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Knowledge building in mathematics: Supporting collaborative learning in pattern problems

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Abstract While it has been suggested that patterning activities support early algebra learning, it is widely acknowledged that the shift from perceiving patterns to understanding algebraic functions—and correspondingly, from reporting empirical patterns to providing explanations—is difficult. This paper reports on the collaborations of grade 4 students (n= 68) from three classrooms in diverse urban settings, connected through a knowledge-building environment (Knowledge Forum), when solving mathematical generalizing problems as part of an early algebra research project. The purpose of this study was to investigate the underlying principles of idea improvement and epistemic agency and the potential of knowledge building—as supported by Knowledge Forum—to support student work. Our analyses of student-generated collaborative workspaces revealed that students were able to find multiple rules for challenging problems and revise their own conjectures regarding those rules. Furthermore, the discourse was sustained over 8 weeks and students were able to find similarities across problem types without the support of teachers or researchers, suggesting that these grade-4 students had developed a disposition for evidence use and justification that eludes much older students.

Keywords Early algebra · Collaborative mathematical discourse · Patterns and generalizing problems · Knowledge building · Knowledge forum · Epistemic agency

Introduction

For the last 2 years we have incorporated a knowledge-building environment—Knowledge Forum (Bereiter & Scardamalia, 1989)—into ongoing research of students' development and learning of patterns and functions. While knowledge-building pedagogy, supported by

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Knowledge Forum, has been shown to support student development of a disposition for knowledge building in the domain of science, little has been reported on student use of Knowledge Forum in the learning of mathematics (Hurme & Jarvela, 2005; Nason & Woodruff, 2002). Thus, one of the goals of our larger ongoing research project has been to investigate whether Knowledge Forum, with its underlying knowledge-building principles, can promote the kind of inquiry orientation towards mathematics learning that has been shown with respect to students' science learning (e.g., Bereiter & Scardamalia, 1989, 2003; Scardamalia, Bereiter, & Lamon, 1996). Specifically, we have been investigating how Knowledge Forum can support students in working collaboratively to find algebraic rules for mathematics generalizing problems.

Generalizing problems, also known as numeric sequences or geometric growing sequences, present patterns of growth in different contexts. Students are asked to find the underlying structure and express it as an explicit function or "rule." Students often have difficulty finding underlying functional relationships for generalizing problems, in part because developing useful strategies to discern algebraic rules is challenging, but also because students lack the interest or ability to justify their conjectures.

In this paper, we present verbatim Knowledge Forum notes that were created by students (9–10 years old) from three grade-4 classrooms connected through Knowledge Forum. We incorporated Knowledge Forum in our study to address specifically the kinds of difficulties reported in the mathematics education literature. Our hypothesis was that appropriate support for knowledge building would help students generalize their understanding of functions and provide them with a context to offer justifications. In previous papers we have reported on the overall success—in terms of gains in scores on pre- and post-test measures—of our larger research project to foster improvements in students' abilities to work with patterns and to find underlying functional rules (e.g., Moss, Beatty, & London McNab, 2006). The focus of this study was specifically on students' problem solving on Knowledge Forum. Our analyses for this paper focused on the kinds of multiple rules that students found as well as the degree to which students provided justifications for their conjectures of rules.

We begin this paper with a discussion of both the anticipated benefits of generalizing problems in mathematics learning and the problems that have been widely reported. Next, we present an overview of the larger project and discuss our decision to incorporate Knowledge Forum in our work. We also present a brief account of the procedures employed in this study. Finally, we present our analyses of the students' collaborative problem solving on Knowledge Forum in order to illustrate the effectiveness of the use of this discourse space to support students' abilities to work with generalizing problems.

Patterns for early algebra

In recent years, patterning and algebra have become part of the elementary curriculum in many countries (e.g., Greenes, Cavanagh, Dacey, Findell, & Small, 2001; National Council of Teachers of Mathematics (NCTM), 2000; Ontario Ministry of Education and Training (OMET), 2003, 2005; Warren, 1996, 2000). From a mathematics perspective, the introduction of patterns in the early years has many potential benefits for students. Patterns offer a powerful vehicle for understanding the dependent relations among quantities that underlie mathematical functions (Ferrini-Mundy, Lappan, & Phillips, 1997; Lee, 1996; Mason, 1996), as well as a concrete and transparent way for young students to begin to grapple with the notions of abstraction and generalizations (Blanton & Kaput, 2004; Carraher, Schlieman, Brzuella, & Enrnest, 2006; Cuevas & Yeatts, 2001; English &

Warren, 1998; Greenes et al., 2001; National Council of Teachers of Mathematics (NCTM), 2000).

Underlying this new initiative for early algebra is the hope that an early introduction to algebra may serve to diminish the abrupt and often difficult introduction to formal algebra in high school (Kieran, 1992) and thus may afford all students access to wider opportunities in later mathematics and in career choice (Greenes et al., 2001; Kaput, 1998; Moses, 1997). Another claim for including patterns in the curriculum concerns the potential of patterning activities to provide a familiar context for students to begin generalizing (Lee, 1996; Mason, 1996; Zazkis & Liljedahl, 2002), and for supporting justifications as an introduction to algebraic reasoning (Lee & Wheeler, 1987; Radford, 1999, 2000).

The work that we report in this paper focuses on a specific type of pattern problem—generalizing problems. Generalizing problems are usually presented as numeric or geometric sequences, and typically ask students to predict the number of elements in any position in the sequence and to articulate that as a rule. A well-known generalizing problem is the *handshake problem*, which asks students to predict the number of handshakes required to fully introduce a group of people of any number. It has been shown, however, that given traditional instruction, students' approaches to these problems are limited.

Generalizing problems—limited justifications

Mason (1996), who has written extensively on fostering generalizing in classrooms, observes that it is rare that students engage with these kinds of exercises beyond the most limited kinds of mathematical generalizations. As Bednarz, Kieran, and Lee (1996, p. 7) note, it has been widely reported that when students are presented with geometric or numeric sequences the pervasive strategy "is on the construction of a table of values from which a closed-form formula is extracted and checked with one or two examples." This approach, in effect, "short-circuits all the richness of the process" (Mason, 1996, p. 70). These same observations are corroborated by Stacey (1989), who noted that when students are presented with generalizing problems, they tend to construct rules and generalizations too readily with an eye to simplicity rather than accuracy. Cooper and Sakane (1986) further reported that once students selected a rule for a pattern, they persisted in their claims even when finding a counter example to their hypotheses. Students would rather refute the data presented than modify their original rule.

Hoyles (1998) observed that even if students can explain how certain inputs lead to certain results or outputs, their attempts to justify or prove their conjectures of rules appear to be added on rather than inherent to the activity. The propensity of students to develop rules based on few examples, and without a disposition to explain why the rule works, precludes opportunities to develop the ability to move from particular instances of a pattern to a generalized understanding of the underlying mathematical structure. Certainly these findings put current practices and expectations for pattern learning into question.

Analyses of difficulties

Researchers who have studied patterns and generalizing suggest that it is not generalizing problems per se that are difficult, but rather the way that they are presented to students and the limitations of the teaching approaches used. Generalizing problems are presented in a variety of contexts—geometric, tabular, and narrative—but the tendency of most instruction is to prioritize the numeric aspect of patterning (Noss, Healy, & Hoyles, 1997; Noss &

Hoyles, 1996) with the result that these become data-driven, pattern-spotting activities in which tables of numeric data are constructed using a recursive strategy and a closed-form formula is extracted and checked with only one or two examples. The context and meaning of the variables thus become obscured, which severely limits students' ability to conceptualize the functional relationship between variables, explain and justify the rules that they find, and use the rules in a meaningful way for problem solving.

