On the robustness of risk-based asset allocations

Thorsten Poddig · Albina Unger

Abstract   Since the subprime crisis, portfolios based on risk diversification are of great interest to both academic researchers and market practitioners. They have also been employed by several asset management firms and their performance appears promising. Since they do not rely on estimates of expected returns, they are assumed to be robust. The same argument holds for minimum variance and equally weighted portfolios. In this paper, we consider a Monte Carlo simulation, as well as an empirical global portfolio dataset, to study the effect of estimation errors on the outcomes of two recently proposed asset allocations, the equally weighted risk contribution (ERC) and the principal component analysis (PCA) portfolio. The ERC portfolio is more robust to changes in the input parameters and has a smaller estimation error than the Markowitz approaches, whereas the PCA portfolio is even more unstable than the classical approaches. In the worst-case scenario, neither approach delivers what it promises. However, in every case the resulting return–risk relationship is dominated by the Markowitz approaches.

T. Poddig (✉) · A. Unger
Chair of Finance, University of Bremen, Hochschulring 4, 28359 Bremen, Germany
e-mail: poddig@uni-bremen.de

A. Unger
e-mail: aunger@uni-bremen.de

1 Introduction

Ever since the recent financial crisis, there has been a great deal written, by both academic researchers and market practitioners, on the necessity of building portfolios that are more diversified. Most of this literature is based on risk management rather than estimating returns (Lee 2011). Another new approach to portfolio optimization aimed at achieving true diversification based on "uncorrelated bets" is proposed by Meucci (2009). Critics argue that the classical mean variance framework proposed by Markowitz (1952) results in the underdiversified portfolios that failed in the financial crisis (Allen 2010). Numerous studies are cited in support of these approaches based on their apparent outperformance versus passive market-capitalization weighting or static, fixed-weight portfolios (Lee 2011).

In this paper, we concentrate on the so-called equally weighted risk contribution (ERC) and principal component analysis (PCA) portfolios. The concept of the ERC portfolio is rooted in risk budgeting. It is "a process of measuring and decomposing risk, using the measures in asset-allocation decisions, assigning portfolio managers risk budgets and using these risk budgets in monitoring the asset allocations and portfolio managers," according to Pearson (2002, p. 7). However, the optimization procedure of the equal risk contribution portfolios is not only about risk monitoring, but also involves allocating the capital among the assets according to equal risk contributions in the portfolio. So the concept of ERC portfolios goes one step further and defines the asset allocation procedure based on the risk decompositions. The ERC portfolios allocate the market risk equally across asset classes, including stocks, bonds, and commodities (Qian 2005).

We also investigate the so-called PCA portfolio approach suggested by Meucci (2009). In his methodology, the variance–covariance matrix is decomposed in uncorrelated principal portfolios (as shown in Partovi and Caputo 2004). Based on these principal portfolios, a diversification measure is built (Meucci 2009, p. 10). For the optimal portfolio, the entropy based on the contributions to uncorrelated bets in the portfolio is maximized.

It is a known fact that the classical Markowitz portfolios tend to concentrate on a small subset of the available securities, which does not appear to be consistent with the idea of risk diversification. Moreover, the asset weights are very sensitive to changes in the input parameters, such that a small variation of the input parameters results in very different portfolios. The input parameters, i.e., expected returns and correlations, are typically estimated from historical data and are often imprecise (Michaud 1989, 1998; Scherer 2004; Broadie 1993; Chopra and Ziemba 1993; Herold and Maurer 2004; Tütüncü and Koenig 2004; Fabozzi 2007). This leads to a large estimation error with respect to the estimated ex ante portfolio performance, as developed by Broadie (1993).

In light of the problems of the classical Markowitz approach described above, we explore the robustness of these new risk-based asset allocation techniques. To compare the results, we also study the following portfolio constructions: efficient Markowitz portfolio, minimum variance, and equally weighted. Thus we combine two current streams of the literature and examine the sensitivity of the equally weighted contribution and maximum entropy portfolios through a simulation and an empirical

study. This issue is highly relevant in practice as no one can perfectly forecast the future and thus there are always estimation errors. The question is how the estimation errors influence the resulting portfolio structure and portfolio performance.

The ERC approach is the topic of several recent articles (Qian 2005, 2009, 2010; Neukirch 2008; Maillard et al. 2010; Allen 2010; Foresti and Rush 2010; Little 2010; Lee 2011). However, none of this work studies the influence of estimation errors on portfolio performance. To the best of our knowledge, no evaluation of the performance of the PCA approach has yet been undertaken. These gaps in the research pose a real obstacle to the practical use of portfolio optimization.

We show that by investigating the effect of the true resulting outcomes, the performance of the approaches is different from their performance based on estimated parameters. The results show that the ERC portfolio is far less sensitive to estimation errors than the Markowitz approaches, whereas the PCA approach has the worst performance in all cases. However, all in all, neither approach provides real diversification when it is most needed, namely, in times of crisis.

2 Literature review

2.1 Mean-variance optimization and weaknesses

Markowitz (1959) mean-variance optimization is the classical technique for allocating capital among a set of assets (Michaud 1998, p. 1). Since the return is measured by the expected value of the random portfolio return, while the risk is quantified by the variance of the portfolio return, it is called a mean-variance framework (Recchia 2010, p. 14). Given the returns, variances, and correlations of the assets, the mean-variance approach allows determination of efficient portfolios. A portfolio is called efficient if there is no other portfolio that has a higher expected return for the same level of risk or, alternatively, the portfolio demonstrates lower risk for the same level of expected return. The efficient frontier represents all efficient portfolios.

2.1.1 Weaknesses

The mean-variance optimizer treats the inputs as "true parameters" and does not adjust for uncertainty within the estimated parameters (Michaud 2009, p. 18; Drobetz 2001, p. 59). In reality, however, the inputs are not known in advance, but can only be forecasted or estimated, usually with large errors. These estimation errors are known to result in not well-diversified "optimal" portfolios, which tend to concentrate on a small subset of the available securities and are also often very sensitive to changes in input parameters (Tütüncü and Koenig 2004, p. 2).

As demonstrated by Black and Litterman (1992), a small change in the expected asset returns can result in large changes in the optimal portfolio allocation. The asset weights are extremely sensitive to variations in expected returns so that adding a few observations to the estimation sample may change the asset allocation completely (Jorion 1985, p. 261).

Since the true "future" returns, variances, and covariances are unknown in advance, they are usually estimated from historical data. As shown in Merton (1980) and Jorion (1985), the sample mean is far from being a precise estimate of the expected return. Simultaneously, returns have the largest effect on the optimizer, which results in portfolio structures that are far from being the true optimal allocation (Jobson and Korkie 1980; Michaud 1989; Chopra and Ziemba 1993; Broadie 1993; Jagganathan and Ma 2003). Michaud (1989) focuses on the limitations of the Markowitz approach and calls the MV optimizer an "estimation-error maximizer" due to its tendency to maximize the effects of errors in the input assumptions (Michaud 1989, p. 33).

Broadie (1993) analyzes the effect of estimation error by distinguishing between the true efficient frontier, the estimated frontier, and the actual frontier. Using the true (but unknown) parameters, the "true" efficient frontier is computed. Next, using the estimated input parameters, a similar frontier, the "estimated" frontier, is determined. Finally, by using the portfolio weights derived from the estimated parameters, and then applying the true parameters, the "actual" frontier is obtained. In short, the estimated frontier is what appears to be the case based on the data and the estimated parameters, but the actual frontier is what really occurs based on the true parameters (Broadie 1993, p. 24). In a simulation study he concludes that the errors of the estimated returns using historical data are so large that these parameter estimates should always be used with caution.

In this paper, we follow Broadie's method to investigate the real outcome of the portfolios.

2.2 Robustness

The problems of the Markowitz approach described above have given rise to a large and growing body of literature that deals with estimation errors in order to create "robust portfolios". The literature contains two standard methods for dealing with estimation errors: the robust estimation method, which incorporates uncertainty sets directly in the optimization process, and the Bayesian method, which incorporates uncertainty to generate robust inputs (for an overview, see Fabozzi 2007, 2010; Scutellà and Recchia 2010).

However, the term "robustness" is not well defined in the literature (Jen 2001 for at least 17 different definitions of robustness in different contexts and Brinkmann 2007, p. 18). As stated by Tütüncü and Koenig (2004): "Generally speaking, robust optimization refers to finding solutions to given optimization problems with uncertain input parameters that will achieve good objective values for all, or most, realizations of the uncertain input parameters." This concept is called "solution robustness" as the value of the objective function remains stable.

Mostly, robustness is associated with such stable solutions in the case of varying input parameters, as most techniques provided in the literature try to reduce the sensitivity of the outcome of the Markowitz-optimal portfolios to input uncertainty or the impact of estimation errors (Goldfarb and Iyengar 2003, p. 2; Fabozzi 2007, p. 10). However, robustness of the solution (mean and variance of the portfolio returns) does not imply structural robustness, i.e., the stability of the asset weights.

Tütüncü and Koenig (2004) suggest an alternative definition of robustness, suggesting that a solution is robust when it "has the best performance under its worst case". The worst case is defined as a situation where the input parameters are poor, i.e., low returns and high standard deviations of the assets. The question, then, is, given the same worst-case scenario, will some portfolio constructions have a better performance than other portfolio optimization techniques?

Here, we follow Brinkmann (2007) and define and study robust portfolios with respect to two characteristics:

- Solution robust: these are portfolios whose return and variance remain stable with respect to uncertain varying input parameters;
  - worst-case robust: these are portfolios that provide good outcomes even in bad circumstances (low returns and high standard deviations of the assets; obviously, this is simply a special case of solution robustness);
- Structure robust: these are portfolios whose structures are insensitive to changing input parameters (or, more precisely, portfolios that do not have extreme allocations and sensitive weights).

In Brinkmann (2007), the two characteristics are incompatible and "robustness" is dependent on the position of the portfolio on the efficient frontier. Portfolios that are near the minimum-variance portfolio are more structure robust, whereas portfolios that lie "near the investor-optimal mean-variance portfolio" are more worst-case robust. However, the results vary with the underlying data set and none of the approaches investigated show a good worst-case performance (Brinkmann 2007, p. 281). In this paper, we explore whether this is also the case for the ERC and the PCA portfolio.

