	[<u>the small</u> (.) <u>the small one is too</u> probably equilateral=	

45		**Ayla:**	=let's measure right away	Mete selects the points D, E, and F. ((Point D seems to be located in the middle of side AB)).
46	0:02:29;18	**Ayla:**	(1.4) what did you do=	
47		**Ayla:**	=oh you're measuring the one inside=	((The measure of DEF is shown as 60° on the screen)).
48		**Mete:**	=is there a shortcut for angle	
49		**Researcher:**	there isn't just like you do	Mete measures the angle EFD.
50				((The measure of EFD is shown as 60° on the screen)).
51	0:02:39;27	**Mete:**	(3.7) this is too equilateral	((He concludes that triangle EDF is also equilateral, without measuring the angle EDF)).
52		**Mete:**	hhh (.) t:these are also equilateral (.) naturally	Mete moves the mouse over points B, C and A very fast.
53		**Ayla:**	(1.3) we don't know though if equilateral	Ayla touches the screen over point A.
54		**Ayla:**	now this is sixty	Ayla points at the angle EAD on the screen.
55		**Ayla:**	this is also sixty	Ayla points at the angle FED (with her finger).
56	0:02:51;24	**Ayla:**	did you measure that	Ayla points at the angle FDE.
57		**Mete:**	didn't but sixty obviously	
58		**Mete:**	these two are sixty	He moves the mouse over angles FED and EFD.
59		**Mete:**	this is also sixty	He moves the mouse over FDE.
60		**Ayla:**	[uh huh	
61		**Mete:**	[inside	
62	0:02:58;06	**Mete:**	let's look at that (.) that angle=	He selects points F, D, and B ((to measure angle FDB)).
63		**Ayla:**	[if you check that too it will be done	
64				((The measure of FDB is shown as 58° on the screen)).
65	0:03:04;25	**Mete:**	(4.1) ahh not sixty (.) why not	
66		**Researcher:**	you measured FDB	Mete starts dragging the point D along AB.
67		**Mete:**	[sixty-	He moves the point D back on somewhere in the middle of AB.
68		**Researcher:**	[cause FDB changes=	
69		**Ayla:**	=it is something that changes yes=	
70		**Mete:**	=sixty we could make it though	
71	0:03:16;22	**Mete:**	(1.7) when this is exactly parallel to this	He points at line segments AB and then EF with his finger.
72		**Ayla:**	that would be sixty yes=	

73		**Mete:**	=uh huh	
74		**Ayla:**	=by those alternate interior angles	Ayla makes a “Z” with her hand.
75		**Mete:**	[t:that yes=	
76	0:03:21;20	**Ayla:**	=uh huh	