Lee (1996, p. 95) notes that students working with geometric patterns often have difficulty perceiving "algebraically useful patterns." She uses the term "perceptual agility" to characterize the ability to see multiple patterns coupled with a willingness to abandon those that do not prove useful for rule making. Mason (e.g., 1996), who uses the term "multiple seeings," suggests that students be given opportunities to find multiple kinds of patterns and that visualization and manipulation of the figures on which the generalizing process is based can facilitate rule finding and formula making.

The larger study—overall goals

The research we have been conducting has been designed to address these issues, identified in the mathematics education literature. Our goal is to provide learning contexts in which students can have the opportunity for "multiple seeings" and to develop "perceptual agility" (Lee, 1996). Our research has involved us in designing and assessing particular contexts both to foster students' development and learning of the functional relationships between two sets of data, and to support student motivation and interest in providing justifications. In our view, it is important to find ways of maximizing the potential of patterning activities, since not only are patterns and functions part of the curriculum, but we know they are widely enjoyed (Seo & Ginsburg, 2004), at least by young children, and they have an appeal and points of entry for older students that cross ability levels (Moss, Beatty, Barkin, & Shillolo, to publish in 2008).

The interventions that we have designed and implemented have two distinct parts emanating from two different theoretical frameworks. Case's theory of mathematical development (e.g., Case & Okamoto, 1996; Moss & Case, 1999) served as the basis for the design of a set of carefully sequenced lessons intended to foster connections between geometric growing sequences and numeric representations as a means of developing students' conceptions of linear functions. The second theoretical framework that has informed our research—the major focus for this paper—is Scardamalia and Bereiter's knowledge-building theory and pedagogy, particularly as it is evidenced in their knowledge-building software, Knowledge Forum.

Knowledge Forum and mathematics

Knowledge Forum has not been used extensively in student work in mathematics, particularly at the elementary level. There are, however, studies that reveal that elementary school students who use Knowledge Forum as part of science learning engage in inquiry and become part of a knowledge-building culture. As mathematics education researchers, we wondered if the collaborative nature of Knowledge Forum and the knowledge-building principles that underlie it—particularly the concepts of epistemic agency and idea improvement—might support young students working with generalizing problems and help avoid the kinds of difficulties that are described in the mathematics education research literature.

Our decision to use Knowledge Forum was based on our recognition of the potential of this collaborative discourse platform—and the knowledge-building principles that underlie it—to provide a context for students to generalize the understanding of functions acquired during the initial lesson sequence. We believed that a collaborative knowledge space would provide an authentic platform for student discussion. In addition, the software would offer students access to multiple perspectives on a variety of rules for different generalizing problems. We were also interested to discover if Knowledge Forum and the knowledge-building principles would provide a setting of inquiry in which students would be motivated to try to provide justifications for their conjectures of rules. On a more general level, we had questions about whether students would use the software collaboratively for solving generalizing problems or whether they would be more inclined to use the database as a repository for their own individual efforts. In addition, we wondered if students' contributions would focus on mathematics.[1]

Knowledge-building principles and mathematics

While we anticipated that students might benefit from the chance to contribute their theories on Knowledge Forum, it was the theoretical framework underpinning Knowledge Forum, particularly the emphasis on student agency and the centrality of student's ideas, that influenced our decision to incorporate Knowledge Forum in our research.

Scardamalia and Bereiter use the term "epistemic agency" to characterize the responsibility that the group assumes for the ownership of ideas that are given a public life in Knowledge Forum (Bereiter, 2002; Scardamalia, 2002). In this discourse structure, it is not the teacher who asks for justifications and evidence of the conjectures of rules that students contribute, but rather the students themselves who, with an eye towards moving the theorizing forward, take on this responsibility (Moss et al., to publish in 2008).

A related concept central to knowledge-building pedagogy is the principle that *ideas are improvable*. Students are good at generating ideas, but working deliberately to improve them does not come naturally or easily (Scardamalia, 2002). An important feature of Knowledge Forum is that notes can be revisited and revised at any time. This, coupled with the asynchronisity of the discussion, provides students with an extended time to think. Therefore, students working on Knowledge Forum have both the software capability and the time to engage in sustained idea improvement.

Studies of students collaborating on science investigations on Knowledge Forum reveal that, as a result of their participation, students become engaged in sustained efforts to improve their ideas/theories/solutions (Scardamalia, 2004; Scardamalia et al., 1996). We wondered if this same orientation to idea improvement would be found in students working on challenging generalizing problems. Specifically, we wondered if Knowledge Forum would support students (1) to find multiple rules; (2) to revise their own conjectures of rules; (3) and to provide evidence and justification for their conjectures of rules.

In the next section we briefly describe the procedures used in this study and go on to show the analyses that we conducted and the results found in answer to the questions above.

[1] In a recent study that incorporated Knowledge Forum in a high-school math program, Hurme and Jarvela (2005) reported that a significant proportion of notes contributed were "trivial," that is, the notes did not address mathematical issues but rather were social in nature.

Materials and methods

Participants

The data for the present study come from the second year of our project of early algebra and patterning (see, for example, Beatty & Moss, 2006; London McNab & Moss, 2006; Moss, 2005; Moss & Beatty, 2005; Moss et al., to publish in 2008). The students in this study were in grade 4 ($n=68$) and were from three classrooms. One of the classrooms was from a university laboratory school and the other two classrooms were from an inner city public school serving an at-risk population (high ESL, low SES). All of the students in the study had used Knowledge Forum for at least one science project and were familiar with the software and processes.

Procedures

The overall intervention took place over a 4-month period and consisted of two parts. First, the students in the three classrooms participated in a specially designed instructional sequence consisting of 12 lessons taught at a rate of approximately 2/week. The scope of this paper does not allow for a description of this instructional intervention (for details of the lessons see Moss et al., 2006.) Students were introduced briefly to linear functions of the form $y=mx$ and $y = mx + b$ both numerically, through "Guess My Rule" activities (e.g., Carraher & Earnest, 2003; Rubenstein, 2002; Willoughby, 1997), and visually, through pattern building with blocks and ordinal number cards placed under each position of geometric growing patterns (see Fig. 1).

This instruction, based on growing elements in a sequence, provided students with a basic understanding of simple multiplicative and composite functions. At no time were the students taught any formal algebraic notation; however, as part of the Guess My Rule activities, students were taught to use the words *input* and *output* to stand for the independent and dependent variables, respectively.

