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REGULAR ARTICLE

Long-run consequences of debt

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Abstract Empirical evidence suggests that both public and private debt may have long-run detrimental effects on the economy. However, theoretical works have not provided a unique explanation to the issue. In this paper, therefore, we propose a framework that is able to describe the long-run effects of different kinds of debt. We introduce a stock-flow consistent dynamic model where the economy is represented as a network of trading relationships among agents. Debt contracts are one of such relationships. The model is characterized by a unique and stable steady-state and predicts that: (i) aggregate income is always limited from the above by the money supply; (ii) debts cause in the long-run a redistribution of borrowers' wealth and income in favor of lenders; (iii) the redistribution is magnified by the level of the interest rate and (iv) by the degree of debt persistence. In the aggregate this may also lower the average marginal propensity to spend and nominal income, providing therefore a clear-cut explanation to the empirical evidence.

Keywords Debt · Wealth distribution · Networks · Stock-flow consistency · Dynamic systems

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1 Introduction

The 2008–2009 crisis has produced among the economists a renewed interest in the role of debt and financial variables in general. One important research question refers to the

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long-run consequences of a regime of persistent debt. Recent empirical investigations such as the controversial Reinhart and Rogoff (2010), Checherita and Rother (2010) and Cecchetti et al. (2011), find in general a negative relationship between growth and high levels of both public and private debt (for a survey see Panizza and Presbitero 2013). Strong evidence is also found on the negative role of high private debt levels on macroeconomic stability (Sutherland et al. 2012). Evidence is even stronger in the case of developing Countries (see, among the others, Pattillo et al. 2002; Clements et al. 2003).

The theoretical literature on public debt offers different answers according to the perspective assumed: while at least since Keynes' *General Theory* there is an acknowledgment for the short-run benefits brought about by public deficits during recessions (DeLong and Summers 2012), macro models tend to predict long-run negative effects for high and persistent public debts. For example, Modigliani (1961) and Diamond (1965) are classical studies relating public debt to a lower pace of capital accumulation because of crowding-out effects and raising taxes. More recent models lead to similar conclusions (Elmendorf and Mankiw 1999; Checherita-Westphal et al. 2012). When considering private debts, models emphasize the role of balance sheet conditions (debt and leverage, in particular) in transmitting and amplifying shocks from and to real sectors. Debt-deflation theory (Fisher 1933), the Financial Instability Hypothesis by Minsky (1982) and the Financial Accelerator theory by Bernanke and Gertler (1989, 1990) are foremost examples of this strand of research. Along the same lines are Greenwald and Stiglitz (1993) and Kiyotaki and Moore (1997). The adverse impact of external debt in developing Countries has been also highlighted (Krugman 1988). The role of debt and its complex interactions with the real sector has also been analyzed in recent agent-based macroeconomic literature (e.g. Raberto et al. 2012; Riccetti et al. 2013; Assenza et al. 2015).

However, a unified theory fitting with any kind of debt does not exist. In fact, each of the above models focuses on one single typology of debt and relies upon very different hypotheses. Moreover, while they reveal how debt negatively impacts on the economy through its interactions with other mechanisms (such as the external finance premium), in general they do not consider its *direct* effects. The aim of this paper, therefore, is to fill in these gaps by developing a model that is able to describe the long-run direct effects of several kinds of debt abstracting from possible interactions with other factors such as asymmetric information or crowding-out effects. However, we will focus our analysis only on those typologies of debt that do not cause the money stock to increase, such as for example corporate bonds, Government bonds sold on secondary markets and commercial paper. Consequently, our framework is not suitable to analyze bank loans.

We will model the economy as a network of interconnected agents characterized by state and control variables. The nodes of the network represent agents whereas the links represent trading relationships between pairs of agents. Debts are just special cases of such relationships. In addition, we introduce a form of stock-flow consistency assuring the closure of the model. This is particularly important when considering inter-temporal relationships such as debts.

The steady-state solution yields three testable predictions, qualitatively in accordance with the empirical evidence. First, in the long run debts determine a redistribution of income and wealth from debtors to creditors. Thus, highly indebted agents will experience a lower capacity to spend: firms will reduce investment, households will reduce consumption and Government will reduce public expenditure. Second, the magnitude of the redistribution is amplified both by the level of the interest rate and by debt duration. This suggests that regimes of persistent debt should be avoided. Third, if debtors have a higher marginal propensity to spend than creditors, then debts will also reduce aggregate spending. This result, which straightforwardly explains the empirical evidence, may not come as a surprise to many, but we want to stress the strength and the added value of our modeling approach: generality and parsimony of assumptions.

The rest of the paper is organized as follows. Section 2 introduces the baseline model without debts, proves the existence of its equilibrium and presents Lemma 1, which states that individual steady-state wealth stock is decreasing in one's spending propensity. Section 3 shows that debts increase individual spending propensity. Thus, the main results are derived as a consequence of Lemma 1. Section 4 concludes.