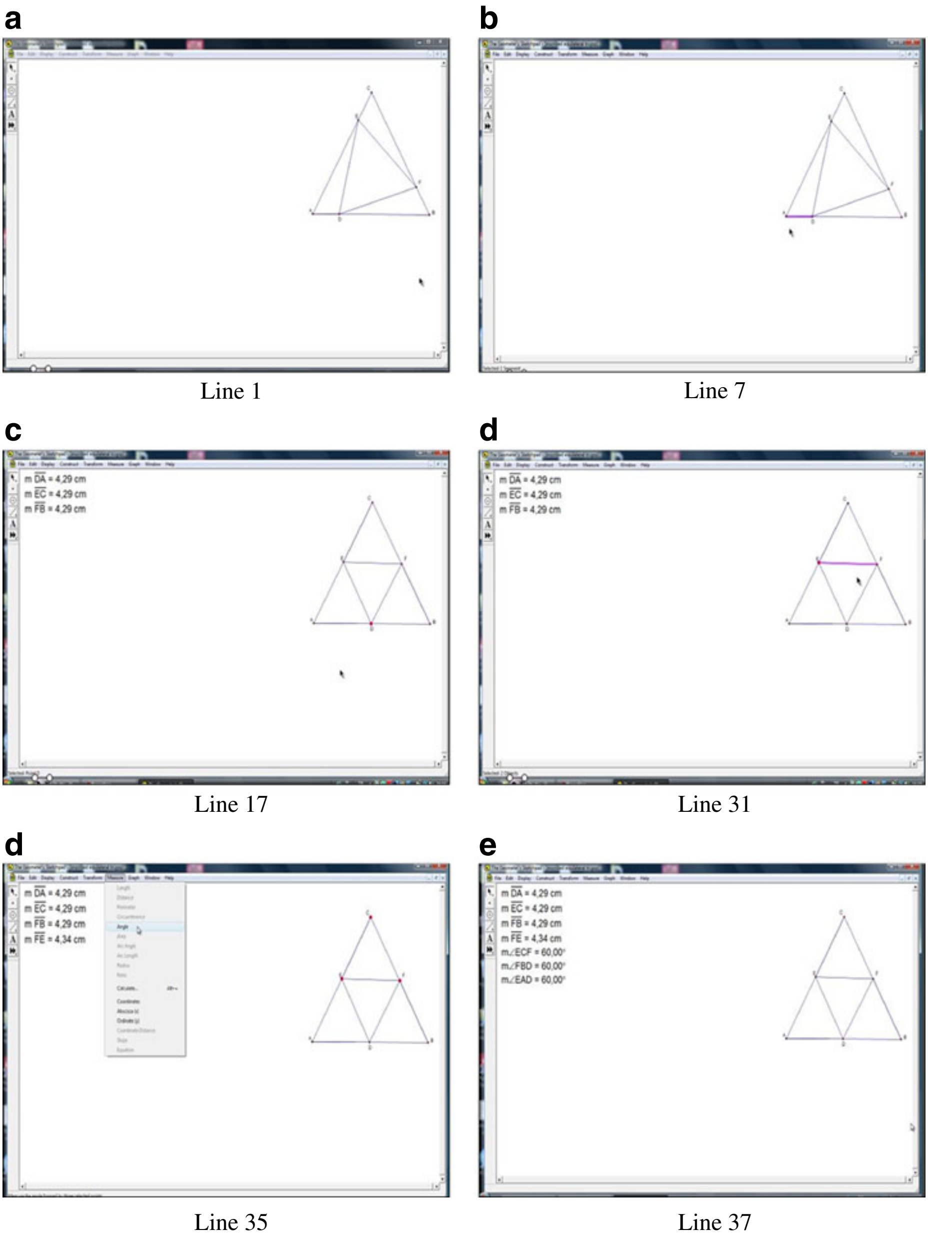

Fig. 2 Screens from the first excerpt

The excerpt starts with Ayla's unfinished sentence "ECF, ED ((inaudible)), inside, the ones inside also." This utterance overlapped with that of Mete, who conjectured that the small triangle was also equilateral, which was a move from SG to T (#2, Table 2). Mete was referring to the triangle EDF (based on his subsequent actions), while Ayla perhaps wanted to say something about other small triangles. However, she immediately accepted what Mete said and suggested measuring and thus verifying Mete's conjecture. When Mete started to select points D, E, and F to measure angle DEF, Ayla detected a discrepancy, and asked "what did you do?" (Excerpt 2, Line 46). Then without giving Mete any chance to explain, she realized, "oh you are measuring the one inside." Here it can be argued that Ayla just acted cooperatively, even though she at first did not understand what Mete was referring to as the "small one." Yet as soon as Mete started measuring, she realized the misalignment. However, no explanation was necessary; she instantly inferred Mete's meaning by looking at the GSP screen, where the measure of DEF was displayed as 60°. Here Mete was doing *validation measuring* (Olivero and Robutti 2007); that is, he used measurements to verify his conjecture. As this move took place on the GSP screen, it helped Ayla to track Mete's actions.

In what followed, while Ayla was still busy inferring meaning from Mete's actions, Mete was occupied with something else. He asked the researcher if there was a shortcut to measure all the angles at once (Excerpt 2, Line 48) while measuring the angles of triangle EDF one by one. He measured only two of its interior angles and concluded that triangle EDF was also equilateral. Here Mete must have used his theoretical knowledge of the sum of the interior angles of a triangle, thus operating at the T level (#1, Table 2). Meanwhile he used measurements, as well as his geometrical knowledge, to verify his conjecture (EDF is equilateral) and thus moved from T to SG (# 2, Table 2). However, he did not explain any of these actions to Ayla explicitly, which prompted her inquiry (more on this below). He simply announced that "this is too equilateral" (Excerpt 2, Line 51).

It is evident that during these actions that Mete was mostly interacting with the software and his geometrical knowledge and neglecting his partner. He made a further conjecture that "these are also equilateral naturally" (a move from SG to T, #3, Table 2). While saying this, he moved the mouse very quickly over the points B, C, and A, which suggested that he was referring to triangles EAD, CEF, and FDB. At this point Ayla disagreed by stating that they did not know if those triangles were equilateral (Excerpt 2, Line 53), even though she was about to make a similar conjecture in the beginning (see Excerpt 2, Line 43). Thus it could be argued that with her utterance in line 53 her main goal was to slow her partner down. Next she reminded him that the measures of the angles EAD and FED were 60°, and asked: "did you measure that?" pointing at the angle FDE with her finger. This was the angle whose measure Mete had just inferred by the theorem about the sum of the interior angles of a triangle. Mete then replied that he did not measure it, but it was obvious (Excerpt 2, Line 57). He next stated that "these two are sixty" while moving the cursor over the angles FED and EFD. Thus he only implied that he derived it by knowing the two angles of the triangle (Excerpt 2, Lines 58–59). This episode shows that Ayla was not able to follow Mete's actions when they took place at the theoretical level.

Nevertheless, Ayla sounded content with this explanation and played along when Mete wanted to measure the angle FDB, saying, "if you check that too it will be done" (Excerpt 2, Line 63). The purpose of the measurement should have been to verify one of Mete's conjectures, namely that FDB (one of the small triangles formed) was also equilateral. However, Mete was disappointed when he found out that the measure of the angle FDB was 58° rather than 60 (Excerpt 2, Line 65). He was expecting that angles EFD and FDB would be congruent and 60° by the theorem that alternate interior angles are congruent when formed by two parallel lines (which was evident in the following talk). Two aspects of this section are worth mentioning.

First, Mete was operating at the SG level (#3, Table 2). That is, he ignored the fact that D was a draggable point, and points E and F were dependent on it. At that moment on the screen, D was located somewhere in the middle of side AB, and side EF seemed parallel to side AB (Fig. 3b). That is, he did not construct the figure as such, but placed the elements perceptually. Meanwhile he made a prediction at the SG level based on his geometrical knowledge, which represented a move from T to SG (#4, Table 2). However, he did this while treating the figure as it appeared on the screen at that particular moment.

The second point was that none of these actions was visible to Ayla at first; they were taking place within Mete's *personal channel* (Sfard 2001). If Mete had verified his prediction and proceeded, this might have gone unnoticed by his partner, as before. Luckily GSP provided the necessary feedback, which was also visible to his partner. When the researcher pointed out that the angle FDB was changeable, Ayla noted that "it is something that changes yes" (Excerpt 2, Line 69). Next Mete decided to explain his reasoning, adding that they could have made it 60 when line segment EF became parallel to side AB. When the geometrical reason was made clear, Ayla confirmed the statement (Excerpt 2, Line 72) and articulated it with a "*Z*" gesture (Excerpt 2, Line 74).

The analysis shows that in the first two phases some of Mete's actions were opaque to Ayla compared to the other phases of their problem solving session discussed below. Within