The Knowledge Forum activities made up the second part of the intervention. For this part of the study the students from the three classrooms in the two schools were linked electronically and invited to collaborate on solving six generalizing problems in which they were required to discern a functional relationship between two sets of data in order to find and express a generality. The problems were comprised of linear and quadric functions

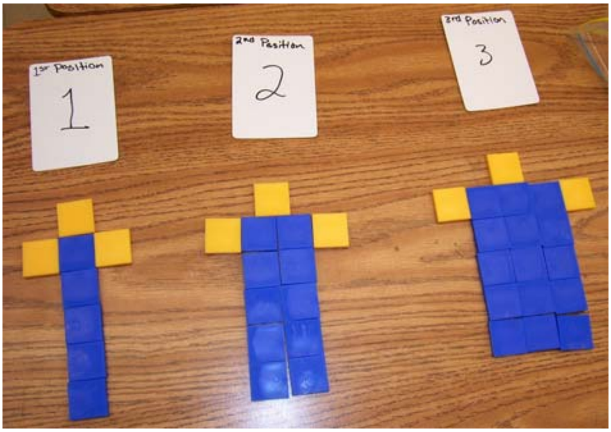


Fig. 1 Geometric pattern with position cards. This pattern follows the functional rule $y = 5x + 3$

embedded in various contexts. (Please see Appendix for presentation of all six problems). The generalizing problems were chosen for this study as a means of extending the students' reasoning about functional relationships. These generalizing problems were more complex in that not all rules conformed to the $y = mx + b$ structure that was used for all of the patterns in the instructional sequence. Two of the problems chosen followed the form $y = mx + b$ (Linear Pair 1). We also included two problems of the form $y = mx - b$ (Linear Pair 2), and two problems of the form $y = (x^2 - x)/2$ (Quadratic Pair). In addition, there were multiple potential rules for each problem.

Each problem comprised its own Knowledge Forum view. Figure 2 presents one of the views of this study (the Perimeter Problem, also known as the Shaded Square Problem [e.g., Steele, 2005; Steele & Johanning, 2004]). The small squares shown in this figure represent notes that students created and the connecting lines show the links between notes created as students read and responded to each other's contributions.

Figure 3 is an example of a student note as it appears in the database (the content of this note will be discussed in a later section of this paper). Each note contains a space for composing text. The *metacognitive scaffolds*[2] appear at the far left of each note. Students can also use Knowledge Forum's graphics palette to create illustrations, or they can scan drawings, function tables or photographs to support their explanations.

Below is the text of the student note shown in Fig. 3. Each note that we present is taken verbatim from the Knowledge Forum database. Each note was given a code based on the problem view to which it was contributed—all notes were numbered in chronological order as they were posted to the six different views (notes were numbered starting from 1 for each of the views). In the example note, PP19 refers to the fact that this note was the 19th note contributed to the Perimeter Problem View. Revisions to notes were coded with a lowercase 'r' followed by the revision number. So, for example, all revisions to the note below would be coded as PP19r1, PP19r2, etc. All notes were given a title, indicated by bold text, by the student(s) who posted them—the note in the example below is titled "Relationships." Authorship is indicated by the student's first and last initial. Any *metacognitive scaffolds* are indicated through the use of italics. The scaffold selected by NS in this note is "my theory."

PP19 **Relationships**—NS

> My theory is that the Output # is = to the input # times 4-4. My evidence is that $3 \times 4 = 12$ and −4 is 8 which is the output. You need to multiply the input×4[only one side] because without multiplying you wouldn't get 12. Then when you minus another # besides 4 the output wouldn't be 8. The same rule applies for all the other numbers. Like: $100 \times 4 = 400 - 4 = 396$ $10 \times 4 = 40 - 4 = 36$ $14 \times 4 = 56 - 4 = 52$

Prior to working on the database, students were instructed that they would be working on difficult mathematics problems, and that they would have to work together in order to solve them. Once students began working on the problems on Knowledge Forum, the database was entirely student managed. Students were not provided with answers, nor were they told if their answers were correct or incorrect. The researchers and teachers did not post any notes on the database. The Knowledge Forum database was available to students over an 8-week period. On average, each student had approximately half an hour to 45 min/week to work on the database.

[2] Metacognitive scaffolds for knowledge building include *my theory*, *I need to understand*, *new information*, *a better theory*, and *putting our knowledge together* They are designed to encourage students to engage in theory building while they write their notes (Scardamalia, 2004).

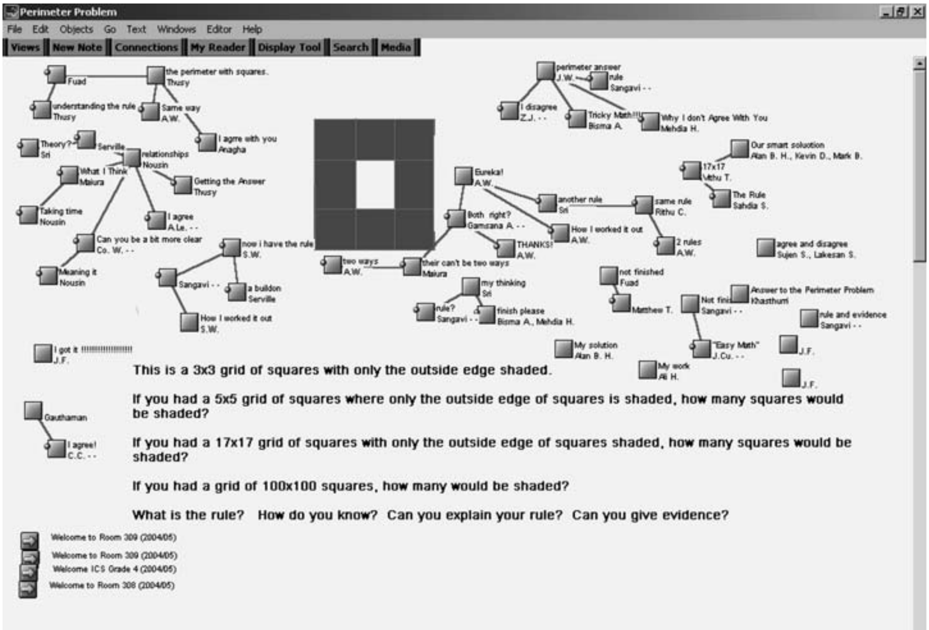


Fig. 2 Perimeter problem view

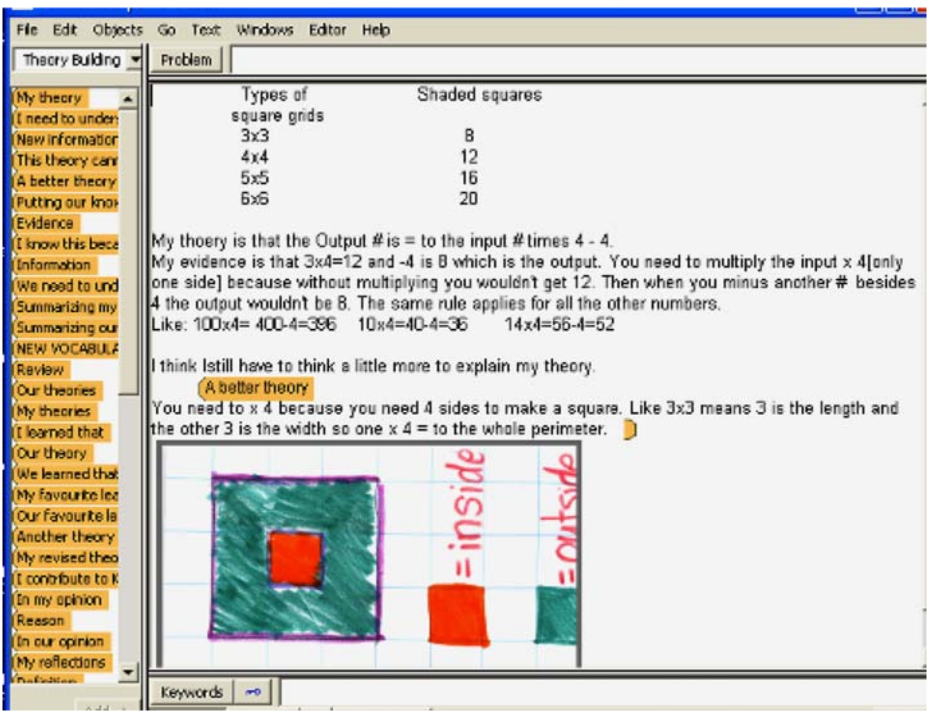


Fig. 3 Student note

Results

Participation on the database

We began our analyses by conducting frequency counts of all of the notes that the students from all of the three classrooms contributed to the series of six generalizing problems. Each note was read and coded by the researcher (second author) and one or more research assistants.