2 The baseline model

Consider an economy operating in continuous time that is structured as a network of n generic infinitely lived economic agents. Each agent i is characterized by the state variable $W_i(t)$, standing for its current stock of monetary wealth, and by the flow variables $E_i(t)$ and $I_i(t)$ denoting respectively current expenditure and income. Expenditure consists of a monetary flow from one agent to another when implementing some market transaction. For the time being financial transactions are ruled out, but they will be explicitly considered in Sect. 3 once debts will have been embodied into the model. As a consequence, we can assume that the money supply is a constant M such that $\sum W_i = M$ for each t.

2.1 Stock-flow consistency

As anticipated, crucial assumption for our model is the consistency between stocks and flows. Basically, we require that flows originate from stocks and that they accumulate in stocks without leakages or undue additions of money. Formally, two variables $(x(t), y(t))$ are said to be stock-flow consistent if

$$dx(t)/dt = y(t). \tag{1}$$

For example, if y denotes investments then x is the capital stock, or if y is savings then x is the stock of wealth. In our setting Eq. (1) turns into the law of motion for the wealth of the generic agent i:

$$\dot{W}_i(t) = I_i(t) - E_i(t). \tag{2}$$

However, Eq. (2) satisfies our requirement only in part as it defines where flows are going but not where they are coming from. In order to complete the implementation of stock-flow consistency, therefore, we still need to determine how $E_i(t)$ is financed (while $I_i(t)$ will be automatically obtained by the aggregate identity between income and expenditure). In what follows we explain how.

From a logical point of view a time interval must separate the flows of income and expenditure. When an agent receives income, this cannot be simultaneously used to finance expenditure: first it accrues to the wealth stock [as in Eq. (2)], and then it can be spent. Consequently, $I_i(t)$ cannot be a *direct* financing source for $E_i(t)$. At first glance this statement would seem to clash with the established view according to which income determines expenditure, and in fact it does if we do not consider the proper time interval. If we observe an agent for a long period of time, for example one year, then its expenditure $E_i(t)$ relative to this period can hardly be assumed as independent of its contemporaneous income $I_i(t)$. Indeed, almost all expenditure would be financed out of income. But if we consider a shorter period (say one quarter), then it is reasonable to think expenditure as largely but not totally financed out of the income received during the same quarter. And if we consider one month, the share of monthly expenditure paid out of income would be even smaller. Thus, considering shorter and shorter periods, we can arrive to conceive a time interval that is small enough to regard individual expenditure as totally independent, within this period, of individual income. Call this time lapse dt. Of course, real agents are in general characterized by different dt's. For example, a worker receives his income on monthly basis, so in this case dt would correspond to one month, while if we consider a seller making money on daily basis, then dt would correspond to one day. In order to cope with this heterogeneity it is sufficient to us figuring out that the chosen dt corresponds to the smallest among all the agents. To convince the reader, we suggest an analogy with the physical system of a reservoir full of water ($W_i(t)$) provided with an outlet for inflows ($I_i(t)$) and an outlet for outflows ($E_i(t)$): observed for small time intervals, the water entering in the reservoir ($I_i(t)$) is not the same water that is flowing out ($E_i(t)$).

By the above discussion we have argued that for small time periods current income cannot be regarded as the financing source of current expenditure. So, how might agents finance their spending? The only possible answer is that each agent finances expenditure by its buffer of wealth available at the beginning of the period dt. Basically, we are stating that a cash-in-advance constraint must hold for each agent (remember that they cannot resort to debt). As a consequence, even though from a behavioral point of view expenditure may be any function $f_i(t)$, wealth must always provide an upper bound such that

$$E_i(t) = \min\{f_i(t), W_i(t)\}. \tag{3}$$

In what follows we will simply assume that current expenditure be proportional to wealth:

$$E_i(t) = c_i W_i(t), \tag{4}$$

where the parameter c_i is the marginal propensity to spend.[1] Since agents can spend at most what they posses (until we do not introduce debts), the condition $c_i \leq 1$

[1] To be precise, this is a marginal propensity to spend out of wealth (or liquidity) and not out of income.

must hold true. Equation (4), therefore, satisfies the cash-in-advance constraint (3) and, moreover, is an economically reasonable behavioral rule. We choose this rule essentially for convenience, but we can justify it, besides invoking common sense, also on the basis of some theories and empirical evidence showing that individuals tend to consume according to buffer-stock rules that ultimately lead to a constant individual income-wealth ratio (Deaton 1991; Carroll 1997). This is exactly what happens in our model when at equilibrium expenditure equals income (see next section), so that $I_i = c_i W_i$ and, consequently, the individual income-wealth ratio becomes constant.