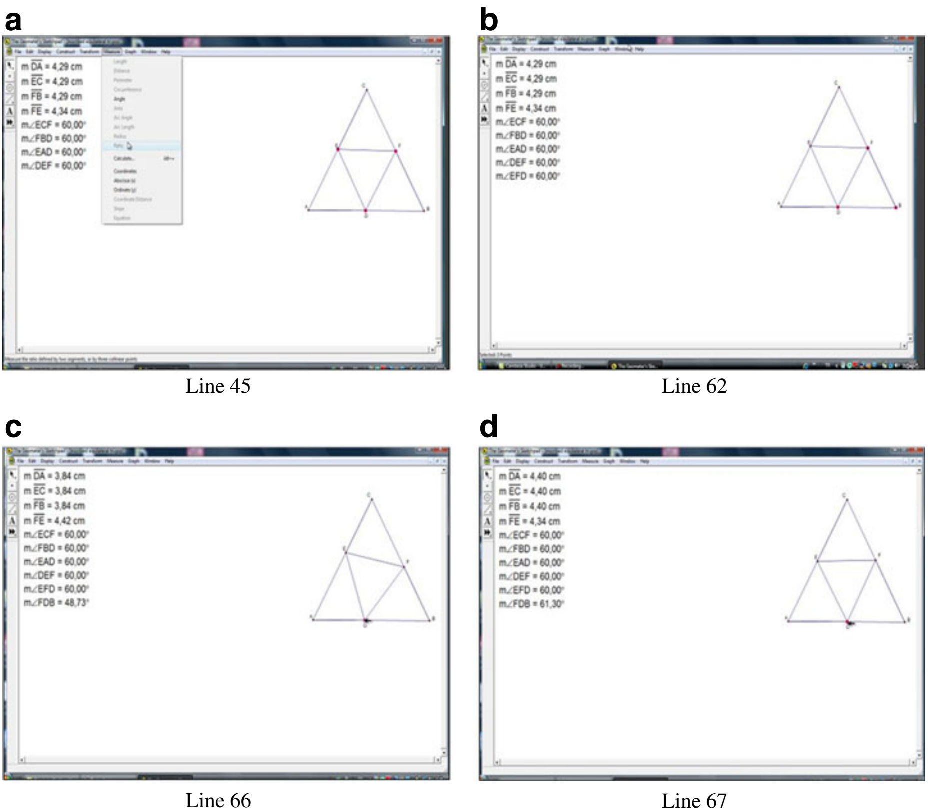

Fig. 3 Screens from the second excerpt

the second phase, when Mete's GSP actions represented a move from the T to the SG level, Ayla was able to infer the meaning of his actions through the GSP screen. However, when he was simply operating at the theory level and no related SG level actions were taking place on the GSP, Ayla was not able to follow him. Another event that led Mete to slow down and explain his thinking was when there appeared to be a discrepancy between his prediction about the figure and the measurements shown on the GSP screen. This move from T to SG also made visible to Ayla that there was an inconsistency between theory and figure. Unlike in the first phase, this prompted Mete to explain his thinking about the figure.

Phase 3: Suggesting solution strategies

In the third phase of their problem solving session, the participants started to think about a solution for the given problem. Unlike in the first two phases, both partners addressed each other closely. They did not put any content-related resources into use in this phase; rather, their interaction primarily reflected the relational resources of which they took advantage. These included a*greeing on a strategy, overlapping talk, and explaining the figure,* which are exemplified in the following excerpt. More specifically, in the episode below one can see that the participants agreed on a strategy without knowing whether it was feasible with GSP. While they did that, there was overlapping talk. In addition, Mete explained the figure to his partner with demonstrations and gestures on the GSP screen.

Excerpt 3

Line	**Time**	**Speaker**	**What is said**	**What is done**
10	0:03:38;10	**Ayla:**	now how can we draw this h:h first we'll draw an equilateral triangle?=	
11		**Ayla:**	what property does the one to be drawn inside have	
12		**Mete:**	(3.7) any one-	
13		**Mete:**	(1.5) one-	
14	0:03:53;07	**Mete:**	(2.5) we will determine the sixty degree (.) and go to the other side	His hand moves around the point F.
15		**Mete:**	I mean (.) we'll determine a po- ((self repair)) one point on the side	
16		**Ayla:**	two to make it sixty degrees	She makes an angle with her hand.
17		**Mete:**	[to make it like we will go to the other side=	He makes some gestures with his hands also.
18		**Ayla:**	=things we'll draw=	
19		**Mete:**	=and then connect with the other so it will be sixty=	
20	0:04:07;05	**Ayla:**	[we'll connect them so it is sixty	
21		**Mete:**	[where they intersect are these three points	He traces the triangle EDF with his hands on the screen.
22	0:04:11;08	**Mete:**	for instance (2.0) we can't change F=	He tries to drag point F.
23		**Ayla:**	=we can't	
24	0:04:14;14	**Ayla:**	we'll do it first from D then	
25		**Mete:**	[uh huh yes	
26		**Mete:**	(2.2) like this=	He drags D until it coincides with point B.

27		**Ayla:**	=yes	
28		**Mete:**	(1.6) it becomes the whole triangle the other way	He drags point D back to the middle of line segment AB.
29		**Mete:**	cause this point D takes any value between A and D- B ((self repair))	He points at points A and B with the mouse.
30	0:04:27;04	**Mete:**	when D comes to B sharp, the two triangles become equal like this=	He points at points A and B with his finger.
31		**Ayla:**	=yes yes=	
32		**Mete:**	=become like this=	He drags D towards B but does not overlap them.
33		**Ayla:**	=okay	
34	0:04:34;10	**Mete:**	or like this	He drags D towards A but does not overlap them.

In this conversation, one can see that the participants did not leave many gaps between each other's utterances; indeed, their talk overlapped. Between lines 17–21 (Excerpt 3), one can observe a series of overlaps, which may be characterized as *recognitional* (Jefferson 1983). That is, the participants seemed to recognize what the other was trying to utter and their overlapped talk was meant to demonstrate this recognition. Thus rather than interrupting the talk, the participants were establishing the ground that they understood what the other was trying to say.