In all, 247 notes were posted with individual contributions ranging between 3 and 18 notes per student. Furthermore, these notes were evenly distributed across the three classrooms. In our interest to discover if the students were motivated to read each others' contributions, we coded the notes to differentiate between those that were originals—notes created by a student or pair of students that presented a thought or conjecture that was not in response to another student's note—and those that were written in response to the note of another student. Our analyses revealed that 30% of all of the notes were *original* notes, and the remaining 70% of notes were responses. This indicated to us that students were reading and building on the ideas of others rather than simply posting individual ideas. In addition, we discovered of the responding notes that only 28% were social-support notes containing comments such as "good job," or "I like your theory." Thus, we could see from these initial counts that the students, without teacher input, were motivated and interested to work on the mathematics of the generalizing problems. In the next sections we consider the content of student notes as they relate directly to our central research questions: the finding of multiple rules, the revision of rules, and the provision of evidence and justifications.

Multiple seeings: Proposing and negotiating multiple rules

Throughout the database we found evidence of students finding multiple rules. For each of the six problems posted, the students came up with a minimum of two different rules.

For the Perimeter Problem view, students generated and posted five different rules, the three most common of which will be discussed in this paper. The Perimeter Problem has been used in studies by other investigators and has been shown to be challenging for even high-achieving older students (Steele & Johanning, 2004). In this problem, students are asked to find a rule that will allow them to predict the number of squares in the perimeter of a square grid of any size.

Algebraically, the functional rule for this problem is represented as $f(x) = 4x - 4$. Based on multiple possible visual interpretations, however, there are a number of different ways of expressing a generalized rule for this problem. Figure 4 illustrates three different rules that students in our study discovered, based on three different perceptions of the grid figure. The first and most frequently posted strategy is the *Overlapping Corners Strategy*, which can be written algebraically as $y = 4x - 4$ and is based on a perception of the perimeter as four sides composed of x number of squares ($4x$), with each side "overlapping" or "sharing a square" at all four corners (−4). The *Area Strategy* involves finding the area of the grid in terms of the total number of squares, and subtracting the number of squares that make up the inner area, leaving the perimeter. Algebraically this is represented as the rule $y = x^2 - (x - 2)^2$. The *Doubling Sides Strategy* (which will be illustrated in a later section) is based on doubling the number of squares along one side of the grid, and then doubling the number of squares along the remaining sides, $y = 2x + (2x - 2)$.

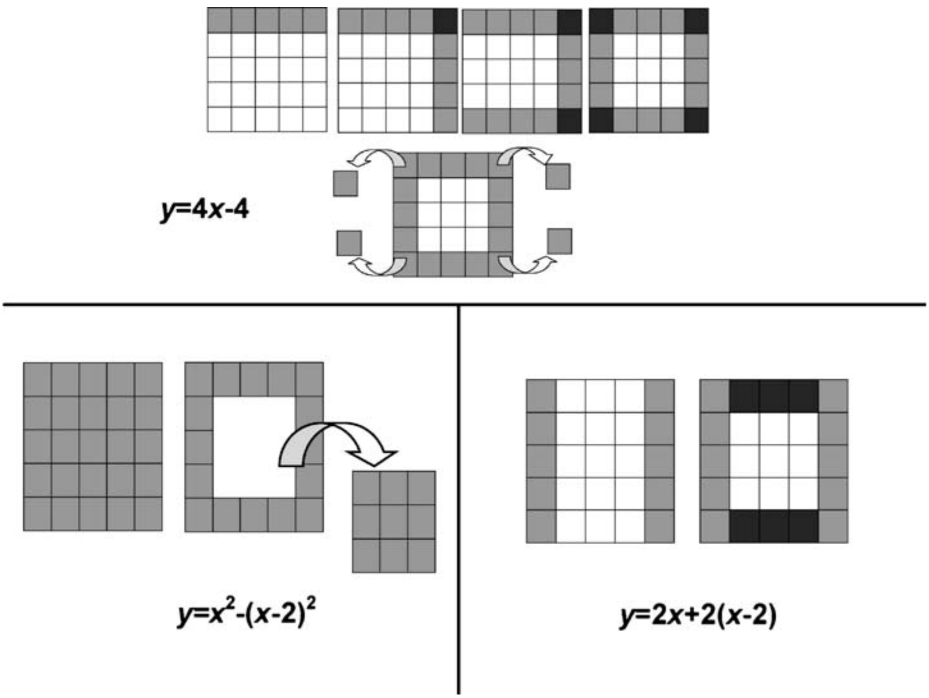


Fig. 4 Three solutions to the perimeter problem

As we read through the notes for this problem we discovered that initially most of the students came up with the rule $y = 4x - 4$. Notes PP16 and PP17 are representative of the way students offer their ideas about rules.

PP16 **Now I have the rule**—SW

I figered out the rule and it is the number times 4 (the four sides) minus 4 (becuase you use one twice at each corner)

PP17 **A buildon**—SRV

I agree with you s.w because i know the rule is times 4−4 and ×4 is the 4 sids and−4 is when you−4

However, as work on this problem proceeded, the students became aware of different ways of "seeing" the problem. In the discussion below, a student posted a rule based on the Area Strategy. What followed was a negotiation, during which students had to work to understand the perspectives of other students in order to accept the idea of multiple problem perceptions and the associated multiple (correct) solutions. The notion that there could be more than one solution was taken up by students in all three classrooms in a sustained discussion that broadened their understanding of mathematical problem solving. This discussion began with a solution posted by a student, AW, (PP38) which she titled 'Eureka!'[3]

[3] Students were not taught symbolic notation, or how to represent variables, and so used the letter *n* to represent *any number*.

PP38 **Eureka!** A.W.

for the 5×5 question you do $5 \times 5 = 25$ the square of 25 is 5 and you minus two from the square and square that then minus it from your original number and you have your answer! First i drew the five by five grid and there was nine in the middle to take away—$3 \times 3 = 9$

so then i figured out a 6×6 square was 36 and i know that inside there would be a 4×4 square to take away so the difference between 6 and 4 is 2 —so it was $36 - 16 = 20$ then $17 \times 17 = 289$ the square root is 17 then minus 2 from 17 which is 15 (because before there was 2 difference like between 6 and 4) and then $15 \times 15 = 225$ then i minused 225 from 289 and got 64

$n \times n = n$ squared $-$ $(n - 2)$squared - so minus (n−2)squared from nsquared

In response to AW's note, SI posted a note (PP40) in which he offered a different rule based on the Overlapping Corners Strategy.

PP40 **Another rule**—SI

I have another rule for you and it is the output ×4−4. In the rule it is ×4 because there is 4 sides in a square. It is −4 because when you multiply 4 you are repeating the corners twice so you −4.

In her response to SI, (PP42), AW speculated that there may be more than one solution to the problem.

PP42 **2 rules**—AW

but there might be two rules because we got the same answer for both so i think there is more then 1 way to figure the problem out

Another student, GA, then questioned AW's rule, not on the basis of whether it yields the correct answer, but rather on the basis of elegance. She titled her note "Both right?"