2.3 ERC portfolios

Recent years have witnessed a growing interest in portfolio construction approaches focused on risk and diversification, called equal risk contribution portfolios or, more simply, risk parity. The first term is used more often by academics, whereas the second term is popular among practitioners. While the traditional "efficient frontier" portfolio construction involves allocating capital, thus resulting in individual asset risk contributions, the equal risk contributions approach turns that concept on its head, with risk allocation now determining the capital allocation (Morris and Haeusler 2010). With equal contributions to risk, risk diversification within the portfolio is achieved and no asset class dominates the total portfolio risk. For example, whereas an allocation of 50/50 in terms of weights between stocks and bonds is assumed to be diversified, it is known that the volatility for equities is considerably larger than that for bonds (e.g., for the Swiss market, the stock volatility was about four times higher than that of the bonds; Young and Johnson 2004). Many asset management firms claim the benefits of the ERC portfolio, but academics tend to be skeptical of these claims since it is a new approach to asset allocation and not all consequences are well known yet.

Although this investment idea dates back to the 1990s, it has only been the focus of a lot of attention in the last 6 years (Schwartz 2011). This seems related to the global

financial crisis in 2008, which caused investors to question what had gone wrong with many portfolios that were believed to be diversified (Lee 2011).

The ERC portfolio is a tradeoff between the naive and the MV strategy. The naive strategy splits the capital equally, the MV strategy splits the capital with respect to equal marginal risk contributions, and the ERC strategy splits the risk equally across the assets (equally absolute risk contributions). However, in a portfolio with many assets and quite similar variances and covariances, there is no significant difference between the ERC and the naive approach. On the other hand, when the risks of the assets are very different, the MV and the ERC approach strongly dominate the naive approach. However, whereas the ERC portfolio is always diversified in terms of weights and risk contributions, the MV approach is much more concentrated (Maillard et al. 2010, p. 11).

As the ERC portfolio technique is a heuristic approach, it does not lie on the ex ante estimated efficient frontier. Thus, there are other mean-variance portfolios with the same risk but a higher return or with the same return but a lower risk. Because of this, in practice, the ERC technique is mostly used with leverage.

2.4 Portfolio construction via principal component analysis

Another new idea based on risk diversification is from Meucci (2009), who states that whereas "the qualitative definition of diversification is that the portfolio is not heavily exposed to individual shocks, there exists no broadly accepted, unique, satisfactory methodology to precisely quantify and manage diversification" (Meucci 2009, p. 2) (for an overview of different approaches to the measurement of diversification, see Frahm and Wiechers 2011). Meucci's idea is based on the concept of principal components introduced by Partovi and Caputo (2004). The return data of the n assets is transformed into a set of i linear combinations of these assets, called principal components or principal portfolios. The attractive feature of principal portfolios is that they are uncorrelated with each other. The i principal portfolios can be interpreted as factors, which explain "most" of the variance in the data (Frahm and Wiechers 2011, p. 15). The principal portfolios are the risk drivers of the portfolio, which, by definition, are uncorrelated. As outlined by Frahm and Wiechers (2011), "Meuccis's definition of a well-diversified portfolio can be stated as a portfolio, the risk drivers of which are invested into equally. Put another way, the portfolio would be not well diversified if not all risk drivers are invested into equally."

The PCA approach appears to be an interesting approach to risk diversification but, to the best of our knowledge, the performance of this approach has not yet been studied. It is discussed by Frahm and Wiechers (2011) and by Stefanovits (2010), but is only used to measure the diversification of a portfolio. Stefanovits (2010) concludes that "it would be interesting to compare the allocation and the performance of the two principles (ERC and PCA) in some examples."

As the ERC and PCA portfolio allocations claim to provide real risk diversification, they should also be robust. Because of the same underlying idea of these recent portfolio construction techniques, in this paper we compare the performance as well

as the robustness with respect to the two characteristics of ERC and PCA portfolios discussed above.

The code for calculation of the PCA (maximum entropy) portfolio in MATLAB is taken from Meucci (see: http://www.symmys.com/node/199).

3 Description of the asset-allocation models considered

In this section, we briefly describe the portfolios that are investigated with respect to their robustness. To obtain a fair comparison of the robustness of the different approaches, the following constraints are applied for every optimization technique:

$$\sum_{i=1}^{n} w_i = 1,$$
$$w_i \geq 0, \quad i = 1, \ldots, n.$$

3.1 Classical Markowitz' mean-variance optimization

The mean-variance framework of Markowitz (1952) is the major model used today in asset allocation and active portfolio management (Fabozzi et al. 2010). Mean-variance optimal portfolios deliver the highest expected return given the levels of risk, where risk is typically measured by the standard deviation or volatility of returns. If the input parameters, such as forecasts of returns, risks, and risk aversion, are defined, the mean-variance portfolios deliver the optimal allocation of wealth (Lee 2011).

The classical Markowitz mean-variance optimization model can be formulated as follows:

Minimize the variance subject to a lower limit on the expected return:

$$\begin{aligned} \min \quad & w^{\mathrm{T}} V w \\ \text{s.t.} \quad & \mu^{\mathrm{T}} w \geq r* \end{aligned} \tag{1}$$

where μ and V denote the estimated expected return vector and the covariance matrix of the given assets, respectively and $r*$ indicates a specified target return.

3.2 Equally weighted portfolio (1/n)

The equally weighted portfolio assigns to each asset the same weight. With respect to the asset weights, this strategy is well diversified. However, this may not be the case due to covariation between different asset returns (Lindberg 2009). There is no objective function associated with the EW portfolio; the strategy involves holding a portfolio weight $w = 1/n$ in each of the n risky assets (DeMiguel et al. 2007, p. 1922). Obviously, this approach ignores both the return and risk prospects of the investments (Lee 2011).

3.3 Minimum-variance portfolio (MV)

Due to the fact that the estimation error in the sample mean is much larger than in the sample covariance matrix, the minimum-variance portfolio has garnered a great deal of interest, since it relies solely on estimates of the covariance matrix (DeMiguel and Nogales 2009). It is also the only portfolio that is on the efficient frontier without expected returns as inputs (Lee 2011). The optimization is similiar to the efficient portfolio, but without considering the expected returns of the assets:

$$\min \quad w^{\mathrm{T}} V w, \tag{2}$$

where V denotes the covariance matrix of the assets.

3.4 Equal risk contribution portfolio

There are various approaches to equal risk contribution portfolios, but we focus on the one proposed by Maillard et al. (2010). The ERC portfolio is obtained by equalizing risk contributions from the assets of the portfolio. The risk contribution of one asset is the share of total portfolio risk attributable to that asset. It is computed as the product of the asset weight with its marginal risk contribution, the latter being given by the change in the total risk of the portfolio induced by an increase in holdings of the asset. The principle can be applied to different risk measures, but we follow the approach of Maillard et al. (2010) and restrict ourselves to the volatility of the portfolio as a risk measure. The marginal risk contributions are defined as follows:

$$\mathrm{MRC} = \frac{\partial \sigma(w)}{\partial w_i} = \frac{\mathbf{V} \times \mathbf{w}}{\sigma(w)},$$

where $\sigma(w) = \sqrt{(\mathbf{w}^{\mathrm{T}} \mathbf{V} \mathbf{w})}$ is the portfolio standard deviation, $\mathbf{V}$ the variance–covariance matrix and $\mathbf{w}$ the vector of the portfolio weights. The portfolio risk can be viewed as the sum of the absolute risk contributions:

$$\sigma(w) = \sum_{i=1}^{n} \mathrm{ARC}_i,$$

with $\mathrm{ARC}_i = w_i \times \mathrm{MRC}_i$ as the absolute risk contribution of asset i.

Starting from the definition of the risk contribution, the equally risk contribution strategy finds a risk-balanced portfolio such that the risk contribution is the same for all assets of the portfolio. Thus the following optimization problem is considered:

$$\min \quad f(w)$$
$$\text{where } f(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathrm{ARC}_i - \mathrm{ARC}_j)^2. \tag{3}$$

3.5 Principal-portfolio (PCA)

Partovi and Caputo (2004) propose a new methodology for building the efficient frontier: the original set of assets is decomposed in uncorrelated portfolios, the so-called principal portfolios. Meucci (2009) proposes a diversification number based on the principal portfolios, namely, the entropy: the effective number of uncorrelated bets in a portfolio. To optimize diversification, the entropy is maximized under a set of investment constraints (Meucci 2009, p. 11): PCA makes use of the spectral decomposition theorem from linear algebra, which ensures that the covariance matrix Σ can be written as the product: $E^{\mathrm{T}} \Sigma E \equiv \Lambda$. In this expression, the diagonal matrix $\Lambda \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_i), i = 1, \ldots, n$ contains the eigenvalues of Σ, sorted in decreasing order, and the columns of the matrix $E \equiv (e_1, \ldots, e_i), i = 1, \ldots, n$ are the respective eigenvectors. The eigenvectors define a set of the n principal portfolios, whose returns $\tilde{R} \equiv E^{-1}R$ are decreasingly responsible for the randomness in the portfolio. The eigenvalues Σ are the variances of the principal portfolios (Meucci 2009, p. 3). The original portfolio w is now reconstructed as a linear combination of the principal portfolios via $\tilde{w} \equiv E^{-1}w$. Finally, Meucci introduces the *diversification distribution*:

$$
p_i \equiv \frac{\tilde{w}_i^2 \lambda_i}{\sigma_w^2}, \quad i = 1, \ldots, n
$$

with $\tilde{w}_i^2 \lambda_i$ as the variance due to the ith principal portfolio and σ_w^2 as the variance of the portfolio w. The p_i's add up to one and measure the risk that arises from the ith weighted principal portfolio. The portfolio is well diversified when its total variance is not concentrated in a few p_i's, but instead the p_i's are approximately equal. The diversification of a portfolio can then be measured by the entropy based on the diversification distribution:

$$
N_{\mathrm{Ent}} \equiv \exp\left(-\sum_{i=1}^{n} p_i \ln p_i \right).
$$

This entropy measures the number of truly independent sources of risk that are apparent in the portfolio. A higher value of N_{Ent} means a more diversified portfolio, whereas a lower value indicates concentration in only a few independent sources of risk (Frahm and Wiechers 2011, p. 19).

In the style of (Bera and Park 2008), Meucci proposes a new heuristic for asset selection based on maximizing the entropy (Meucci 2009, p. 11):

$$
w_\varphi \equiv \max_{w \in C} N_{\mathrm{Ent}}(w), \tag{4}
$$

where C is a set of investment constraints (e.g. short-sell and budget constraint).

4 The effect of estimation errors on the portfolio composition

Now we study the effect of estimation errors on the portfolio allocations by comparing the performance measures to be introduced in Sect. 4.1. We first conduct a simulation study based on unrealistic simple true parameters of the assets so as to examine the pure effect of the estimation errors on the portfolio allocations when all assets have constant true returns as well as constant true correlations. Next, we examine the portfolio performances in an empirical dataset to observe the real outcome.