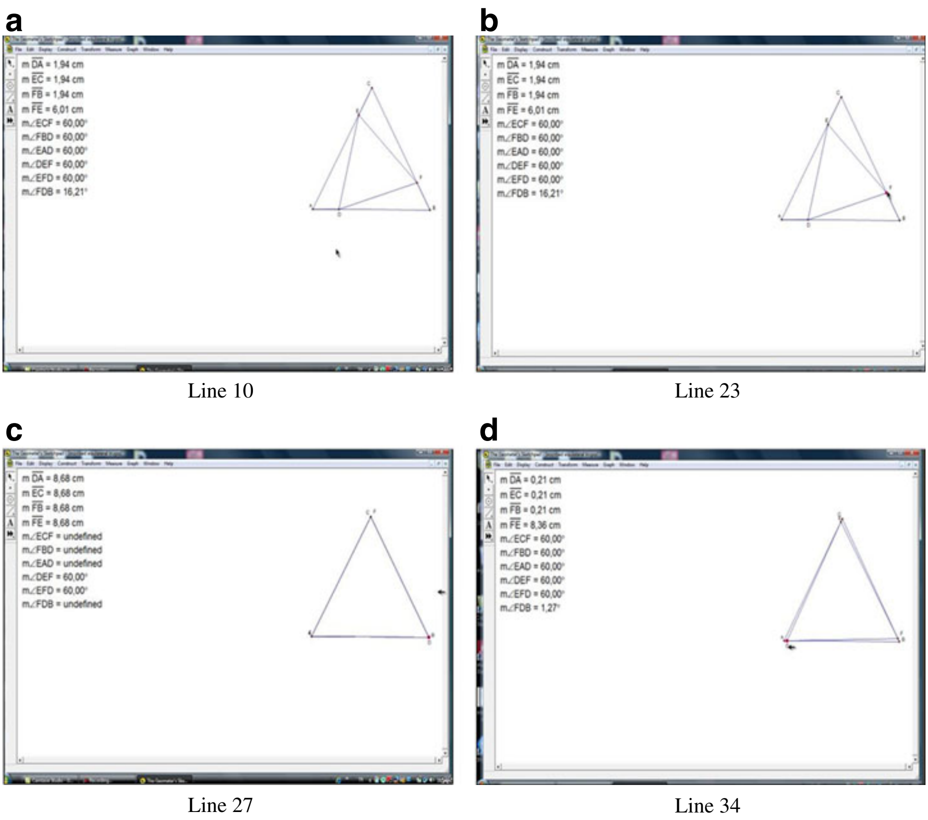

Fig. 4 Screens from the third excerpt

The episode starts with Ayla's vocalization of the problem statement. Namely, they needed to draw an equilateral triangle first, and their problem was to figure out the property of the figure which they were supposed to draw inside the equilateral triangle (Excerpt 3, Lines 10–11). Mete accepted this strategy and built on it by suggesting that they proceed from angles. He said that they should construct line segments intersecting at 60° inside the triangle ABC. However, as will become clear below, he did not know how to construct such line segments with the GSP. Nevertheless, Ayla seemed to understand Mete's strategy very well, which was evident from the recognitional overlaps in lines 17–21. At this stage, the participants seemed satisfied with their strategy without knowing whether it was feasible within the GSP environment.

Another critical feature evident here was Mete's explanation to Ayla of the two important aspects of the figure which happen to be crucial in replicating it. Furthermore, he demonstrated what he meant either by manipulating the figure on GSP or pointing things out with his finger. The first was that point F cannot be changed (i.e., dragged) (Excerpt 3, Line 22), which led Ayla to state that they should start from point D (Excerpt 3, Line 24). The second was that while D moves along the line segment AB and reaches the end points A and B the two triangles become congruent (Excerpt 3, Lines 29–34, Fig. 4d). One can also argue that Ayla was already aware of these two aspects, as she immediately suggested starting from point D and acknowledged Mete's explanations instantly (Excerpt 3, Lines 31 and 33) without showing any sign of trouble in the interaction.

In addition, Mete incorporated self-initiated self-repairs in lines 15 and 29. That is, he immediately corrected his utterances that could be problematic in the communication. In line 15, he specified the "a point" by repairing it with "one point on the side." In line 29, he corrected his slip of tongue—he first said point D, yet he meant point B.

Thus, this episode primarily exemplifies the relational resources that the participants used. More specifically, participants agreed on a strategy through overlapping talk, explained the figure and its properties with demonstrations and gestures, and engaged in self-repair when they recognized that their utterances could be problematic.

Phase 4: Constructing the outer equilateral triangle

In the fourth phase, the participants managed to reproduce the outer equilateral triangle, which was a task they had completed before, yet did not immediately recall. One striking aspect of the third phase that the analysis brought forward was that while the participants were putting the content-related resources into use, they were also relying on the relational resources. This mainly took the form of *narrating and performing GSP actions together,* which is demonstrated in the following excerpt.

Excerpt 4

Line	Time	Speaker	What is said	What is done
37	0:05:52;14	**Ayla:**	yes, control l	Mete constructs a radius by connecting the control point and the center using CTRL+ L—the shortcut to construct a line segment in GSP.
38		**Ayla:**	<u>now</u> select the point=	
39		**Mete:**	=select the point	Mete selects the control point and drags along the circle.
40		**Mete:**	select the line ((he means radius))	He selects the radius.
41		**Ayla:**	well (.) <u>construct</u>	
42	0:06:02;22	**Ayla:**	circle by ° center radius ° okay	

43		**Mete:**	[uh huh okay	
44		**Mete:**	we'll determine those points they in- intersect=	He selects both circles and goes to "construct intersections" in the menu.
45		**Ayla:**	=yes=	
46		**Mete:**	=intersection	They find the intersection points of two circles.
47		**Ayla:**	(1.9) uh huh=	
48	0:06:11;24	**Mete:**	=now two isosceles oops equilateral ((repair)) triangles are formed here=	
49		**Ayla:**	=when we connect this	She points at three points that would form a triangle with finger.
50		**Ayla:**	ha that ((inaudible))	Mete selects the other three points.
51		**Mete:**	connect	
52	0:06:24;11			They construct the equilateral triangle by connecting the points.
53		**Mete:**	(7.7) okay now	
54		**Ayla:**	[okay	
55	0:06:27;24			Mete applies the drag test.

As in the previous excerpt, there were few intervals between the utterances of the participants. Both Mete and Ayla narrated and performed GSP actions together using the keyboard shortcuts and the mouse simultaneously. Although the dyad had constructed equilateral triangles before, at first neither of the participants recalled how to do it. Prior to the given talk, Mete was playing with some circles, constructing, deleting, and dragging them randomly. After some trial and error, Ayla remembered that they were supposed to intersect two circles. The episode starts with Ayla's suggestion of using a shortcut to construct a line segment (ctrl + L) in GSP, rather than using the menu item, and for Mete to construct a radius by connecting the control point and the center of a circle. From then on, Ayla kept both giving directions and narrating Mete's actions (Excerpt 4, Lines 38–42). Starting with line 46, Mete also began to narrate his actions. As in the previous episode, Mete self-initiated self-repair of his slip of the tongue in line 48, when he mistakenly uttered "isosceles" while he meant "equilateral."

Throughout this episode, the participants' actions and utterances represented a form of collaboration that consisted of giving directions and narrating their own and each other's actions. They also made use of content-related resources. In line 48, Mete interpreted the

Table 3 Descriptions of SG and T levels and moves between them in the fourth phase

Sequence	Reference fields	Explanation
1	$(T)_{\#3}$	Referring to geometrical knowledge while constructing the equilateral triangle: "We should set the angle in between at 60°" (4th phase, Line 24).
2	$(T \text{ to } SG)_{\#5}$	Making a prediction about the resulting figure when two radii are connected: "That would be isosceles" (4th phase, Line 28).
3	$(SG \text{ to } T)_{\#4}$	Interpreting the construction in geometrical terms—about the equilateral triangles formed in the figure (4th phase, Line 48).
4	$(T \text{ to } SG)_{\#6}$	Applying the drag test to check the equilateral triangle construction (4th phase, Line 55).