PP43 **Both right?**—GA

I agree with you and disagree with you because you've got the answer but in a complicated way. I disagree with you because there's an easier way than taking the square of 25, subtracting 2 from it and square that and then subtract that from your original answer. I got the rule times 4 −4 because a square has 4 sides and you don't count the corners twice. I agree with you because for the first few questions you got it right.

At this point in the discussion AW became firm in her conjecture that there is more than one solution to the problem.

PP46 **Two Ways**—AW

Why can't there be two ways. There are different ways to do lots of different problems i think you can have two ways $n \times n = n$ squared $-$ $(n - 2)$squared − so minus (n−2) squared from n squared works and ×4−4 works

While it was in their discussion on the Perimeter Problem that students first encountered the idea of multiple rules, our analysis indicated that as the students gained experience and posted answers for more difficult problems, they developed a general acknowledgement of the validity of having more than one correct solution, and the offering of multiple rules became a regular part of their work on these problems. As illustrated in the note below (contributed as a solution to the Pattern Kingdom problem, which is similar to the Handshake Problem), AN acknowledged the rules of two other students and the fact that both rules work for every case.

PK4 **Rule**—AN

My theory is that there are two rules sri's and mine and gauthman's. So the rule is $n \times (n-1)/2$ or $n \times n - n/2$. it works for everyone.

Revision of notes: Improving ideas

There are examples throughout the literature that reveal how students resist changing initial conjectures of rules even in the face of contradictory evidence. In this section of the results we present analyses of students' work on revising their ideas. An important feature of Knowledge Forum is that notes can be revisited and revised at any time. To answer our second research question we counted the number of times that students revised their own notes in each problem view. Each time a student added or modified an idea in their note it was counted as a revision. In all, for the 6 problems in the database, there were 194 revisions, with a spread of 0–11 revisions per note.

In the first example, taken from the Handshake Problem view, we present a discussion in which a student, GN, was prompted to revise his conjecture. In this discussion GN posted his original idea, which was then questioned by two other students.

HP15 **Handshake Problem**—GN

My theory is that the rule is the input + the output − (minus) the input. I will make a t-table and see if my rule applies.

# of people	# of handshake
2	1
3	3
4	6

HP16 **I think**—AM

thats a good theory but you need to figure out the output. this pattern does not really explain how to figure out the output

HP17 **right**—AH

I agree with you because GN's rule needs the output but that is why we need the rule. It is like a rule has to be done with the input to find the output.

The handshake problem is challenging for both students and adults because it is based on a quadratic function. Typically, students solve this problem using a recursive strategy of noting a pattern of differences in the output (number of handshakes) column. In his initial pattern spotting, GN described the numeric pattern he discovered moving from input to output column—"the input plus the output minus the input equals the output"—without seeming to realize that his rule is essentially $a + b - b = a$. The limitation of this rule was recognized by other students. They pointed out that GN's "rule" depends on knowing the values for both the input and output number, and therefore it cannot be applied to find unknown output numbers from known input numbers. In response, GN then posted another theory, which he labeled "the true rule." He used the metacognitive scaffold "this theory does not explain" referring to his original theory, and then the scaffold "a better theory" as he expressed his new rule. Our analysis of his contributions revealed that GN revised this second note five times over the course of 2 weeks.

> HR15R5 I found it (the true rule)—GN
>
> *This theory does not explain* because I need to know the output and this rule uses the output when I need to know the output.
>
> *A better theory* is the input times input −input divided by 2. I multyply the input by the input because 2 times 2 equals 4 and when i − the input it will equal 2 and 2 divided by 2=1. This is how I worked it out:
>
> $$2 \times 2 = 4 - 2 = 2/2 = 1$$
> $$3 \times 3 = 9 - 3 = 6/2 = 3$$
> $$4 \times 4 = 16 - 4 = 12/2 = 6$$
> $$5 \times 5 = 25 - 5 = 20/2 = 10$$
> $$6 \times 6 = 36 - 6 = 30/3 = 15$$
> $$7 \times 7 = 47 - 7 = 42/2 = 21$$
> $$8 \times 8 = 64 - 8 = 56/2 = 28$$
> $$9 \times 9 = 81 - 9 = 72/2 = 36$$
> $$10 \times 10 = 100 - 10 = 90/2 = 45$$
> $$11 \times 11 = 121 - 11 = 110/2 = 55$$

In his response, GN acknowledged the problem with his original rule, and offered a new rule that, when applied to the input number, yields the output.

While GN was prompted by other students to refine his ideas, there were also examples in the database of students revising their own ideas without prompting. The note below, entitled "Relationship," was written by a student, NS, who posted a solution to the Perimeter Problem but appeared to recognize that her solution was not fully explained. As she commented in her note, "I think I still have to think a little more to explain my theory."

> PP19 **Relationships**—NS
>
> *My theory* is that the Output # is = to the input # times 4 − 4. My evidence is that $3 \times 4 = 12$ and −4 is 8 which is the output. You need to multiply the input×4[only one side] because without multiplying you wouldn't get 12. Then when you minus another # besides 4 the output wouldn't be 8. The same rule applies for all the other numbers. Like: $100 \times 4 = 400 - 4 = 396$ $10 \times 4 = 40 - 4 = 36$ $14 \times 4 = 56 - 4 = 52$
>
> I think I still have to think a little more to explain my theory.

In this note, NS's solution is based exclusively on a numeric analysis of the data given, with the conclusion that only the operations of ×4 −4 would lead to the correct output numbers. She stated that her rule would apply for all the other numbers, but is unclear as to why this is true. When we analyzed all of NS's contributions to this note we could see that she revised this original note a total of five times over the course of 3 weeks. In her final revision (PP19r5), posted 3 weeks after her original entry, she contributed a solution that linked her original rule to the structure of the problem itself, which numerous researchers argue reveals a deeper and more conceptual understanding of the problem (e.g., Steele & Johanning, 2004; Swafford & Langrall, 2000). Her final revision, similar to GN's, began with the metacognitive scaffold "a better theory."

PP19r5 *A better theory*

You need to × 4 because you need 4 sides to make a square. Like 3×3 means 3 is the length and the other 3 is the width so one length or width×4=to the whole perimeter.

Whereas NS's first explanation, based on testing her rule with a series of numbers, was more in the spirit of "guess and check," in this final explanation NS revealed that she was aware of why the rule works. She realized that the four in her rule ($4n$) represents the 4 sides of a square grid and, as she demonstrated, she can generalize how this would work for grids of other dimensions—"Like 3×3 is the length and the other 3 is the width." This kind of revision and rethinking was typical of many of the efforts of other students. Our analyses of revisions revealed that the students made use of the time and opportunity to revise their notes, and worked on the problems over an extended period of time.

Justifications and epistemic agency

In knowledge-building communities, members make progress not only in improving their personal knowledge, but also in developing collective knowledge through progressive discourse (Bereiter & Scardamalia, 2003; Muukkonen, Lakkala, & Hakkarainen, 2005; Oshima et al., 2006). In the preceding sections, while our analyses focused on multiple rules and revisions, throughout these examples there is also clearly evidence of students providing justifications and evidence for their rules. In this final section of the results we focus particularly on the students' provision of evidence and justifications. Since the students were solely responsible for the contents of the database, we were interested in determining the extent to which they worked to ensure that their reasoning was fully explained.