4.1 Performance measures

Comparison of robustness is based on the following different performance measures.

To compare the sensitivity of the asset weights to changes in input parameters and the extreme allocations, the minimum, the maximum, and the standard deviation of each portfolio asset weight is computed across the simulation runs.

To measure the distance of the estimated portfolio asset weights to the true portfolio asset weights, we compute the (unnormalized) Euclidean distance of the resulting weight vectors for each simulation run. To quantify the robustness of the portfolio weights, we also compute the standard deviation of the particular distances. As the performance measures are computed for the simulation as well as for the empirical study, the running index k is taken either for the simulation run or the month of the optimization.

These measures are given by

$$\text{Mean distance (MD}_w)$$
$$= \frac{1}{\text{NumSim}} \sqrt{(w_{\text{estimated}}(k) - w_{\text{true}})^{\text{T}}(w_{\text{estimated}}(k) - w_{\text{true}})} \quad (5)$$

$$\text{Standarddeviaton (SD}_w) = \sqrt{\frac{1}{\text{NumSim}} \sum_{k=1}^{\text{NumSim}} (d_k - md)^2}, \quad (6)$$

where d_k is the distance for the optimization in k and md is the mean distance.

To quantify the difference between the estimated portfolio and the actual portfolio, the average difference of the μ- and σ-values is computed. This shows the "large degree to which estimated frontiers are optimistically biased predictors of actual portfolio performance" (Broadie 1993, p. 45). The average differences are given by

$$\text{AV}\mu = \frac{1}{\text{NumSim}} \sum_{k=1}^{\text{NumSim}} (\mu_{\text{estimated}}(k) - \mu_{\text{actual}}(k)) \quad (7)$$

$$\text{AV}\sigma = \frac{1}{\text{NumSim}} \sum_{k=1}^{\text{NumSim}} (\sigma_{\text{estimated}}(k) - \sigma_{\text{actual}}(k)). \quad (8)$$

Generally the AVμ is positive whereas the AVσ is negative as the mean is overestimated and the risk underestimated.

We now examine different appraisal factors applied to the portfolios: the Sharpe ratio, the turnover, the Herfindahl index for the weights and the risk contributions, and the entropy, as well as the value at risk (VaR) and the maximum drawdown (MDD).

The Sharpe ratio and the Herfindahl indices are given by

$$\text{Sharpe ratio} = \frac{1}{\text{NumSim}} \sum_{k=1}^{\text{NumSim}} \frac{\mu_{\text{actual}}(k)}{\sigma_{\text{actual}}(k)} \tag{9}$$

$$h_k = \sum_{k=1}^{\text{NumSim}} w_{\text{estimated},k}^2 \quad \text{or} \quad h_k = \sum_{k=1}^{\text{NumSim}} CR_{\text{actual},k}^2, \tag{10}$$

where w are the estimated weights and CR are the actual percental risk contributions: $CR_i = \frac{\text{ARC}_i}{\sigma_{\text{actual}}}$ and $\sum_{i=1}^{n} CR = 1$.

This index takes the value 1 for a perfectly concentrated portfolio and $1/n$ for a portfolio with uniform weights. To scale the statistics onto [0, 1], we consider the modified Herfindahl index (Maillard et al. 2010, p. 22):

$$\text{Herfindahl}_{w \text{ or rc}} = \frac{1}{\text{NumSim}} \frac{h_k - 1/n}{1 - 1/n}. \tag{11}$$

We also calculate the entropy (see Sect. 3.5) for the actual parameters in order to compare the PCA approach to the other techniques. The 0.95 VaR and the maximum drawdown (defined as the cumulated largest drop from a peak to a trough) for the actual portfolio return are also computed to compare the risk of the different portfolio approaches.

4.2 Simulation study

4.2.1 Structure robustness

To study and compare the effect of the estimation errors on the different portfolio approaches, we perform a simulation study similar to Michaud (1998) and Broadie (1993). It is comprised of predetermination of the true mean returns and true variance–covariance matrix, generation of a sequence of returns in every simulation run that are statistically equivalent to the true returns, subsequent determination of portfolio weights for every simulation run, and, finally, evaluation of robustness.

The specific steps of the simulation study are as follows.

1. Predetermine
   - n the number of securities,
   - μ the vector of true means, i.e., $\mu = (\mu_1, \ldots, \mu_n)$,
   - σ the vector of true standard deviations of returns, i.e., $\sigma = (\sigma_1, \ldots, \sigma_n)$,

- ρ the matrix of true correlations of returns of the securities, i.e., ρ_{ij} is the true correlation of returns of securities j and i, for $j, i = 1, \ldots, n$.

2. Identify the true portfolio composition $w = (w_1, \ldots, w_n)$ from the inputs of (1). The true variance–covariance matrix $\Sigma = \sigma^T \rho \sigma$ is computed.
3. Resample from the inputs of (1) by taking 60 draws from a multivariate normal distribution $\Phi(\mu, \Sigma)$ and estimate new input parameters $\hat{\mu}$ and $\hat{\Sigma}$.
4. Repeat step (3) NumSim times and save the estimated parameters.
5. Identify the portfolio composition $\hat{w} = ((\hat{w}_1), \ldots, (\hat{w}_n))$ with the estimators $\hat{\mu}$ and $\hat{\Sigma}$ from step (4). This results in NumSim different portfolio optimizations. The target return of the efficient portfolio equals the resulting true ERC portfolio return.
6. Compare the robustness of the approaches.

To ensure that every portfolio optimization technique has the same data sample as inputs, we draw the samples and save the estimated parameters. After this, the portfolio optimization for different approaches is performed.

To investigate robustness, at first a simple framework is used, featuring

- 10 assets with
- multivariate normal distributed returns,
- each asset with a return of 1 % and a standard deviation of 8 %,
- a correlation of 0.5 among all assets, and
- NumSim $= 10,000$.

As the naive portfolio is the true asset allocation according to the predetermined dataset, its results are not really comparable. The target return of the efficient portfolio is here 1 %. If it is not possible to attain the efficient portfolio (the target return is too restrictive), the draw is dropped from the analysis. We first focus our analyses on the generated weights of the different portfolio optimization techniques.

For each optimization technique, the average asset weights do not much differ not widely from those of the equally weighted portfolio, in which each asset has a weight of 10 %. This is not surprising due to the assumptions made and therefore is not shown here. However, the spread between the minimum and the maximum asset weights is not the same.

Table 1 shows the minimum and the maximum of the weights across the 10,000 runs for each asset of the portfolio.

As is obvious from Table 1, each approach except the naive and the ERC portfolio provides a minimum asset weight of 0 %. As the naive asset weights do not vary by construction, they can be ignored. The ERC portfolio shows a minimum asset weight of 6 %.

Analysis of the maximum portfolio weights provides evidence that the PCA and the Markowitz efficient portfolio take the largest possible position in each asset. The ERC portfolio shows a maximum weight across all assets of only about 17 %, whereas the MV approach takes a position of about 60 %.

To assess the stability of the asset weights across the 10,000 portfolio formations, we look at the standard deviation of the asset weights. This value is indicative of the redeployment of the portfolio. The higher the variation of the asset weights, the

Table 1 Minimum and maximum portfolio weights for the different portfolios (%)

Asset	Weights (%)	Markowitz	ERC	PCA	MV	1/n
Asset 1	Max	100	16	100	59	10
	Min	0	7	0	0	10
Asset 2	Max	100	16	100	61	10
	Min	0	6	0	0	10
Asset 3	Max	100	16	100	56	10
	Min	0	6	0	0	10
Asset 4	Max	100	16	100	56	10
	Min	0	7	0	0	10
Asset 5	Max	100	17	100	59	10
	Min	0	7	0	0	10
Asset 6	Max	97	16	100	53	10
	Min	0	7	0	0	10
Asset 7	Max	100	17	100	61	10
	Min	0	6	0	0	10
Asset 8	Max	99	19	100	61	10
	Min	0	7	0	0	10
Asset 9	Max	100	16	100	59	10
	Min	0	7	0	0	10
Asset 10	Max	100	16	100	56	10
	Min	0	7	0	0	10
Average	Max	100	17	100	63	10
	Min	0	7	0	0	10

Average minimum and maximum estimated portfolio weights across 10,000 simulation runs per asset. The inputs for the portfolio optimizations in each simulation run are the same and the assets have the same "true" return and volatility. The target return for the efficient portfolio equals the true ERC portfolio return, which is 1 % in this setting

Table 2 Standard deviations of the portfolio weights

Average standard deviations of the estimated weights across 10,000 simulation runs and assets

	σ_{weights} (%)
Markowitz	13.1
ERC	1.1
PCA	29.1
MV	10.1
1/n	0

higher the investor's transaction costs. Thus, this is an important indicator of structure robustness.

Table 2 shows the average standard deviations of the weights across 10,000 simulation runs and assets. The average standard deviations across the simulation runs per asset are given the Appendix. As expected from the spread of the portfolio weights, the same is the case in terms of the standard deviation of the portfolio weights.

Table 3 Performance statistics of the different portfolios for the simulation study

	Markowitz	ERC	PCA	MV	1/n
$\mu_{\text{estimated}}$ (%)	1.000	1.003	0.992	1.000	1.003
μ_{actual} (%)	1.000	1.000	1.000	1.000	1.000
$\sigma_{\text{estimated}}$ (%)	5.890	5.834	7.213	5.519	5.908
σ_{actual} (%)	6.370	5.936	7.890	6.200	5.933
MD_w (%)	39.043	3.429	91.778	31.277	0.000
SD_w (%)	13.854	0.933	6.797	6.104	0.000
AV μ (%)	0.000	0.003	−0.008	0.000	0.003
AV σ (%)	−0.395	−0.102	−0.677	−0.681	−0.025
Sharpe ratio (%)	15.736	16.845	12.687	16.134	16.855
Herfindahl$_w$ (%)	19.069	0.140	94.105	11.283	0.000
Herfindahl$_{rc}$ (%)	21.564	0.168	95.553	12.769	0.000
Turnover (%)	64.018	6.274	89.893	55.151	0.000
VaR (%)	−11.48	−10.76	−13.98	−11.20	−10.76
MDD (%)	0.000	0.000	0.000	0.000	0.000
Entropy	1.644	1.011	3.641	1.522	1.000

The statistics are all given as means across 10,000 portfolio optimizations and are based on actual parameters. For better comparison they are all shown in percentage, except the entropy. The target return for the Markowitz portfolio is 1 %

Surprisingly, we observe the highest standard deviation, of about 29 %, for every asset in PCA portfolios. The Markowitz approaches deliver a standard deviation of about 13 and 10 %. Despite the 1/n portfolio, which by construction delivers a standard deviation of 0 %, the ERC portfolio has the smallest standard deviation of approximately 1 %.