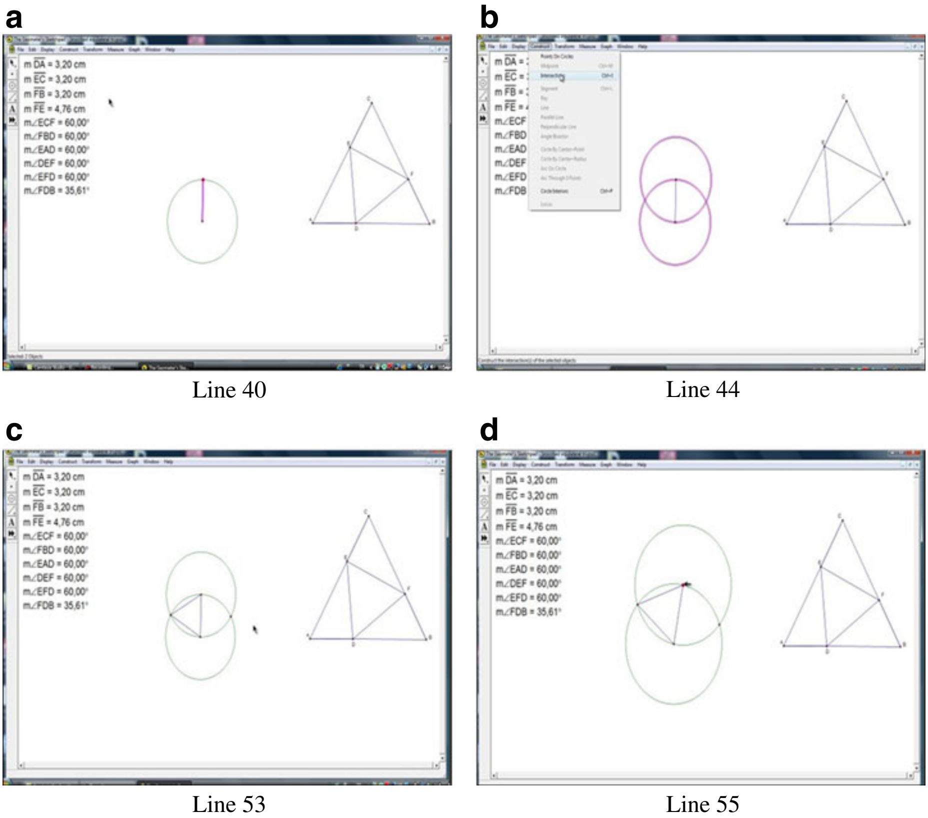

Fig. 5 Screens from the fourth excerpt

figure in geometrical terms by explaining to Ayla that two equilateral triangles were formed by connecting the radii and intersection points of the two circles, thus moved from SG to T (#4, Table 3). He also applied the drag test to their equilateral triangle construction (T to SG, #6, Table 3, Fig. 5d).

Phase 5: Constructing the inner equilateral triangle

In the final phase, the participants managed to construct the dynamic inner equilateral triangle within the equilateral triangle ABC. The most notable aspect of the final phase was the distributed nature of contributions toward solving the problem. The episode below captures what took place in the fourth phase.

Excerpt 5

Line	Time	Speaker	What is said	What is done
1	0:06:45;29	**Mete:**	let's say we have something like this=	Mete drags their triangle so that it looks like the one given, triangle ABC.
2		**Ayla:**	=yes=	

3		**Mete:**	=now let us construct a point on this	He selects a side of their newly constructed triangle B′C′ (side corresponding to BC) and goes to the Construct menu.
4	0:06:50;17	**Ayla:**	shall we also construct here like that	Ayla points at an imaginary point on the side A′B′ first and then point D on the screen with finger.
5		**Mete:**	[haa	
6		**Ayla:**	I mean we are supposed to do it from D yes ehehe °so they are alike°	
7		**Mete:**	[got it okay let it be alike	Mete deselects B′C′ and selects A′B′.
8	0:06:58;27			Mete goes to the Construct menu. Two options are available: "point on segment" and midpoint, and he pauses over the "midpoint" briefly.
9		**Ayla:**	point on segment?=	But then he selects "point on segment" on the menu and constructs a point on A′B′.
10		**Mete:**	=uh huh okay	
11	0:07:2;00	**Ayla:**	[okay °we could have said midpoint though° anyways	Mete moves the point to the left.
12		**Ayla:**	hm	
13		**Mete:**	(2.1) .h::h now the thing we'll do	
14		**Mete:**	(5. 3) hmm	
15		**Mete:**	(1.2) the line we drew-	
16		**Mete:**	(3.2) now let's construct a point on that too =	Mete moves the cursor over C′B′ and selects it again.
17		**Ayla:**	=but i mean th:::at	Ayla swipes the line segment EC with finger.
18		**Ayla:**	hm the distance does not have to be always equal ((more talking to herself))	Mete constructs another point on C′B′, point F′.
19	0:07:22;11	**Ayla:**	does it? (.) look EC and AD and FB (1.2) are always equal in length=	Ayla swipes line segments EC, AD and FB on the screen with finger.
20	0:07:29;11	**Mete:**	= ha then we'll do the thing we'll measure that gap	Mete selects point A′.
21		**Ayla:**	[we should determine something based on that	
22		**Mete:**	[let's measure that gap =	Mete selects point D′.
23		**Ayla:**	=ye:es	
24		**Mete:**	how will we measure?	
25		**Ayla:**	[shall I do ctrl l there?=	
26		**Mete:**	=do once	
27				((The line segment is constructed)).
28		**Ayla:**	huh measure that ((inaudible))	
29		**Mete:**	[wait before that let's name them all	
30		**Ayla:**	okay name them (.) by selecting the points=	
31	0:07:40;16	**Mete:**	=let's delete these first	
32				Mete deletes the measure displays on the screen.
33		**Ayla:**	okay now hm	
34		**Mete:**	A=	Mete names A′ as G.

35		**Ayla:**	=also ((inaudible))=	Mete names C′ as H.
36	0:07:50;02	**Mete:**	[not important it does that cause A and B are there	Mete names B′ as I.
37		**Ayla:**	[uh huh	
38				Mete names points D′ as K and F′ as J.
39		**Mete:**	now we drew ((constructed)) this right? ((referring to GK))	
40		**Ayla:**	G ((inaudible)) ha	Mete selects the line segment GK.
41		**Mete:**	[what's the measure of this?	
42				Mete clicks on "measure length" (of GK) and 2.10 is displayed on the screen.
43	0:08:00;04	**Ayla:**	two point ten	
44		**Ayla:**	now from I	Ayla points at point I on the screen with finger.
45				Mete selects point J.
46		**Ayla:**	will you delete J	Mete selects point I.
47	0:08:10;29	**Ayla:**	cause can we construct a point 2 point 10 cm away from it	Mete deselects I and J and then he deletes point J.
48		**Mete:**	[aha right we'll select this draw this	Mete selects point I and line segment GK.
49		**Mete:**	we could do it with the thing	Mete goes to construct "circle by center and radius."
50		**Ayla:**	[I:ıgh=	Mete constructs a circle with center at I and radius as GK, circle (I, GK).
51	0:08:19;08	**Mete:**	=with the circle also=	
52		**Ayla:**	=hm=	
53		**Mete:**	=intersect this with this	Mete selects C′B′ and the circle (I, GK).
54	0:08:23;11	**Ayla:**	[yes there exactly	Mete goes to construct "intersection."