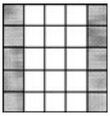
Our analyses of students' justifications took two forms: a rating of notes in terms of levels of justifications offered and a developmental analysis of how students progressed in terms of the kind of justifications offered. Notes designated Level 1 were either those in which only a conjecture was offered—*My theory is that the rule is ×4 −4*—or that offered social support—*good theory!*. The next two levels of notes offered evidence to support conjectures. Notes coded as Level 2 offered a conjecture with a brief explanation—*I figured out the rule and it is the number times 4 (the four sides) minus 4 (because you use one twice at each corner)*. We coded notes as Level 3 If they offered a conjecture and evidence and included multiple representations, and/or a detailed account of why the rule worked. To illustrate the kinds of justifications coded as Level 3 we present three notes composed by students that are taken from three of the different problem views.

The first example (PP9) titled "I got it!" was offered by a student, JF, as an explanation for the Perimeter Problem. This note is as an example of a note coded as Level 3 because it includes a proposed rule and a clear, contextualized explanation based on this student's particular way of perceiving the problem.

PP9 **I got it!**—JF

i got x+x+(x − 2 × 2) and i tried it out for a lot of them and it worked:
5+5+(5 − 2 × 2)
5 + 5 = 3 × 2
5+5+6

$10 + 6 = 16$ so that means the rule is $x+x+(x - 2 \times 2)$ and that's it! if you seperate the question into two you have 5+5 and $+(5 - 2 \times 2)$ so let's focus on 5+5 and that equals 10 and that means that we can do this



So now we can focus on this part: $=(5 - 2 \times 2)$ so $5 - 2 = 3$ and the space left in the square is 6 and 3×2 is 6 so it's $x+x+(x - 2 \times 2)$

The second example of a Level 3 note is taken from the Triangle Dot Problem. The Triangle Dot problem (Steele & Johanning, 2004) presents students with an illustration of two equilateral triangles, one of which is made up of nine dots with four dots per side, and the other made up of 15 dots with six dots per side. The challenge for this problem is to find the number of dots that would be needed to make a triangle with n dots per side. In the note below, a student (KD) explains why his rule ×*3 −3* (times three minus three) works, based on his knowledge of the three-sidedness of a triangle and an understanding that, because the sides of a triangle meet at the corners, the dots overlap at these corners and so the extra three dots are subtracted.

TD7 **3 corners**—KD

Our theory is that 45 dots are in a 16 dot triangle. 48 dots are in a17 dot triangle. The rule is ×3 −3. We figured out it—the rule—by knowing there are 3 sides to a triangle. It always stays 3 no matter what because it doesn't matter how much dots there are to a side. So that means it's ×3. But I thought how could a 4 dot triangle have 9 dots. An answer popped out of my head. 1 corner shares 2 sides so theres no need for the extra dots in 3 corners. So I took them away. I then thought the rule was ×3 −3 and it worked!

As can be seen, this student and his group were committed to generating a rule and illustrating how this rule would work for various sized triangles.

The final example of a Level 3 note that we present, confidently titled "Right Answer," is taken from the Handshake Problem view and offers a detailed explanation of why a rule works.

HP2 **Right answer**—AN

My theory is that the pattern rule is that the #of handshakes are equal to the total#of people. The rule is $n \times (n - 1)/2$ in other words we have to times the number befor the actual input and then divide by 2.we need to minus 1 because we are not shaking hands with ourselves.we have to handshake with everyon else execpt ourselves so it has to be n×(n−1).We have to divide 2 because 2 people make one handshake. It works for 2 like $2 \times (2\,\text{the input} - 1 = 1) = 2/2 = 1$. number 3 is also works $3 \times (3\,\text{the input} - 1 = 2)6/2 = 3$ it works for four $4 \times (4\,\text{the input} - 1 = 3)12/6 = 2$. SO for 5 it should be 10 because $5 \times (5 - 1 = 4)20/2 = 10$. so if we try for 10 it should be $10 \times (10 - 1 = 9)90/2 = 45$. Then we must try $100 \times (100 - 1 = 99)9900/2 = 4950$. I shall tell the handshakes for 10,000 is 49,995,000 in other words 49 million 995 thousand handshakes. Incase some people don't know what n is it is input.

In this note, AN offered a rule of the input number multiplied by one less the input number divided by two, or $n(n-1)/2$. This version of $(n^2-n)/2$ is clearly derived from the context of the problem, as AN relates each component of his rule to the problem situation. As he clearly explains, every person shakes hands with every person less one (themselves), but since two people make a handshake this total number can be divided by two. He then offers proof in the form of examples and has included very large numbers in order to illustrate the robustness of his theory.

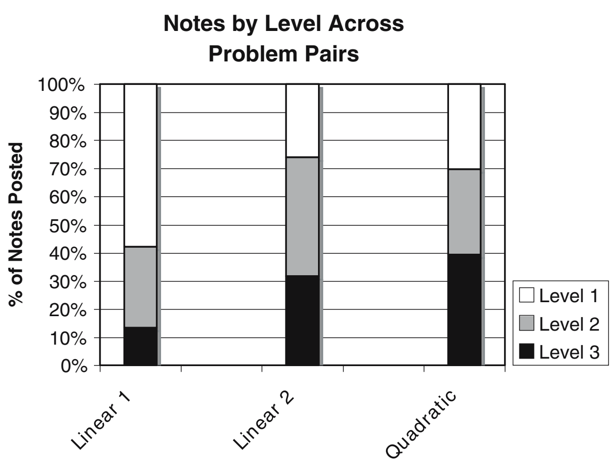
These three examples show the students' desire to explain to others the reasoning behind the rules offered. They explain why the rule works in the context of the problem given and why the rule would work for any instance or iteration of the pattern. We observed that as the students worked on increasingly difficult problems, their commitment to offering detailed explanations also increased. Specifically, we noticed that there was an increase in Level 2 and Level 3 notes posted as students progressed through the database. Our analyses revealed that whereas only 42% of notes written in response to the first two problems posted on Knowledge Forum included justifications, approximately 70% of notes written in response to the last four problems included at least one justification and were coded as either Level 2 or 3. Our analysis of the database revealed that the majority of students moved through the problems in chronological order, starting with the first pair of linear problems and finishing with the quadratic problems. Thus, it appears that as students became more experienced in working in Knowledge Forum, they became more sophisticated and more mathematically oriented in their offering of evidence and justification. Table 1 shows the levels of notes as a function of Problem Pair in chronological order.

In addition (and noteworthy), while our analyses revealed that the students became more committed to more elaborate offerings, further analyses also revealed that with time spent on solving problems there was an unexpected finding; namely, that the students went beyond strictly focusing on individual problems to recognizing the structural similarities across pairs of problems presented.

Structural similarity—meta-rules

As part of our selection of problems for this research study we chose to include pairs of problems that were structurally similar. Steele and Johanning used a similar methodology in

Table 1 Level of justification of note as a function of problem pair



their research study with a small group of high ability 7th grade students. While the goal of their study included determining students' abilities to find the similarities, we did not expect that our young students would achieve this level of generalization as they worked independently on the database.

We believe that it is important to consider this unexpected occurrence as we explore the potential of Knowledge Forum to support students' work on generalizing problems and, perhaps more generally, for Knowledge Forum and mathematics problem solving. Therefore, to conclude our presentation of results, we present examples in which students recognized and articulated structural similarities between problems.

The following note was posted by KD (and friends). In this example we can see that KD recognized that a rule based on multiplying the four sides of a square and subtracting the four overlapping corners is "just like the triangle dot problem" for which he multiplied the three sides of a triangle and subtracted the three overlapping corners.