Table 3 displays the performance measures for comparison of portfolio robustness as well as performance statistics.

The estimation error ($AV \mu$) of the returns is near 0 % for all approaches, which is not surprising in this special setting. The difference is more obvious between the estimated and the actual risk ($AV\sigma$). The MV and the PCA approach underestimate the risk by about 0.7 % and the Markowitz approach does the same by about 0.4. Additionally, the PCA approach has a much higher volatility.

The distance measures (MD_w and SD_w) show evidence that the PCA portfolio has the highest deviation from the true portfolio weights, followed by the efficient and the MV portfolio. The ERC portfolio shows the smallest distance on average and the smallest standard deviation of the distance (except for the naive portfolio, by construction).

The mean Sharpe ratio (based on actual parameters) is the highest for the naive approach, closely followed by the ERC approach. The PCA approach exhibits a poor Sharpe ratio and alarmingly poor weight and risk diversification, as well as a huge turnover. The ERC portfolio is highly diversified in terms of weights, as well as in terms of risk contributions (which is related in this dataset). Notice that although the MV approach is designed to minimize portfolio risk, it is more concentrated in terms

of risk contributions compared to the ERC approach. As to the VaR, the PCA has the highest value, followed by the Markowitz approaches.

4.2.2 Worst-case performance

To compare the performance of the different approaches in a worst-case scenario, we modify the simulation described above so that only undesirable input values are obtained:

1. Same as in Sect. 4.2.1.
2. Resample from the inputs of (1) by taking 60 draws from a multivariate normal distribution $\Phi(\mu, \Sigma)$.
3. Generate a bootstrap sample by taking 1,000 data samples (each data sample has 60 returns for each asset) from the data in step (2) and define the array $\mu =$ mean (Bootstrap sample returns) (dimension: $1,000 x n$ matrix) and the array of covariance matrices $\Sigma =$ cov (Bootstrap sample returns) (dimension: $n x n x 1,000$ matrix).
4. Estimate the best mean $= \max(\mu)$ and the worst mean $= \min(\mu)$ from the mean's in step (3).
5. Estimate the best variance–covariance matrix through taking the lowest entry for each position in the matrix from all available matrices from step (3) and the worst variance–covariance matrix through taking the highest entry.
6. Repeat steps (2), (3), (4) and (5) NumSim times and save the input parameters.
7. Perform the portfolio optimizations with the input parameters from step (6) by taking the worst means and the worst variance–covariances as the true and the best means and the best variance–covariances as the estimated input parameters.
8. Compare the performance (worst-case robustness) of the approaches as described above.

This setting is constructed to examine a worst case in which the market is in a bull phase and the portfolio is estimated with the best parameters and then the situation changes to a bear market and the true parameters are the worst.

Table 4 summarizes the results for the worst-case simulation.

The actual mean portfolio returns in the worst case are negative and much lower than the estimated portfolio returns (by construction) for all the approaches and there is no real difference between the actual returns. However, the actual return of the ERC portfolio is the lowest after the naive approach. The volatility is overestimated by the PCA and slightly by the efficient portfolio, whereas it is underestimated by the other approaches. The estimation error of the portfolio risk is the largest for the PCA and the MV portfolios.

The mean, as well as the standard deviation, of the distance again demonstrate that the ERC portfolio displays the best stability in terms of weights and is thus structure robust.

The Sharpe ratio is negative and thus not informative (in this case, a large negative portfolio return with a large portfolio standard deviation results in a better Sharpe ratio than only a slightly negative portfolio return with a small portfolio standard deviation (see, e.g., Brinkmann 2007, p. 185). The Herfindahl index shows that the ERC portfolio

Table 4 Performance statistics for the worst case simulation

	Efficient	ERC	PCA	MV	Naive
$\mu_{estimated}$ (%)	4.279	4.279	4.221	4.221	4.308
μ_{actual} (%)	−2.215	−2.263	−2.198	−2.205	−2.291
$\sigma_{estimated}$ (%)	8.431	6.850	10.562	6.617	7.046
σ_{actual} (%)	8.262	8.075	8.581	7.985	8.176
MD_w (%)	44.507	6.657	96.453	34.281	0.000
SD_w (%)	17.068	3.337	30.788	10.075	0.000
AV μ (%)	5.574	6.542	6.419	6.425	6.599
AV σ (%)	0.145	−1.225	1.981	−1.368	−1.131
Sharpe ratio (%)	−26.685	−28.051	−25.711	−27.647	−28.047
Herfindahl$_w$ (%)	15.813	0.815	37.543	8.414	0.000
Herfindahl$_{rc}$ (%)	17.014	0.703	41.007	8.561	0.134
Turnover (%)	59.524	14.237	78.280	48.181	0.000
VaR (%)	−11.374	−11.020	−11.916	−10.929	−11.158
MDD (%)	−19,009.030	−22,625.549	−21,982.066	−22,045.837	−22,906.933
Entropy	1.409	1.055	2.102	1.295	1.009

The statistics are given as means across 10,000 portfolio optimizations and are based on actual parameters. The worst case is seen here as a situation, in which the portfolio is estimated with the "best" parameters, but the "worst" parameters are the "true" parameters. The target return for the Markowitz portfolio is equal to the estimated ERC portfolio return

is again diversified in terms of weights (except the naive portfolio) as well as in terms of risk contributions. However, the naive portfolio has the lowest Herfindahl index based on the actual parameters. The PCA portfolio delivers the worst diversification, the highest turnover, and the highest VaR. However, concerning the drawdowns, the naive portfolio displays the largest drop, followed by the ERC, the MV, and the PCA approaches.

4.2.3 Results of the simulation study

The results show that the PCA portfolio delivers the worst results when it comes to robustness. It displays large changes in asset weights, extreme positions, and large estimation errors of portfolio returns and risks in both cases. We suspect that this is attributable to the statistical concept of the PCA itself, as the principal components have to be estimated before portfolio optimization, which results in an additional estimation error. As the PCA portfolio is based on all principal components, but the components are decreasingly responsible for the randomness in the market, change in the input parameters induces a drastic change in the last factor and thus in the asset weights. Based on these assumptions, it would be useful to examine the robustness of a portfolio where the allocation is based on the PCA, but the weight allocation is proportional to the explanation of the factors. This would perhaps reduce the large redeployment of the weights.

The ERC portfolio displays stability in regard to the standard deviation as well as diversification in terms of weights and low estimation errors in the simulation study. However, in the worst case, it provides worse outcomes than the Markowitz approaches and is thus not worst-case robust. We examine in the next section how the portfolios perform under real circumstances by means of a simulation based on empirical, not artificial, data.

4.3 Empirical study

We now examine the robustness of the approaches in an empirical dataset so as to give some practical application to our results.

We perform a historical simulation based on a global diversified portfolio of major asset classes in the period from 31.01.1995–30.07.2010 by using daily returns and rebalancing the portfolio monthly. The first estimated parameters are the mean and the covariance matrix, based on the first 12-month daily returns (31.01.1995–31.01.1996). We construct the true portfolio parameters by using the return vector and the true covariance matrix from the following month. By rolling the dataset 1 month further, we compare the robustness of the approaches across 174 months. We take a global dataset from (Maillard et al. 2010), which provides variation in the asset volatilities and correlations, to examine differences between the portfolio techniques. As described above, in a portfolio with quite similar risks, there is no significant difference between the ERC and the naive approach, but if the risks differ, the differences are obvious. Therefore, we examine the portfolio optimizations in a dataset with diverse asset risks.

As a robustness check, we also examine the case when the market is assumed to be in an equilibrium and the true returns as well as the true covariance matrices are estimated by applying the capital asset pricing model (CAPM) from Sharpe and the Black Litterman implied returns. To estimate the CAPM, the World-Datastream Market Index is taken as the "Market Index" because the portfolio consists of different major indices, and the 3-month T-Bill rate is taken as the risk-free rate. The results are quite similiar and can be found in the Appendix.

4.3.1 Structure robustness

The correlation matrix across the whole sample of 174 months is set out in Table 5. The individual risks, as well as the correlations of the asset classes, are very different.

The average maximum and minimum estimated asset weights across the 174 optimizations across all assets are summarized in Table 6. (The results for every asset in detail are given in the Appendix.) We observe substantial diversity across the different portfolio approaches. For example, the PCA portfolio takes large extreme positions in almost every asset, whereas the ERC portfolio is much more diversified. The ERC portfolio exhibits the highest weights in bonds due to lower volatilities and correlations. The MV approach also focuses on bonds, but to a much greater degree than the ERC approach. The efficient portfolio takes middle-high positions in every asset by showing the highest weights also in the two US bonds and the emerging market bond.

Table 5 Annualized returns, annualized volatilities and correlations of the assets in the empirical study

	Return (%)	Volatility (%)	Correlation matrix (%)													
SPX	3.3	20.5	100	86.6	52.8	48.9	10.6	61.3	31.1	18.5	−22.3	−4.8	21	27.8	16.6
RTY	4.5	23.7		100	49.3	44.5	10	57.6	32.2	20.2	−23.9	−5.3	20.2	23.1	17.7
EUR	3.2	23.8			100	86.4	27.5	55.1	51.7	37.6	−24.7	−13.4	34	28.8	19.5
GBP	1.8	19.7				100	28.2	55.5	52.8	38.4	−22.3	−8.4	35.7	29	22.1
JPY	−4.2	21.6					100	22.1	32.5	56	−6.5	0.4	31.7	16.7	12.2
MSCI-LA	10.3	29.4						100	50.2	35.8	−19.1	4.4	30.6	57.9	26.9
MSCI-EME	8.5	31.1							100	48	−17.7	18.8	36	30.3	26.5
ASIA	1.2	22.6								100	−11.7	2.8	34.8	22.8	16.2
EUR-BND	0.8	5									100	25.1	4.8	9.8	−12
USD-BND	5.7	10.2										100	6.4	7.4	12.7
USD-HY	6.5	4.6											100	32.9	14.1
EMBI	10.6	9.1												100	11.3
GSCI	2.9	23.5													100

Names (codes) of the assets are as follows: S&P 500 (SPX), Russell 2000 (RTY), DJ Euro Stoxx 50 (EUR), FTSE 100 (GBP), Topix (JPY), MSCI Latin America (MSCI-LA), MSCI Emerging Markets Europe (MSCI-EME), MSCI AC Asia ex Japan (ASIA), JP Morgan Global Govt Bond Euro (EUR-BND), JP Morgan Govt Bond US (USD-BND), ML US High Yield Master (USD-HY), JP Morgan EMBI Diversified (EMBI), S&P GSCI (GSCI)

Table 6 Minimum and maximum portfolio weights for the different portfolios in the global dataset

Asset	Weights (%)	Markowitz	ERC	PCA	MV	1/n
Average	Max	35	15	66	21	8
	Min	0	3	0	0	8

Table 7 Average standard deviations of the portfolio weights across 174 months and assets

	Average σ_w (%)
Markowitz	7.31
ERC	2.85
PCA	12.31
MV	5.46
1/n	0

The standard deviations per asset are given in the Appendix

Table 7 displays the average standard deviations of the asset weights across all months and assets (see Table 13 in the Appendix for standard deviations of every asset). The deviations show a large variability of the PCA portfolio in almost every asset. The ERC portfolio exhibits in the USD-BND the sole high standard deviation of 12 %, whereas the MV approach displays approximately 20 and 27 % in the US bonds. The box plot graphs in Figs. 1 and 2 (in the Appendix) show the distribution of the estimated weights and risk contributions. The highest variability is observable for bonds.