As in other episodes, the talk in this phase can also be characterized as including little gaps between successive turns, along with some overlaps. Considering the sequence organization also, this indicated that the participants oriented closely to each other's talk.

The episode began with Mete dragging their triangle so that it is similar to the one given and initiating the problem solving activity with the sentence: "Let's say we have something like this" (Excerpt 5, Line 1). Ayla agreed. Then Mete wanted to construct a point on one of the sides of the triangle and first selected B′C′ (line segment corresponding to BC) (Excerpt 5, Line 3). However, Ayla warned him that they should start by selecting the line segment A′B′, since they "were supposed to do it from D," referring to an insight that took place in the third phase (Excerpt 3, Line 24). Mete agreed and constructed a point on the line segment A′B′ (Excerpt 5, Line 9; Fig. 6c). While he was doing this, Ayla suggested that they could have constructed a midpoint, yet her voice was pitched very low, suggesting that she was not quite sure of this (Excerpt 5, Line 11). Thus she did not know how to proceed from there either. They paused a little here. Mete then started to think out loud to voice his actions (Excerpt 5, Lines 13–16). He decided to construct another point on the line segment C′B′ corresponding to point F (Fig. 6d). Yet as soon as he selected C′B′, Ayla pointed out the crucial relationship that they realized in the first phase, which was the finding that DA, and FB, and EC were all congruent (Excerpt 5, Lines 17–19). Mete immediately took this into consideration by saying that they needed to measure "that gap" while selecting points A′ and D′ (Excerpt 5, Lines 20 and 22; Fig. 6f). This was an instance of

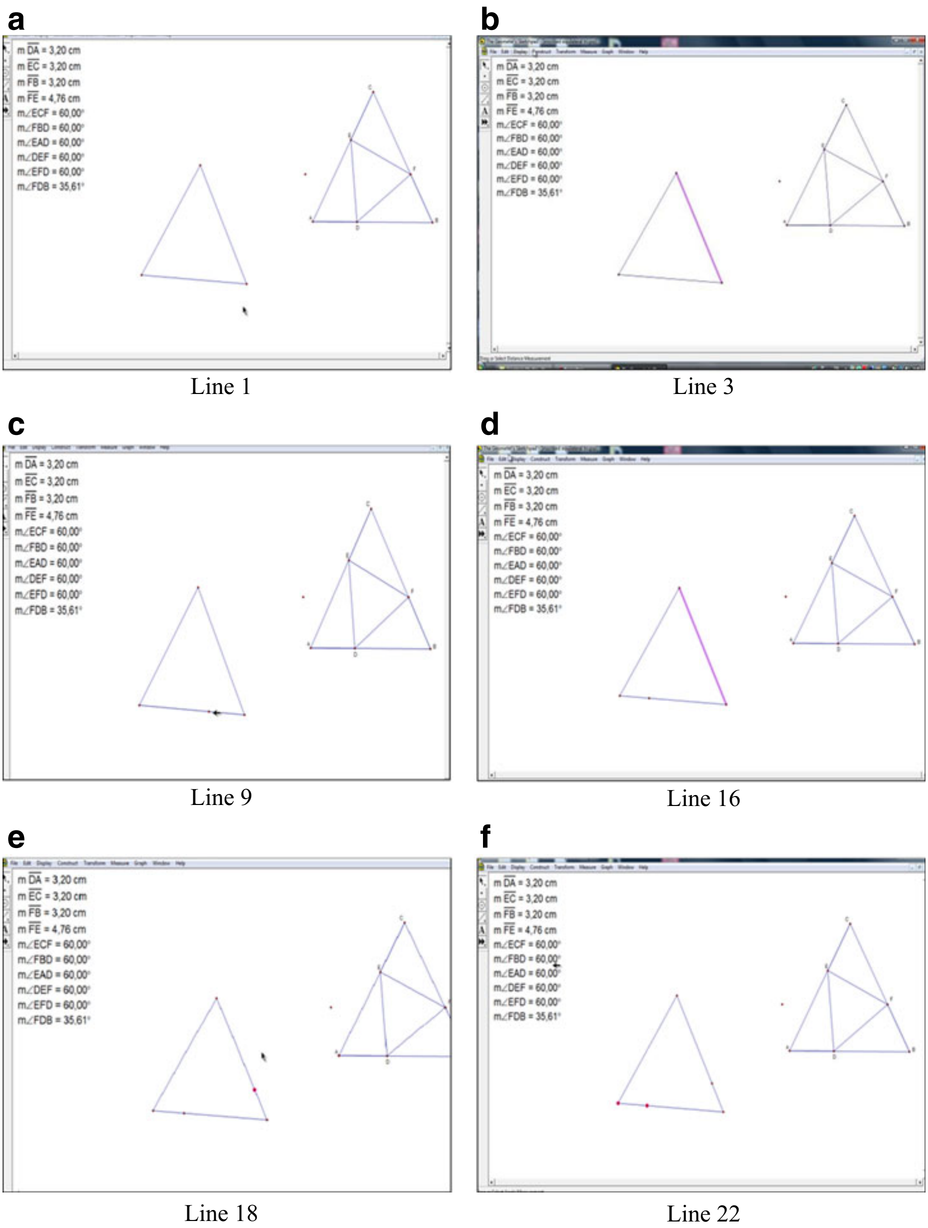

Fig. 6 Screens from the fifth excerpt

transactive dialogue (Berkowitz and Gibbs 1985 as cited in Barron 2000), such that while one partner brought one crucial insight into their JPS, it was engaged in by the other partner.