> PP10 **Our Smart Solution** (to the perimeter problem)—KD, ABH, MB
>
> *Our theory* it's just like the triangle dot problem.....Yes there is a rule and the rule is × 4 −4. We figured it out by using blocks to make the squares we know that there are 4 sides to a square. So whatever the × square it is you times it by 4. but when we made the × squares the corners over lapped so there's no need for the 4 corners. So that means − 4. Our evidence is that we tried it for each × square and it matched the correct number of shaded squares.

Similarly, we present the following two notes from the Pattern Kingdom Problem, a problem that is structurally identical to the Handshake Problem. Both AN and GN, whose answers for the Handshake Problem are in previous sections of this paper, realized that the Pattern Kingdom is essentially the same problem and that, therefore, the rule for both is the same.

> PK5 **same rule**—AN
>
> *My theory* is that this has the same rule as the handshake problem since they are the same numbers. see in handshake problem view, right answer AN.
>
> PK11 **Same as handshake**—GN
>
> *My theory* is that it is the same rule as the handshake problem, which is the input times the input - the input divided by 2.

$$2 \times 2 = 4 - 2 = 2/2 = 1$$
$$3 \times 3 = 9 - 3 = 6/2 = 3$$
$$4 \times 4 = 16 - 4 = 12/2 = 6$$

Summary of the overall results

The notes and discussions that we have presented are representative of the kinds of postings that we found throughout the database for the series of six problems. The literature reporting the limitations of students' work with generalizing problems suggests that it is rare to find the offering of multiple rules and revisions of conjectures. Furthermore, the provision of justifications and evidence not only eludes most students in their work with generalizing problems, but is a general weakness noted throughout the mathematics education literature (e.g., Healy & Hoyles, 1999; Jacobs et al., 2006).

Discussion

Research questions revisited

This study looked at the utility of Knowledge Forum to support students' work in solving generalizing problems. It is well known that while generalizing problems are considered to provide grounding in algebraic reasoning, there is strong evidence that the transition from patterns to algebra is not smooth. Although students may find a rule for a given generalizing problem, they do not typically find algebraically useful rules and lack what Lee refers to as "perceptual agility," which she defines as the ability to abandon rules that do not work and to find new (better) rules. Furthermore, the literature indicates that patterns can offer meaningful contexts for students to begin to grapple with generalizations and abstraction. Again, there is a substantial literature reporting that typically students lack the rigour required to be able to generalize from useful rules for a given problem to an understanding of mathematical structure. This is particularly evidenced by students' lack of commitment to developing justifications for their conjectures of rules.

The specific issues that drove our Knowledge Forum research were directly related to the difficulties that have been identified in the literature about generalizing problems: Would Knowledge Forum support students to find rules (or multiple rules) and would they revise rules that do not work? Would students (with no teacher input) provide evidence and justifications for their conjectures of rules?

Knowledge Forum and generalizing problems

Our decision to use Knowledge Forum in our research on early algebra was based both on what we perceived to be the affordances of the software and on the knowledge-building principles that underlie the pedagogy and design of Knowledge Forum. We anticipated that the collaborative structure of Knowledge Forum would benefit the students by providing them with access to each other's theories and perspectives on the problems posed in the database, and thus support them in finding algebraically useful rules. When we conducted our analyses of the database we found that students did, in fact, find multiple rules for the problems that were posted (ranging from two to five rules per problem), and that they were able to negotiate the idea that there could be more than one rule for a given problem.

In addition, we also anticipated that the capability Knowledge Forum offers for revisions, and the asynchronisity of discussion, would also provide support for students' work with generalizing problems. Given that students typically do not attempt to revise their rules, we wondered if working on Knowledge Forum would encourage them to do so. When we studied the notes that students contributed to the database we discovered that students did, in fact, revise their own notes. Furthermore, students returned to and revised their original theories for up to 3 weeks after the original note was posted. This kind of thoughtful reflection, while part of the goals of current mathematics reform efforts, does not happen in most mathematics classrooms.

The evidence suggests to us that the Knowledge Forum software supported both students' growing awareness of multiple rules and the ability to revise their offered conjectures. However, we believe that the knowledge-building principles of *improvable ideas* and *epistemic agency* underpinned students' developing dispositions to find and revise multiple rules and, in addition, supported the students' commitment to provide evidence and justifications. While it is difficult to find a conclusive way to distinguish between the enabling effects of the software and students' assumption of the knowledge-building principles, our conjecture is that the knowledge-building environment of Knowledge Forum was a significant factor in the results.

Epistemic agency: Evidence, justifications and meta generalizing

Epistemic agency is defined as the responsibility the group assumes for progressively moving the understanding forward and increasing the (mathematical) knowledge base. In our study, we could see this push to move the knowledge base forward both in the increasing commitment and sophistication of justifications offered by the students—a result that has been central to our investigation—as well as in the unanticipated result of students finding structural similarities across problems. When we analyzed the database for instances of justifications we discovered that the students grew increasingly committed to providing evidence and justifications for their conjectures, a commitment that exceeds that reported in older populations of students. The justifications these 9- and 10-year-old students provided went beyond simple number substitutions to explanations given within the context of the specific problem. Students established a community norm of routinely offering notes that included explanations of their answers using language, tables of values, symbols, and/or images in order to justify their conjectures. Furthermore, our analyses of frequency counts revealed that the students' level and commitment to justifying their notes increased as a function of time spent on the database. In our view, this increasing commitment to and sophistication of justifications is indicative of the students' adoption of the principles of epistemic agency and idea improvement.

Perhaps an even more conclusive finding to support our conjecture of students' adopting a knowledge-building culture of practice was the finding that the students spontaneously began to notice structural similarities between the problems in the database. While we specifically selected structurally similar pairs of problems, we did not anticipate that students at this grade level would be inclined to notice or articulate their perceptions of these similarities. Finding structural similarities implies a level of generalization that goes beyond finding functional rules, and went beyond what we asked of the students.

The students in all three experimental classrooms had used Knowledge Forum for classroom projects in science, and thus were not only knowledgeable about the software and processes but also had been part of a culture of inquiry in which student-driven ideas and theories were at the centre. Thus, it seems reasonable to speculate that this culture of knowledge building established in their work in science influenced their use of Knowledge Forum in mathematics. The informal interviews that we conducted with the students at the end of the research project support this conjecture. Many students asserted that working on mathematics on Knowledge Forum was different from "regular mathematics." To quote, "In regular math you don't use 'my theory' and get to write it down and then get new information to help you develop your theory—in regular math you don't ask people to help you find what the rule is." "This was more fun because we all got to participate and stick our ideas together and got to share our thoughts on the computer and we got to know what kids outside our school thought about the problem and we could stick all our ideas together." "If your answer is wrong, on KF someone comes and helps you out...even if your answer is at first wrong it will be leading to the right answer. KF is about improving ideas, you can improve your answer for the rule."

Working on mathematics problems in Knowledge Forum was a new experience for these students, and in our view what they chose to notice revealed something about the culture of knowledge building, idea improvement and epistemic agency in their mathematics learning.

Knowledge Forum and mathematics: Further research

Current ideas of mathematics education call for the establishment of classroom cultures in which students "analyze and evaluate the mathematical thinking and strategies of others; communicating mathematical thinking coherently and clearly to peers; and make and investigate mathematical conjectures" (National Council of Teachers of Mathematics (NCTM), 2000). However, it has been shown that even teachers who have embraced these goals do not necessarily enact them in the classroom (Jacobs et al., 2006). This calls for new approaches to mathematics problem solving. The students in this study appear to have acquired a disposition to mathematics problem solving that adheres to the goals of reform mathematics. Thus, the evidence from this study suggests that it is worth exploring the use of Knowledge Forum software and pedagogy in the teaching of mathematics in the elementary school grades.