Results of the solution robustness as well as of the historical performance are summarized in Table 8.

The estimation error in the mean across the whole sample period (AVμ) is the largest for the Markowitz and the MV approaches, at 0.011 and 0.009 %, respectively. The ERC portfolio has a lower estimation error of 0.007 % and the naive and the PCA approaches have the smallest estimation error of the returns at 0.004 %. The estimated volatility is lower than the actual volatility for the PCA and the MV approaches. The ERC and the MV approaches estimate the actual portfolio risk the best; however, the ERC slightly overestimates and the MV slightly underestimates the risk. The PCA approach underestimates the actual portfolio risk the most (by 0.013 %) and the naive approach highly overestimates the actual risk (by 0.030 %).

The distances between the true and the estimated weights, as well as the variation of them, are the largest for the PCA portfolio. The ERC portfolio exhibits the lowest distance statistics of all the approaches (except the naive approach by construction). According to the Herfindahl indices, the ERC and the naive approach are both diversified in terms of risk, whereas the naive approach is the most diversified in terms of weights. Note that we computed the Herfindahl indices on the actual parameters and therefore the ERC portfolio does not have a value of 0 %. Based on the estimated parameters, the ERC portfolio has a Herfindah lrc of 0 %, whereas the naive approach displays 4.49 %. Surprisingly, the MV is most concentrated in terms of weights as well as in terms of individual risks. The turnover is the highest for the PCA approach, followed by the efficient portfolio. The drawdowns show the worst performance for

Table 8 Performance statistics of the global portfolio

	Markowitz	ERC	PCA	MV	1/n
$\mu_{estimated}$ (%)	0.023	0.023	0.025	0.024	0.020
μ_{actual} (%)	0.011	0.016	0.021	0.015	0.017
$\sigma_{estimated}$ (%)	0.191	0.275	0.468	0.168	0.649
σ_{actual} (%)	0.188	0.273	0.481	0.169	0.618
MD_w (%)	45.851	17.505	69.891	34.675	0.000
SD_w (%)	23.827	7.899	28.510	23.273	0.000
AV μ (%)	0.011	0.007	0.004	0.009	0.004
AV σ (%)	0.003	0.002	−0.013	−0.002	0.030
Sharpe ratio (%)	12.706	11.070	6.888	20.459	9.539
Herfindahl$_w$ (%)	40.091	10.314	37.628	53.518	0.000
Herfindahl$_{rc}$ (%)	44.559	5.860	45.714	55.839	5.870
Turnover (%)	13.706	2.448	41.792	4.671	0.000
VaR (%)	−9.025	−11.145	−12.974	−11.515	−21.478
MDD (%)	−19.406	−22.295	−27.261	−18.954	−58.571
Entropy	1.000	4.027	4.902	6.718	1.759
μ_{annual} (%)	2.745	3.932	5.357	3.763	4.143
$\mu_{cumulative}$ (%)	38.173	58.617	83.680	56.598	60.378
σ_{annual} (%)	2.965	4.317	7.605	2.674	9.775
Sharperatio$_{annual}$ (%)	92.592	91.060	70.441	140.703	42.388

The statistics are given as means across 174 months of portfolio optimizations, which were rebalanced monthly. For better comparison they are all shown in percentage, except the entropy. The annual (and cumulative) return, standard deviation and Sharpe ratio are based on the actual parameters and are calculated as follows: $\mu_{annual} = \mu_{actual} \times 250$, $\sigma_{annual} = \sigma_{actual} \times \sqrt{250}$, Sharperatio$_{annual} = \frac{\mu_{annual}}{\sigma_{annual}}$, as we assume 250 trading days in a year

the naive approach, followed by the PCA and the ERC approaches. The mean entropy has the highest value, 7, for the MV and not for the PCA approach (again due to actual parameters!), whereas the PCA and the ERC approaches display an entropy of approximately 5 and 4, respectively. With respect to the entropy, the Markowitz approach has the lowest diversification of uncorrelated risks in the portfolio, followed by the naive approach.

According to the performance statistics, the ERC and the Markowitz approaches have a similar average Sharpe ratio; however, the Markowitz approach has a higher Sharpe ratio due to lower actual return as well as a lower actual risk (the relation of the annual Sharpe ratio is different, as it is calculated simply through the annualized values and not as the average of all months as is the mean Sharpe ratio). The PCA portfolio demonstrates the highest cumulative return and the MV the highest Sharpe ratio. The naive approach has the rhighest annualized volatility, which results in a low Sharpe Ratio. The PCA displays the highest annualized return of the approaches, but also a high annualized volatility. The ERC and the MV approaches have comparable cumulative returns; however, the ERC approach has a higher annualized return, and the MV approach the lowest annual standard deviation, which results in a high annual Sharpe ratio.

4.3.2 Worst-case performance

The worst-case scenario for the empirical study is performed on the basis of the recent financial crisis. We choose the dataset from 30.05.2008–30.04.2009, which results in 12 months of data. The crisis period is variously defined in the literature, but the peak of the crisis is definitely included in our time frame (see e.g. Chong 2011, Campello et al. 2010). The estimated and the true parameters are obtained as above.

The annualized returns are all negative and the volatilities, as well as the correlations, are all higher than in the whole data sample (see Table 9). This is typical for a crisis period.

According to the estimated weights, all portfolios are concentrated on the bond side and hold the highest positions in the USD-BND and the USD-HY. The target return for the efficient portfolio equals -0.015 %, which is attained by the MV portfolio.

All approaches overestimate the return and underestimate the risk. The PCA approach has the largest estimation errors in the portfolio outcomes. The ERC portfolio has a lower estimation error than the MV portfolio in the mean, as well as a slightly lower estimation error in the portfolio risk. The naive approach is the most diversified in terms of risk contributions in the crisis, but as one can see from the annual volatility and the cumulative return, this is not useful for the performance, as the cumulative return and the volatility are the poorest. Moreover, this approach exhibits a huge drawdown compared to all the others. Moreover, the distances show the worst result for the PCA portfolio. We do not display the extreme positions and the standard deviations for the worst case explicitly, but the PCA approach also displays the largest variability. According to the cumulative return and the drawdown, the ERC approach is worse than the MV approach. The entropy of the MV approach based on actual parameters is again higher than that of the PCA approach (Table 10).

As no one would hold a portfolio if the ex ante estimated return is expected to be negative, we also constructed a "hypothetical" worst case to examine the estimation error when the estimated parameters are positive, but the true parameters are negative and the market "crashes" (sudden, unexpected decline of the prices). For this, we chose the highest estimated returns across all estimated parameters and the lowest true returns across all true parameters and assume that the worst true parameters follow the best estimated parameters. Thus, we shifted the historical time periods and the simulation is based on 17 months.

Table 11 shows the result for the hypothetical, but possible, market crash:

The actual portfolio return of the risk-based asset allocations is lower than the actual returns of the Markowitz approaches and the actual risk of the ERC and PCA portfolio is also higher than that of the efficient and the MV approaches. The estimation errors of the return and the risk are also higher than for the mean-variance approaches (except the estimation error of the ERC portfolio for the actual portfolio risk). Concerning the cumulative return and the drawdowns, the ERC and the PCA display the lowest return and the highest drawdowns after the naive approach.

So when the market crashes, the ERC and the PCA approaches display worse performance statistics and a higher loss than do the mean-variance approaches.

Table 9 Annualized returns, annualized volatilities and correlations of the assets in the crisis period

	Return (%)	Volatility (%)	Correlation matrix (%)												
SPX	−38.8	45.6	100	93.6	61.8	56.8	12.8	73.5	44	32.3	−38.9	5.6	24	34.5	40.6
RTY	−35.8	53.5		100	51.2	45.1	3.8	65.4	38.7	25.4	−34.7	5.3	19.9	26.9	33.5
EUR	−37.9	42.1			100	94.4	43.9	79	70.8	56.1	−35.7	24.3	50.2	55.9	51.1
GBP	−31.1	39.4				100	44.8	77	71.5	54.7	−36.4	22.9	52.3	56.8	53.6
JPY	−40.6	41.5					100	38.4	56.5	74	−7.4	29	54.6	57.2	24.9
MSCI-LA	−51.6	62.6						100	70.3	56.1	−40.5	24.5	43.7	59.3	58.4
MSCI-EME	−65.1	68.6							100	66.6	−37.1	39.5	46.8	60.3	50.2
ASIA	−39.8	42.1								100	−18.6	26.5	52.5	56.9	36.1
EUR-BND	5.4	7.7									100	12.8	−14.3	−13.8	−36.2
USD-BND	−7	16.7										100	14.5	31	25.3
USD-HY	−15.6	12.6											100	64.8	28.9
EMBI	−5.2	14.7												100	38.1
GSCI	−64.7	45.3													100

Names (codes) of the assets are as follows: S&P 500 (SPX), Russell 2000 (RTY), DJ Euro Stoxx 50 (EUR), FTSE 100 (GBP), Topix (JPY), MSCI Latin America (MSCI-LA), MSCI Emerging Markets Europe (MSCI-EME), MSCI AC Asia ex Japan (ASIA), JP Morgan Global Govt Bond Euro (EUR-BND), JP Morgan Govt Bond US (USD-BND), ML US High Yield Master (USD-HY), JP Morgan EMBI Diversified (EMBI), S&P GSCI (GSCI)

Table 10 Performance statistics of the global portfolio in the "worst case"