Next, they dealt with measuring the line segment A′D′. This action was also distributed over both partners. Mete asked "how will we measure?" (Excerpt 5, Line 24). Ayla suggested doing *CTRL + L*, which Mete accepted. Later they went on to name all their points and clean up their screen. In line 36, Mete was explaining to Ayla why the software named their points G, I, and H

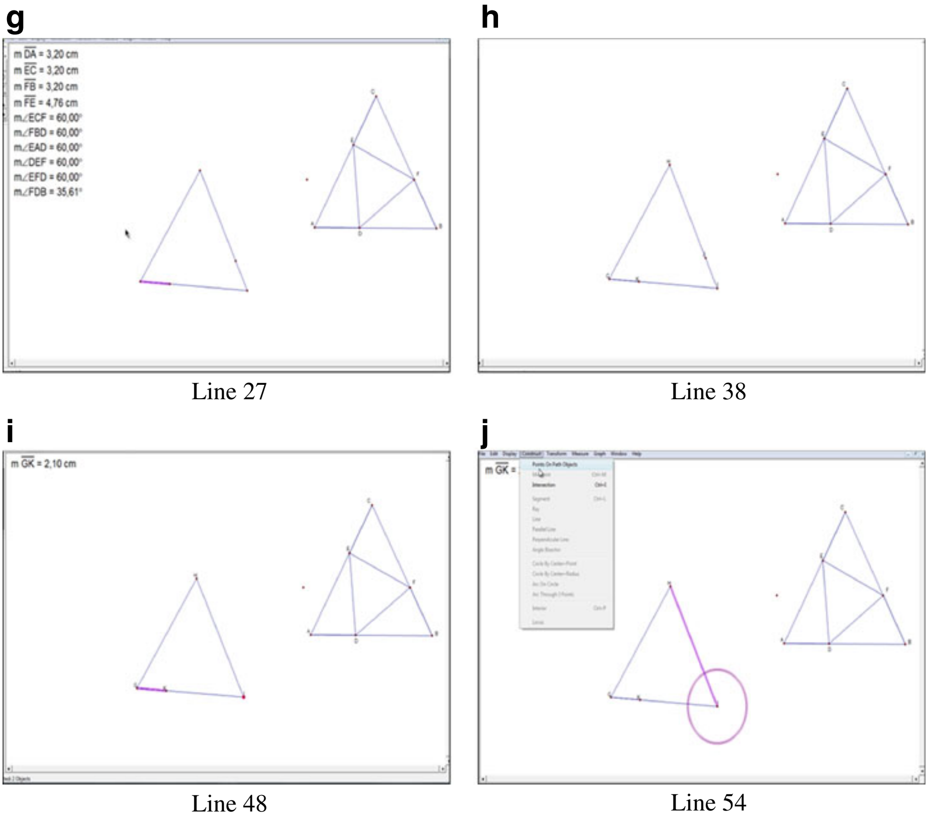

Fig. 6 (continued)

(rather than A, B, and C), without her soliciting such an explanation. Later they measured the length of A′D′ (which is named GK) and found it to be 2.10 cm (Excerpt 5, Line 42).

Then Ayla started to think out loud: "Now, from [point] I" as Mete selected point J, which was constructed independently by Mete earlier (see Excerpt 5, Line 18; Fig. 6e). Ayla then suggested deleting point J and then asked: "Can we construct a point 2.10 cm [the length of GK] away from it [point I]?" thus asking the essential question. This question prompted Mete to use the "construct circle by center and radius" command in the menu. As he explained his actions to Ayla—"we could do it with the thing" (Excerpt 5, Line 49) and "with the circle also" (Excerpt 5, Line 51)—he also carried out the task. He then found the intersection of the circle with the line segment HI, which was also approved by Ayla (Excerpt 5, Line 54; Fig. 6j).

In the rest of the conversation, the group proceeded by constructing a circle (H, GK) around the vertex H by fixing the radius as GK, finding the intersections with the triangle GIH, connecting the intersections, and constructing the inner equilateral triangle (which was named MKL). During these procedures, their actions also included two moves from T to SG: One was to check whether line segment IL (B′F′) was equal to the line segment GK "to be sure" (#7, Table 4) and the other was use the drag test with the triangle MKL to verify that it is equilateral (#8, Table 4).

In this final phase, one can also observe how the participants were taking advantage of both types of resources, both mathematical and tool-enabled, and relational. The relational

Table 4 Descriptions of SG and T levels and moves between them in the fifth phase

Sequence	Reference fields	Explanation
1	(T to SG)$_{\#7}$	Checking whether the line segment IL was equivalent to the line segment GK by measuring “to be sure” (5th phase, Lines 64–66).
2	(T to SG)$_{\#8}$	Measuring the interior angles of triangle MKL to verify that it is equilateral (5th phase, Lines 104–110).

aspect of their interaction included providing explanations for the behavior of the software, even when these were not solicited. More importantly, however, their talk noticeably represented *transactive* dialogue (Berkowitz and Gibbs 1985 as cited in Barron 2000), that is, ideas offered by one partner were taken into consideration by the other.

The analysis additionally revealed another interesting aspect of the participants’ work together. As represented by the excerpt provided, this phase was marked by the evenly distributed contributions provided by the two participants, which were all equally essential to complete the task at hand. It would be quite fair to say that the participants achieved solving the task together, in the sense that neither of their contributions alone was sufficient to complete the task. Specifically, in the excerpt provided, it was Ayla who recalled the essential relationship between line segments DA, FB, and EC, and it was Mete who thought of using circles to make the lengths of the line segments equal and dependent on each other.

Discussion

The purpose of the present study was to suggest a methodological framework to analyze collaborative geometry problem solving. Built on the previous conceptualization of JPS in the literature, this framework aims to operationalize the content-related and relational aspects of JPS within the context of DGS use. The study also presented an in-depth analysis of the work of a dyad, providing an account of adult students’ successful face-to-face collaboration when solving a DGS construction problem.

The participants mainly took advantage of two types of resources within their interaction: collaborative (relevant for the relational or social aspect), and mathematical and GSP-enabled resources (relevant for the content-related aspect). Collaborative resources were investigated using the constructs of conversation analysis. Most of the time the participants managed a productive interaction, orienting themselves towards each other’s talk closely. This was evident in the way they provided explanations to each other about the figure, gave directions, repaired problematic expressions, narrated their own and each other’s actions, performed GSP actions together, and engaged in a transactive dialogue. Most notably, in the final phase of their session, one can observe how contributions from both participants led to the successful completion of the task.

The mathematical and tool-enabled resources included the theoretical geometry knowledge of the participants, the spatio-graphical as well as the theoretical feedback provided by the DGS, and the moves between these two levels. The participants in the study first explored the given figure and discovered its properties. They first realized that the line segments DA, FB, and EC were equivalent (and dependent) and later that the triangles ABC and EDF were equilateral. Exploring and discovering these properties and dependencies required the use of both reference fields (T and SG) as well as the moves between them. More specifically, the participants relied most on these resources in the first two phases, in which they were primarily investigating the properties of the given figure. They were then able to construct two equilateral triangles with the given dependencies.

Thus, the participants' work together reflected expert problem solving in terms of the use of both reference fields and shifts between them, corroborating Laborde's assertion (2004). However, a closer look at the relational aspect of their activity together revealed that Mete was not always responding to Ayla. This particularly took place within the second phase, when he operated at the T level and was not performing any related SG-level actions with the GSP. Thus, it could be argued that when using theoretical resources that are not clearly articulated, there could be periods of discontinuity within the flow of collaborative activity. Nevertheless, the dyad managed to successfully construct the given figure. It is reasonable to argue that such discontinuity is tolerated due to the coordination of not only the content-related resources but also the relational ones put into use throughout the interaction.