Our present study is limited, however, and raises new avenues of exploration. For example, one question that arises directly from our study concerns the levels of collaboration. The students in our study came from two different schools and, therefore, many of the children connected to the Knowledge Forum database had never met. Our analyses revealed that the students displayed a high commitment to collaboration as well as a high degree of commitment to the mathematics. In future studies it would be important to discover if the level of collaboration and the richness of mathematical discussions would be found when Knowledge Forum is used for mathematics learning in single classrooms.

Other, more general issues emerging from this research concern the applicability of Knowledge Forum for mathematics learning. Nason and Woodruff (2002), among others, have identified the potential limitations of school-based mathematical problems as a site for knowledge building. They are concerned with the lack of open-endedness, lack of relevance and lack of authenticity in school-type math tasks. In their view, mathematics textbook problems do not elicit the multiple cycles of testing and refining that occur as part of knowledge building. Our findings indicate that the textbook generalizing problems that we used did, in fact, support students in rigorous inquiry and knowledge building. Thus, in our view, future research should be conducted that investigates whether other domains of mathematics can also be supported by Knowledge Forum.

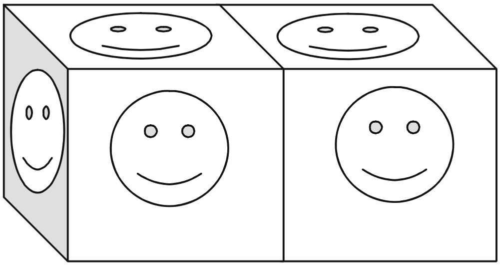
Finally, another concern raised in respect to Knowledge Forum and mathematics learning is that Knowledge Forum is a text-based discourse space, and therefore does not support symbolic representations and tools as a means of communication. Again, our findings do not support these concerns. Our results (similar to those of Hurme & Jarvela, 2005) indicated that the text-based nature of Knowledge Forum may in fact have benefited the students. Our analyses revealed that students made concerted efforts to communicate their ideas by developing a syncopated form of mathematical communication (Sfard, 1995) and naturally interwove words, numbers, and formal symbols in their interactions with each other. We believe that for these students the need to clearly communicate their theories enhanced their mathematical understanding. Further research is needed to analyze the nature of this discourse more deeply.

Appendix

Six generalizing problems

Cube sticker problem

A company makes coloured rods by joining cubes in a row and using a sticker machine to put "smiley" stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker. This rod of length 2 (two cubes) would need ten stickers.



How many stickers would you need for:

A rod of one cube
A rod of two cubes
A rod of three cubes
A rod of four cubes
A rod of ten cubes

How many stickers would you need for a rod of 20 cubes?
How many stickers would you need for a rod of 56 cubes?
What's the rule?

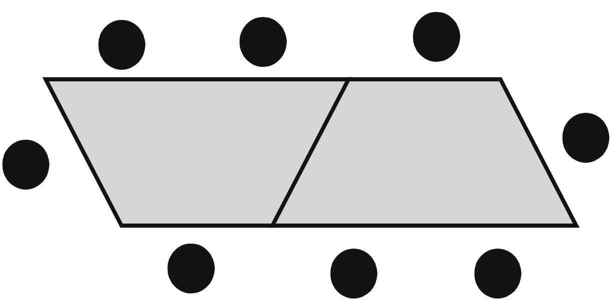
Table and chairs problem

Grenvale Public School has decided to include a lunchroom as part of the school's renovations. Mrs. Chen, the principal, found an amazing sale on trapezoid shaped tables so she decided to buy many of these tables for the new lunchroom.

While Mrs. Chen was waiting for her order to be delivered she thought she would draw a plan for her lunchroom. Mrs. Chen decided she would place the chairs around the table so that two chairs will go on the long side of the trapezoid and one chair on every other side of the table.

This way five students can sit around one table.

Then she found she could join two tables like this:

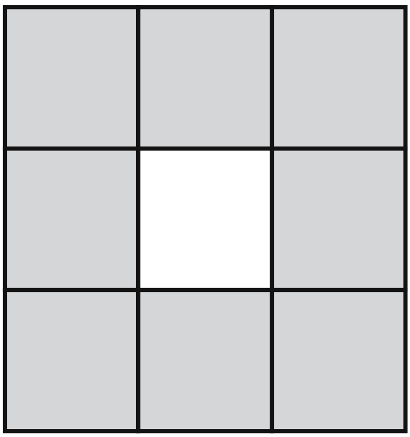


Now eight students can sit around two tables.

How many students can sit around three tables joined this way?

How many students can sit around 56 tables?
What is the rule?
How did you figure it out?
Can you give evidence?

Perimeter problem



This is a 3×3 grid of squares with only the outside edge shaded.

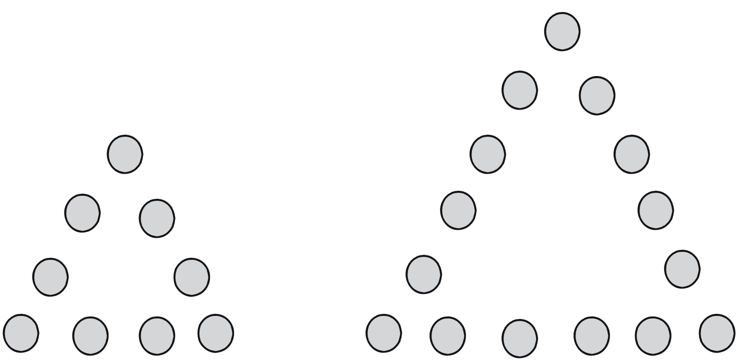
If you had a 5×5 grid of squares where only the outside edge of squares is shaded, how many squares would be shaded?

If you had a 17×17 grid of squares with only the outside edge of squares shaded, how many squares would be shaded?

If you had a grid of 100×100 squares, how many would be shaded?

What is the rule?

Triangle dot problem



Above is a four dot triangle where each side has four dots. It is made using a total of nine dots.

The next triangle is a six dot triangle. It has a total of 15 dots.
How many dots would you need altogether for a 16 dot triangle?
How many dots would you need altogether for a 100 dot triangle?
What is the rule?

Pattern kingdom

In the Pattern Kingdom, each city is connected to the other cities by a road. To make it simple for people to get around, there is a road connecting each city with all of the other cities. When the Pattern Kingdom only had three cities, there were three roads to connect them.

When the Pattern Kingdom grew to four cities, there were six roads to connect them so that there was a direct route from any city to any other city.

Now the Pattern Kingdom has 14 cities. How many roads does it have?
What if there were 32 cities? How many roads would there be?
Is there a rule?

Handshake problem

Imagine that the Maple Leafs won the Stanley Cup and you are at a huge party with everyone in Toronto to celebrate.

Everyone starts to shake hands with other people who are there.

If two people shake hands there is one handshake.

If three people are in a group and they each shake hands with the other people in the group, there are three handshakes.

If four people are in a group and they each shake hands with the other people in the group, there are six handshakes.

How many handshakes would there be if there were ten people in the group?
How many handshakes would there be if there were 100 people in the group?
Can you use a rule to help you figure this out?

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