	Markowitz	ERC	PCA	MV	1/n
$\mu_{\text{estimated}}$ (%)	−0.012	−0.029	−0.013	−0.012	−0.111
μ_{actual} (%)	−0.049	−0.054	−0.073	−0.049	−0.136
$\sigma_{\text{estimated}}$ (%)	0.261	0.397	0.512	0.261	1.221
σ_{actual} (%)	0.371	0.502	0.757	0.371	1.466
MD_w (%)	42.283	18.877	62.352	42.283	0.000
SD_w (%)	19.700	9.977	44.214	19.700	0.000
AV μ (%)	0.037	0.025	0.060	0.037	0.025
AV σ (%)	−0.109	−0.106	−0.245	−0.109	−0.246
Sharpe ratio (%)	−4.677	−10.199	−6.113	−4.677	−9.035
Herfindahl$_w$ (%)	38.902	21.176	48.851	38.902	0.000
Herfindahl$_{rc}$ (%)	49.245	5.732	60.553	49.245	4.669
Turnover (%)	10.135	3.498	49.489	10.135	0.000
VaR (%)	−11.515	−11.145	−12.974	−11.515	−21.478
MDD (%)	−17.602	−20.370	−21.505	−17.602	−53.031
Entropy	6.615	3.620	5.272	6.615	1.338
μ_{annual} (%)	−12.208	−13.523	−18.369	−12.208	−33.885
$\mu_{\text{cumulative}}$ (%)	−13.070	−14.072	−18.532	−13.070	−34.611
σ_{annual} (%)	5.860	7.945	11.964	5.860	23.187
Sharperatio$_{\text{annual}}$ (%)	−208.334	−170.204	−153.540	−208.334	−146.138

The statistics are given as means across 12 months of portfolio optimizations (30.05.2008–30.04.2009), which were rebalanced monthly. For better comparison, they are all shown in percentage, except the entropy

4.3.3 Results of the empirical study

In the empirical case, also, the PCA portfolio shows the worst results for structure and the solution robustness. The effects of the estimation errors have an even larger influence on the portfolio allocation and performance than in the case of the Markowitz portfolios, which are known to be unstable. The PCA portfolio also has a poor outcome in regard to worst-case robustness, and thus is not worst-case robust.

The ERC portfolio is much more structure and solution robust than the efficient and the MV portfolios due to the lower turnover and estimation errors and it is also much more diversified both in terms of weights and in terms of risk contributions. Surprisingly, this does not help in terms of portfolio performance as it cannot outperform the Markowitz approaches when it comes to cumulative return, the Sharpe ratio, or the drawdown. In the crisis period it displays a better performance than the naive portfolio, but the MV portfolio dominates it regarding the resulting return, risk, and drawdown.

Concerning the hypothetical worst case, i.e., the market crashes, the risk-based asset allocations are even worse than those of the Markowitz and the MV approaches and cannot prevent loss, thus failing to perform as designed. In this case, they both display higher estimation errors and drawdowns and lower cumulative returns.

Table 11 Performance statistics of the global portfolio in the "hypothetical worst case"

	Markowitz	ERC	PCA	MV	1/n
$\mu_{\text{estimated}}$ (%)	0.059	0.059	0.063	0.052	0.106
μ_{actual} (%)	−0.056	−0.104	−0.096	−0.043	−0.234
$\sigma_{\text{estimated}}$ (%)	0.206	0.284	0.481	0.172	0.665
σ_{actual} (%)	0.256	0.324	0.538	0.223	0.682
MD_w (%)	61.887	19.361	72.209	42.975	0.000
SD_w (%)	22.778	8.493	23.569	18.741	0.000
AV μ (%)	0.115	0.163	0.159	0.095	0.339
AV σ (%)	−0.050	−0.040	−0.057	−0.051	−0.017
Sharpe ratio (%)	−21.166	−29.622	−15.028	−16.362	−33.476
Herfindahl_w (%)	40.365	10.783	40.902	52.455	0.000
$\text{Herfindahl}_{\text{rc}}$ (%)	42.099	6.314	49.450	59.420	6.774
Turnover (%)	18.594	4.986	43.682	11.621	0.000
VaR (%)	−3.238	−9.659	−8.670	−3.851	−16.622
MDD (%)	−19.741	−35.232	−31.670	−13.941	−76.749
Entropy	5.911	3.990	5.057	6.761	1.743
μ_{annual} (%)	−14.062	−25.938	−24.051	−10.661	−58.461
$\mu_{\text{cumulative}}$ (%)	−20.596	−36.209	−33.336	−12.444	−80.250
σ_{annual} (%)	4.047	5.128	8.513	3.519	10.786
$\text{Sharperatio}_{\text{annual}}$ (%)	−347.467	−505.833	−282.522	−302.955	−542.014

The statistics are given as means across the portfolio optimizations, which were rebalanced monthly. For better comparison, they are all shown in percentage, except the entropy. The hypothetical worst case is seen here as a situation, where the market is in a bull period and high positive returns are expected, but then the market turns suddenly in a bear period with high negative returns

4.3.4 Robustness check

We estimated the "true" returns in the empirical study through the sample mean and the covariance matrix in the month following portfolio optimization. There are, of course, many different ways of estimating "true" returns. To verify our results, the true returns were estimated by using the CAPM market returns and the Black Litterman implied returns.

The CAPM returns were estimated as follows:

$$r_i = \alpha_i + r_f + \beta_i(r_M - r_f) + \epsilon_i,$$

where r_i is the return of the asset i, r_f the risk free rate and ϵ_i the residual. The true covariance was calculated as the covariance of the residuals. The World-Datastream Market Index is taken as the market return and the 3-month T-Bill rate is taken as the risk-free rate.

The Black Litterman implied returns were estimated as follows:

$$\Pi = \lambda \Sigma w_{\text{market}},$$

where Π is the vector of the is the implied returns, λ is the risk aversion parameter, Σ the covariance matrix and w_{market} the market weights.

As the portfolio consists of global diversified indices and the market capitalizations are not available for every index, the market weights are taken as the naive weights.

Results show that the relations and the main findings remain the same, no matter what model is used for estimating the returns. Detailed results can be found in the Appendix.

5 Conclusion

In this paper we study the robustness of two recently proposed approaches to risk-based portfolio optimizations: equal-risk contribution and PCA portfolios. Both focus on the diversification of risk. The ERC portfolio concentrates on the individual risk contributions of the assets; the PCA portfolio determines uncorrelated sources of risk through a PCA.

Through a simulation and an empirical study we show that the outcomes of the ERC portfolio are far less influenced by the estimation errors than are the efficient Markowitz and the MV portfolios when it comes to structure and solution robustness. Regarding worst-case performance, the two Markowitz portfolios perform better with respect to the estimation error and the maximum drawdown. However, the performance of the portfolio in terms of actual return and actual risk remain worse than the Markowitz approaches.

The PCA portfolio delivers the worst results in all cases through dramatically changing parameters and thus is far from being robust. Moreover, the poor performance of the PCA portfolio shows that it is not useful for maximizing entropy based on uncorrelated risk sources so as to improve risk diversification.

Although the objective of the ERC and PCA portfolios is to improve upon the poor diversification of the Markowitz portfolios, both fail to lower loss, especially during periods of crisis (worst case). Surprisingly, the PCA approach delivers the worst performance in all circumstances despite its intuitive attractiveness. We assume that this is due to the estimation of the principal components so that the PCA approach suffers from a double estimation error problem by relying on all principal components. ERC portfolios might be an alternative to MV portfolios; but they are by no means a universal remedy to the estimation error problem in portfolio optimization. It would be interesting to extended this study by including more sophisticated estimation techniques for the covariance (e.g. robust estimation or GARCH models, see e.g. Jochum 1998; Pojarlev and Polasek 2003). It appears that a better solution to portfolio optimization still awaits discovery.

6 Appendix

6.1 Detailed table (Tables 12, 13, 14) and figures (Figs. 1, 2) for the global portfolio dataset

Table 12 Standard deviations of the portfolio weights in % per asset

σ_{weights}	A 1	A 2	A 3	A 4	A 5	A 6	A 7	A 8	A 9	A 10
Markowitz	12.9	13	13.5	13	12.9	12.9	13.6	13	13.1	13.2
ERC	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
PCA	29.2	28.9	29.6	29.2	29	28.5	29.3	28.7	28.9	29.7
MV	10.1	10	10.1	10	10.2	10.1	10.1	9.9	10	10.3
1/n	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Average standard deviations of the estimated weights across 10,000 simulation runs per asset (see Sect. 4.2.1)

Table 13 Minimum and maximum estimated weights across the 174 empirical portfolio optimizations in % per asset

Asset	Weights (%)	Markowitz	ERC	PCA	MV	1/n
SPX	Max	9	7	81	6	8
	Min	0	2	0	0	8
RTY	Max	5	6	100	3	8
	Min	0	2	0	0	8
EUR	Max	6	6	79	3	8
	Min	0	2	0	0	8
GBP	Max	10	7	47	5	8
	Min	0	2	0	0	8
JPY	Max	8	8	88	4	8
	Min	0	2	0	0	8
MSCI-LA	Max	2	4	27	1	8
	Min	0	1	0	0	8
MSCI-EME	Max	5	9	55	5	8
	Min	0	1	0	0	8
ASIA	Max	9	8	59	2	8
	Min	0	2	0	0	8
USD-BND	Max	90	57	81	72	8
	Min	0	11	0	0	8
EUR-BND	Max	39	21	90	15	8
	Min	0	5	0	0	8
USD-HY	Max	94	38	38	98	8
	Min	0	9	0	11	8
EMBI	Max	83	21	51	59	8
	Min	0	2	0	0	8
GSCI	Max	26	9	62	4	8
	Min	0	2	0	0	8
Average	Max	30	15	66	21	8
	Min	0	3	0	0	8

Table 14 Standard deviations of the estimated portfolio weights across 174 months in % per asset

$\sigma_{weights}$	SPX	RTY	EUR	GBP	JPY	MSCI-LA	MSCI-EME	ASIA	USD-BND	EUR-BND	USD-HY	EMBI	GSCI
Markowitz	2.3	0.8	1.1	2.1	2.0	0.4	1.1	2.3	20.6	9.7	28.4	19.8	4.8
ERC	1.1	0.9	1.2	1.5	1.1	0.7	1.8	1.3	11.5	4.3	6.3	3.4	1.6
PCA	11.6	19.2	7.0	5.7	15.0	4.2	11.0	7.3	30.2	23.6	6.1	7.7	12.1
MV	1.5	0.7	0.5	1.2	0.8	0.2	0.9	0.5	19.7	4.2	26.7	11.7	1.0
1/n	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