Barron (2000) argued that participants do not need to maintain joint attention at all times, but should be able to regain it at solution-critical times. Evidently, in order to regain it, partners need to rely on collaborative resources. This, however, does not mean to suggest that communication skills should be taught independent of the content-related aspects of the task. Doing so is "not likely to enhance disciplinary-specific, joint thinking practices" (Barron 2003, p. 354). Thus, in collaborative contexts, a two-way caution is necessary: rather than treating the mathematical knowledge of participants as the determinant of their success, one should be more concerned with how that knowledge was brought to the fore when working with a specific tool and collaborating with a peer. In the same way, simply focusing on the relational aspects and overlooking the content will not be likely to create a context for productive collaboration. Thus, when designing tools for collaborative work for geometry problem solving, it is important to support collaborators with both content-related and relational resources. In order to support the use of content-related resources, these tools should enable users to take advantage of both mathematical and DGS resources. In addition, such software should simultaneously support users in making use of relational resources.

Also, in order to analyze collaborative work, we need theoretical lenses to examine both aspects simultaneously. The proposed framework is an attempt to provide tools for studying the content-related and relational aspects of collaboration within the geometry problem-solving context with DGS. In the framework suggested, the constructs of the SG and T fields were helpful, as they are treated as referents, rather than as mental images or processes, as suggested by Laborde (2004). They further enable one to take into account both aspects of the specific subject domain (geometry problem solving) and the specific tool (DGS). However, they were not sufficient to investigate group problem solving in which interactive phenomena play an essential role. The suggested framework attempts to combine this level of analysis with an eye on conversation structures. This framework could be used to trace the content-related and relational resources used in either face-to face or computer-mediated collaborative interaction. It helps to reveal how the two aspects are entangled, which makes it possible to elaborate on what makes the participants' problem-solving session successful, as well as to identify what could make the interaction problematic.

The two-level analysis could be especially helpful for examining the sequentiality of interactions within group problem solving with DGS. The problem of accounting for the sequentiality of the interactions was evident in Laborde (2004). She pointed out that sometimes it was not possible to clearly identify the domain in which students were working. Laborde called these situations mixed units. There were two types of these: units referring to SG or T based on interpretation, and units containing terms of both domains. The second case was observed in another study (Jones 1998) involving two English university students who are asked to construct a circle that was tangent to two lines and passed through a point P lying on one of these lines. This was a case in which students' geometrical knowledge was brought to the fore through interaction with the software. Although the students knew the properties of a circle and tangent line, they did not use them right away. This knowledge emerged through their interaction with the software. Thus, geometrical knowledge was recalled in the context of dynamic exploration of the problem. If one focuses merely on the content aspect of the collaborative discourse, it may become difficult to explain the unfolding interaction within its terms. Adding another lens to the analysis could better explain how these mixed units came into being, as well as explaining the sequentiality of interaction.

Acknowledgments The author would like to thank the six anonymous reviewers for their comments on an earlier version of this manuscript.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments. *ZDM, 34*(3), 66–72.

Barron, B. (2000). Achieving coordination in collaborative problem-solving groups. *The Journal of the Learning Sciences, 9*(4), 403–36.

Barron, B. (2003). When smart groups fail. *The Journal of the Learning Sciences, 12*(3), 307–59.

Berkowitz, M., & Gibbs, J. (1985). The process of moral conflict resolution and moral development. In M. Berkowitz (Ed.), *Peer conflict and psychological growth* (pp. 71–84). San Francisco: Jossey Bass.

Cakir, M. P., Zemel, A., & Stahl, G. (2009). The joint organization of interaction within a multimodal CSCL medium. *International Journal of Computer-Supported Collaborative Learning, 4*(2), 155–90.

Clark, H. H., & Schaefer, E. F. (1989). Contributing to discourse. *Cognitive Science, 13*, 259–294.

Evans, M., Feenstra, E., Ryon, E., & McNeill, D. (2011). A multimodal approach to coding discourse: Collaboration, distributed cognition, and geometric reasoning. *International Journal of Computer-Supported Collaborative Learning*. doi:10.1007/s11412-011-9113-0.

Goldenberg, E. P., & Cuoco, A. A. (1998). What is dynamic geometry? In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 351–68). Mahwah: Lawrence Erlbaum.

Hutchby, I., & Wooffitt, R. (1998). *Conversation analysis*. Cambridge: Polity Press.

Jackiw, N. (2001). *The geometer's sketchpad (Version 4) [Computer software]*. Berkeley: Key Curriculum Press.

Jefferson, G. (1983). Notes on some orderliness of overlap onset. *Tilburg papers in language and literature (No. 28)*. Tilburg, The Netherlands.

Jones, K. (1998). Deductive and intuitive approaches to solving geometrical problems. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 78–83). Dordrecht: Kluwer.

Laborde, C. (2004). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in mathematics education* (pp. 1–21). Dordrecht: Kluwer Academic Publishers.

Lipponen, L. (2002). Exploring foundations for computer-supported collaborative learning. Proceedings of the Computer Supported Collaborative Learning Conference (CSCL 2002). Retrieved August 22, 2012, from http://repository.maestra.net/valutazione/MaterialeSarti/articoli/Lipponen.pdf.

Malone, T. W., & Crowston, K. (1990). What is coordination theory and how can it help design cooperative work systems? Proceedings of the Computer Supported Collaborative Work (CSCW 1990), pp. 357–370.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning, 12*(2), 135–56.

Oxford University Press. (2012). oxforddictionaries.com. Oxford University Press. Retrieved July 28, 2012, from http://oxforddictionaries.com/definition/english/coordination?q=coordination.

Roschelle, J., & Teasley, S. (1995). The construction of shared knowledge in collaborative problem solving. In C. O'Malley (Ed.), *Computer-supported collaborative learning* (pp. 69–197). Berlin: Springer Verlag.

Sacks, H., Schegloff, E. A., & Jefferson, G. (1974). A simplest systematics for the organization of turn-taking for conversation. *Language, 50*, 696–735.

Sarmiento, J. W., & Stahl, G. (2008). Extending the joint problem space: Time and sequence as essential features of knowledge building. Paper presented at the *International Conference of the Learning Sciences* (ICLS 08), Utrecht, Netherlands.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*, 13–57.

Sidnell, J. (2010). *Conversation analysis: An introduction*. Chichester: Wiley-Blackwell Press.

ten Have, P. (1999). *Doing conversation analysis. A practical guide*. Thousand Oaks: Sage.