The standard deviations of the estimated weights show the highest variability of the weights in the asset class of bonds. The PCA approach displays a high standard deviation in almost every asset
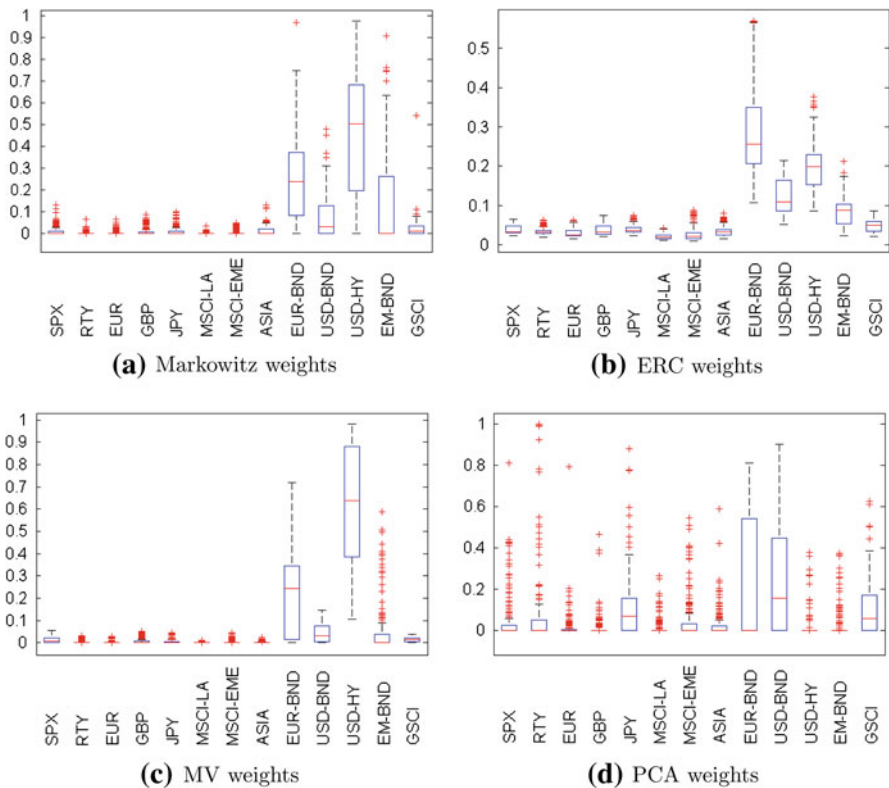
Fig. 1 *Boxplot* of the estimated weights for the empirical study (174 portfolio optimizations)

6.2 Results for the CAPM estimation of the "true" returns (Tables 15, 16)

The true returns are estimated by using the following formula:

$$r_i = \alpha_i + r_f + \beta_i (r_M - r_f) + \epsilon_i,$$

where r_i is the return of the asset i, r_f the risk free rate and ϵ_i the residual. The true covariance was calculated as the covariance of the residuals. The World-Datastream Market Index is taken as the market return and the 3-month T-Bill rate is taken as the risk free rate.

6.3 Results for the Black Litterman estimation of the "true" returns (Tables 17, 18)

The true returns are estimated by using the following formula:

$$\Pi = \lambda \Sigma w_{\text{market}},$$

(a) Markowitz risk contributions

(b) ERC risk contributions

(c) MV risk contributions

(d) PCA risk contributions

(e) naive risk contributions
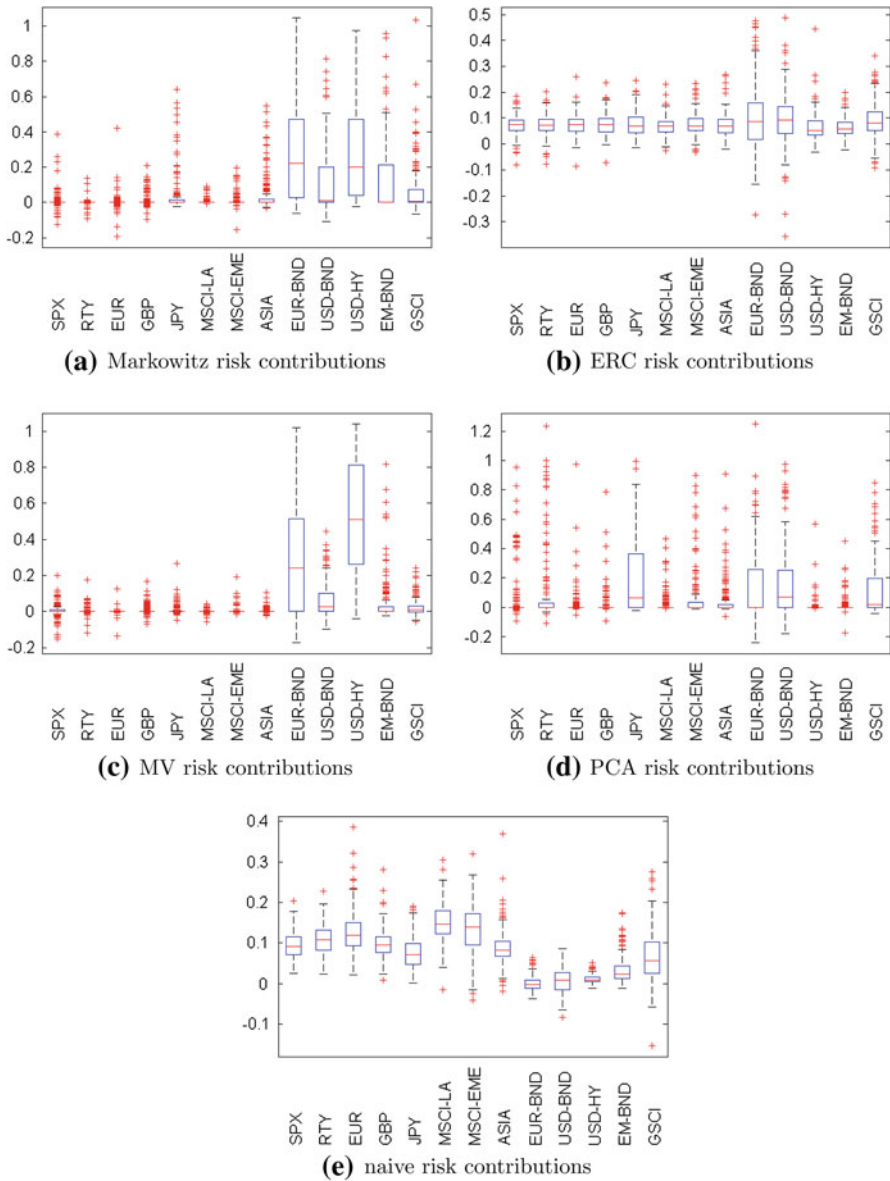
Fig. 2 *Boxplot* of the actual risk contributions for the empirical study (174 portfolio optimizations)

where Π is the vector of the is the implied returns, λ is the risk aversion parameter, Σ the covariance matrix and w_{market} the market weights. As the portfolio consists of global diversified indices and the market capitalizations are not available for every Index, the market weights are taken as the naive weights.

Table 15 Performance statistics of the global portfolio, when the "true" returns are estimated by using the CAPM

	Markowitz	ERC	PCA	MV	1/n
$\mu_{estimated}$ (%)	0.023	0.023	0.025	0.024	0.020
μ_{actual} (%)	0.012	0.015	0.012	0.020	0.016
$\sigma_{estimated}$ (%)	0.191	0.275	0.467	0.168	0.649
σ_{actual} (%)	0.706	0.591	0.720	0.725	0.336
MD_w (%)	83.963	41.883	68.463	85.308	0.000
SD_w (%)	18.745	9.580	14.270	13.344	0.000
AV μ (%)	0.010	0.007	0.013	0.005	0.005
AV σ (%)	−0.486	−0.316	−0.252	−0.558	0.312
Sharpe ratio (%)	7.243	9.596	6.358	7.824	14.740
Herfindahl$_w$ (%)	40.091	10.318	37.634	53.518	0.000
Herfindahl$_{rc}$ (%)	45.981	25.203	58.840	56.657	13.492
Turnover (%)	13.706	2.455	41.479	4.671	0.000
VaR (%)	−9.025	−11.145	−12.974	−11.515	−21.478
MDD (%)	−19.406	−22.295	−27.261	−18.954	−58.571
Entropy	1.000	3.140	3.213	3.671	3.442
μ_{annual} (%)	3.062	3.793	3.088	4.857	3.953
$\mu_{cumulative}$ (%)	38.173	58.617	83.680	56.598	60.378
σ_{annual} (%)	11.160	9.342	11.384	11.468	5.318
Sharperatio$_{annual}$ (%)	27.438	40.602	27.129	42.352	74.336

The statistics are given as means across 174 months of portfolio optimizations, which were rebalanced monthly. For better comparison, they are all shown in percentage, except the entropy

Table 16 Performance statistics of the global portfolio in the empirical "worst case", when the "true" returns are estimated by using the CAPM

	Markowitz	ERC	PCA	MV	1/n
$\mu_{estimated}$ (%)	−0.012	−0.029	−0.013	−0.012	−0.111
μ_{actual} (%)	−0.139	−0.147	−0.155	−0.139	−0.161
$\sigma_{estimated}$ (%)	0.261	0.397	0.512	0.261	1.221
σ_{actual} (%)	1.346	1.140	1.267	1.346	0.456
MD_w (%)	74.760	53.287	78.196	74.760	0.000
SD_w (%)	9.343	8.011	10.968	9.343	0.000
AV μ (%)	0.127	0.117	0.142	0.127	0.050
AV σ (%)	−1.085	−0.743	−0.755	−1.085	−0.765
Sharpe ratio (%)	−10.128	−11.720	−10.268	−10.128	−20.684
Herfindahl$_w$ (%)	38.902	21.196	48.850	38.902	0.000
Herfindahl$_{rc}$ (%)	49.244	47.575	73.015	49.244	23.505
Turnover (%)	10.135	3.516	49.490	10.135	0.000
VaR (%)	−11.515	−11.145	−12.974	−11.515	−21.478
MDD (%)	−17.602	−20.370	−21.505	−17.602	−53.031
Entropy	3.697	3.612	3.850	3.697	4.400

Table 16 continued

	Markowitz	ERC	PCA	MV	1/n
μ_{annual} (%)	−34.804	−36.659	−38.808	−34.804	−40.307
$\mu_{cumulative}$ (%)	−13.070	−14.072	−18.532	−13.070	−34.611
σ_{annual} (%)	21.283	18.018	20.028	21.283	7.202
Sharperatio$_{annual}$ (%)	−163.526	−203.459	−193.771	−163.526	−559.659

The statistics are given as means across 12 months of portfolio optimizations (30.05.2008–30.04.2009), which were rebalanced monthly. For better comparison, they are all shown in percentage, except the entropy

Table 17 Performance statistics of the global portfolio, when the "true" returns are estimated by using the Black Litterman implied returns

	Markowitz	ERC	PCA	MV	1/n
$\mu_{estimated}$ (%)	0.023	0.023	0.025	0.024	0.020
μ_{actual} (%)	0.001	0.005	0.006	0.002	0.015
$\sigma_{estimated}$ (%)	0.191	0.275	0.468	0.168	0.649
σ_{actual} (%)	0.188	0.274	0.481	0.169	0.618
MD$_w$ (%)	71.798	17.467	69.703	34.675	0.000
SD$_w$ (%)	20.979	7.918	28.708	23.273	0.000
AV μ (%)	0.020	0.018	0.020	0.022	0.006
AV σ (%)	0.003	0.002	−0.013	−0.002	0.030
Sharpe ratio (%)	0.584	1.452	0.863	0.585	1.700
Herfindahl$_w$ (%)	40.091	10.316	37.654	53.518	0.000
Herfindahl$_{rc}$ (%)	44.559	5.860	45.530	55.839	5.870
Turnover (%)	13.706	2.454	41.930	4.671	0.000
VaR (%)	−9.025	−11.145	−12.974	−11.515	−21.478
MDD (%)	−19.406	−22.295	−27.261	−18.954	−58.571
Entropy	1.000	4.026	4.913	6.718	1.759
μ_{annual} (%)	0.347	1.288	1.365	0.415	3.677
$\mu_{cumulative}$ (%)	38.173	58.617	83.680	56.598	60.378
σ_{annual} (%)	2.965	4.318	7.605	2.674	9.775
Sharperatio$_{annual}$ (%)	11.717	29.836	17.950	15.511	37.612

The statistics are given as means across 174 months of portfolio optimizations, which were rebalanced monthly. For better comparison, they are all shown in percentage, except the entropy

Table 18 Performance statistics of the global portfolio in the empirical "worst case", when the "true" returns are estimated by using the Black Litterman implied returns

	Markowitz	ERC	PCA	MV	1/n
$\mu_{estimated}$ (%)	−0.012	−0.029	−0.013	−0.012	−0.111
μ_{actual} (%)	0.014	0.027	0.03	0.014	0.085
$\sigma_{estimated}$ (%)	0.261	0.397	0.512	0.261	1.221
σ_{actual} (%)	0.371	0.503	0.757	0.371	1.467
MD$_w$ (%)	42.283	18.825	62.355	42.283	0.000

Table 18 continued

	Markowitz	ERC	PCA	MV	1/n
SD$_w$ (%)	19.700	10.024	44.212	19.700	0.000
AV μ (%)	−0.026	−0.057	−0.043	−0.026	−0.196
AV σ (%)	−0.109	−0.106	−0.245	−0.109	−0.246
Sharpe ratio (%)	2.028	3.860	2.441	2.028	4.400
Herfindahl$_w$ (%)	38.902	21.196	48.850	38.902	0.000
Herfindahl$_{rc}$ (%)	49.245	5.738	60.549	49.245	4.669
Turnover (%)	10.135	3.516	49.491	10.135	0.000
VaR (%)	−11.515	−11.145	−12.974	−11.515	−21.478
MDD (%)	−17.602	−20.370	−21.505	−17.602	−53.031
Entropy	6.615	3.619	5.272	6.615	1.338
μ_{annual} (%)	3.376	6.808	7.489	3.376	21.268
$\mu_{cumulative}$ (%)	−13.070	−14.072	−18.532	−13.070	−34.611
σ_{annual} (%)	5.860	7.947	11.964	5.860	23.187
Sharperatio$_{annual}$ (%)	57.616	85.671	62.600	57.616	91.723

The statistics are given as means across 12 months of portfolio optimizations (30.05.2008–30.04.2009), which were rebalanced monthly. For better comparison, they are all shown in percentage, except the entropy

References

Allen, G.C.: The Risk Parity Approach to Asset Allocation. Callan Investments Institute, Callan Associates (2010)

Bera, A.K., Park, S.Y.: Optimal portfolio diversification using maximum entropy. Econom. Rev. **27**(4–6), 484–512 (2008)

Black, F., Litterman, R.: Global portfolio optimization. Financ. Anal. J. **48**(5), 28–43 (1992)

Brinkmann, U.: Robuste Asset Allocation. PhD thesis, Universität Bremen, Bremen (2007)

Broadie, M.: Computing efficient frontiers using estimated parameters. Ann. Oper. Res. **45**, 21–58 (1993)

Campello, M., Graham, J.R., Harvey, C.R.: The real effects of financial constraints: evidence from a financial crisis. J. Financ. Econ. **97**(3), 470–487 (2010)

Chong, C.Y.: Effect of subprime crisis on U.S. stock market return and volatility. Glob. Econ. Financ. J. **4**(1), 102–111 (2011)

Chopra, V.K., Ziemba, W.T.: The effect of errors in means, variances, and covariances on optimal portfolio choice: good mean forecasts are critical to the mean-variance framework. J. Portfolio Manag. **19**(2), 6–11 (1993)

DeMiguel, V., Garlappi, L., Uppal, R.: Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy? Rev. Financ. Stud. **22**(5), 1915–1953 (2007)

DeMiguel, V., Nogales, F.J.: Portfolio selection with robust estimation. Oper. Res. **57**(3), 560–577 (2009)

Drobetz, W.: How to avoid the pitfalls in portfolio optimization? Putting the Black–Litterman approach to work. J. Financ. Mark. Portfolio Manag. **15**(1), 59–75 (2001)

Fabozzi, F.J.: Robust Portfolio Optimization and Management. Wiley, New York (2007)

Fabozzi, F.J., Huang, D., Zhou, G.: Robust portfolios: contributions from operations research and finance. Ann. Oper. Res. **176**(1), 191–220 (2010)

Foresti, S.J., Rush, M.E.: Risk-Focused Diversification: Utilizing Leverage within Asset Allocation. Wilshire Consulting, Santa Monica (2010)

Frahm, G., Wiechers, C.: On the Diversification of Portfolios of Risky Assets. Discussion Papers in Statistics and Econometrics, Seminar of Economic and Social Statistics, University of Cologne (2/11) (2011)

Goldfarb, D., Iyengar, G.: Robust portfolio selection problems. Math. Oper. Res. **28**(1), 1–38 (2003)

Herold, U., Maurer, P.: Tactical asset allocation and estimation risk. Financ. Mark. Portfolio Manag. **18**(1), 39–57 (2004)

Jagganathan, J., Ma, T.: Reduction in large portfolios: why imposing the wrong constraints helps. J. Financ. **58**(4), 1651–1683 (2003)

Jen, E.: Working Definitions of Robustness. SFI Robustness, (RS-2001-009) (2001)

Jobson, J.D., Korkie, B.: Estimation for Markowitz efficient portfolios. J. Am. Stat. Assoc. **75**(371), 544–554 (1980)

Jochum, C.: Robust volatility estimation. Financ. Mark. Portfolio Manag. **12**(1), 46–58 (1998)

Jorion, P.: International portfolio diversification with estimation risk. J. Bus. **58**(3), 259–278 (1985)

Lee, W.: Risk-based asset allocation: a new answer to an old question? J. Portfolio Manag. **37**(4), 11–28 (2011)

Lindberg, C.: Portfolio optimization when expected stock returns are determined by exposure to risk. Bernoulli **15**(2), 464–474 (2009)

Little, P.: Risk Parity 101. Research Note, Hammond Associates (2010)

Maillard, S., Roncalli, T., Teiletche, J.: On the properties of equally-weighted risk contributions portfolios. J. Portfolio Manag. **36**(4), 60–70 (2010)

Markowitz, H.: Portfolio selection. J. Financ. **7**(1), 77–91 (1952)

Markowitz, H.M.: Portfolio Selection: Efficient Diversification of Investments. John Wiley, New York (1959)

Merton, R.C.: On estimating the expected return on the market: an exploratory investigation. J. Financ. Econ. **8**, 323–361 (1980)

Meucci, A.: Managing diversification. Risk **22**(5), 74–79 (2009)

Michaud, R.O.: The Markowitz optimization Enigma: is 'optimized' optimal? Financ. Anal. J. **45**(1), 31–42 (1989)

Michaud, R.O.: Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation. Harvard Business School Press, Boston (1998)

Michaud, R.O.: Are Good Estimates Good Enough? No, Investment Management Consultants Association (2009)

Morris, D., Haeusler, F.: Engineering a safer investment. FT Mandate (2010)

Neukirch, T.: Portfolio optimization with respect to risk diversification. Working Paper, HQ Trust GmbH (2008)

Partovi, M.H., Caputo, M.: Principal portfolios: recasting the efficient frontier. Econ. Bull. **7**(3), 1–10 (2004)

Pearson, N.D.: Risk budgeting: Portfolio Problem Solving with Value-at-Risk. Wiley, New York (2002)

Pojarlev, M., Polasek, W.: Portfolio construction by volatility forecasts: does the covariance matter? Financ. Mark. Portfolio Manag. **17**(1), 103–116 (2003)

Qian, E.: Risk Parity Portfolios: Efficient Portfolios Through True Diversification. PanAgora Asset Management (2005)

Qian, E.: Risk Parity Portfolios: The Next Generation. PanAgora White Paper (2009)

Qian, E.: Risk Parity: The Solution to the Unbalanced Portfolio. PanAgora Asset Management (2010)

Recchia, R.: Experiments with robust asset allocation strategies: classical versus relaxed robustness. PhD thesis, Università di Pisa, Pisa (2010)

Scherer, B.: Resampled Efficiency and Portfolio Choice. Financ. Mark. Portfolio Manag. **18**(4), 382–398 (2004)

Schwartz, S.: Taking the long view. Special Report Risk Parity, Investments and Pensions, Europe, April (2011)

Scutellà, M.G., Recchia, R.: Robust portfolio asset allocation and risk measures. 4OR. **8**(2), 113–139 (2010)

Stefanovits, D.: Equal Contributions to Risk and Portfolio Construction. PhD thesis, ETH Zurich, Zurich (2010)

Tütüncü, R.H., Koenig, M.: Robust asset allocation. Ann. Oper. Res. **132**, 157–187 (2004)

Young, P.J., Johnson, R.R.: Bond market volatility vs. stock market volatility: the Swiss experience. Financ. Mark. Portfolio Manag. **18**(1), 8–23 (2004)

Author Biographies

Thorsten Poddig is a Professor of Finance at the University of Bremen. He studied Economics and Computer Sciences and works on asset and risk management, especially on financial forecasting, portfolio optimization and empirical capital market research.

Albina Unger studied Economics at the University of Bremen and is a research assistant at the chair of finance. She works on asset allocation and portfolio optimization.