2.2 The trading network

Since n agents populate the economy, we may imagine that expenditure E_i is allotted among different agent i's partners. So, with a slight abuse of notation, it can be generalized by a vector representation:

$$E_i = (E_{i1}, E_{i2}, \ldots, E_{in}) \equiv (c_{i1}, c_{i2}, \ldots, c_{in}) W_i, \tag{5}$$

where the generic element E_{ij} represents a non-negative flow of money from agent i to agent j such that the sum of the elements is equal to E_i and the sum of the $c's$ is equal to c_i. Obviously, we have $c_{ii} = 0$. Grouping the agents all together, we can define the $n \times n$ matrix $E(t)$ of the expenditure flows generated among all the agents during the period dt:

$$E = \begin{pmatrix} E_1 \\ E_2 \\ \ldots \\ E_n \end{pmatrix} = \begin{pmatrix} 0 & E_{12} & E_{13} & \ldots & E_{1n} \\ E_{21} & 0 & E_{23} & \ldots & E_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ E_{n1} & E_{n2} & E_{n3} & \ldots & 0 \end{pmatrix} \tag{6}$$

Matrix E defines the network describing the interaction structure among the agents and is based on the $n \times n$ matrix of coefficients C:

$$C = \begin{pmatrix} 0 & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & 0 & c_{23} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n1} & c_{n2} & c_{n3} & \ldots & 0 \end{pmatrix} \tag{7}$$

We can now obtain the income matrix $I(t)$. Given E, consider for example its element E_{21}. E_{21} is an outflow from agent 2's point of view, but at the same time is an inflow for agent 1. Thus, while each row represents a profile of expenditure by definition, each column represents a profile of income by construction. From this we can deduce agent i's income profile:

$$I_i(t) = (E_{1i}, E_{2i}, \ldots, E_{ni}) \equiv (c_{1i} W_1, c_{2i} W_2, \ldots, c_{ni} W_n). \tag{8}$$

Consequently, the income matrix is straightforwardly defined as the transposed $I = E'$.

Equations (5) and (8) allow to define the dynamics of the system as a whole. Denoting the $n \times 1$ vector of ones by $\mathbf{1}$, Eq. (2) becomes

$$\dot{W}_i = (I_i - E_i)\mathbf{1} = \sum_j c_{ji} W_j - \sum_j c_{ij} W_i.$$

Rearranging the above expression we get

$$\dot{W}_i = c_{1i} W_1 + c_{2i} W_2 + \cdots - \left(\sum_j c_{ij}\right) W_i + \cdots + c_{ni} W_n \equiv \tilde{c}_i W \tag{9}$$

where W is the $n \times 1$ vector of wealth stocks and $\tilde{c}_i = (c_{1i}, c_{2i}, \ldots, -c_i, \ldots, c_{ni})$. If we define the matrix $\tilde{C}$ as

$$\tilde{C} = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \ldots \\ \tilde{c}_n \end{pmatrix} = \begin{pmatrix} -c_1 & c_{21} & c_{31} & \ldots & c_{n1} \\ c_{12} & -c_2 & c_{32} & \ldots & c_{n2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{1n} & c_{2n} & c_{3n} & \ldots & -c_n \end{pmatrix} \tag{10}$$

our model can be described by the homogeneous system of n differential equations

$$\dot{W} = \tilde{C} W, \tag{11}$$

with the additional constraint of a constant stock of money, that is

$$\mathbf{1}' W = M. \tag{12}$$

It is sufficient for our purpose to concentrate ourselves on the steady-state solution. Thus, we should find the vector W^* which satisfies the n conditions $\dot{W} = \mathbf{0}$, that is $\tilde{C} W = \mathbf{0}$, and the money constraint $\mathbf{1}' W = M$. Apparently this task seems impossible to accomplish because we have a system of $n+1$ equations with n unknowns. However, our economy is a closed system and aggregate expenditure is always equal to aggregate income. Consequently, when $n - 1$ equations are satisfied also the last one is.[2] Thus, if we drop the last equation $\dot{W}_n = 0$, the system to be solved reduces to n equations in n unknowns

$$\Gamma W \equiv \begin{pmatrix} \tilde{c}_1 \\ \ldots \\ \tilde{c}_{n-1} \\ \mathbf{1}' \end{pmatrix} \begin{pmatrix} W_1 \\ \ldots \\ W_{n-1} \\ W_n \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ M \end{pmatrix} \equiv \tilde{\mathbf{0}} \tag{13}$$

[2] Algebraically, matrix $\tilde{C}$ is singular, what can be easily verified by summing up its rows.

As now the $n \times n$ matrix Γ is non-singular, the required unique solution can be immediately found:

$$W^* = \Gamma^{-1}\, \tilde{\mathbf{0}}. \tag{14}$$

This solution is also stable. In fact, from Eq. (9) wealth velocity is negatively affected by its own level and positively affected by the level of the other wealth stocks. Consequently, if for instance $\dot{W}_i$ is positive, W_i increases and the other wealth stocks decrease because the system is closed (so total wealth is constant), thus causing $\dot{W}_i$ to decrease and W_i to slow down. For a formal proof see "Appendix 2".

In the simplest case with two agents we have $W_1 + W_2 = M$ and $\dot{W}_1 = c_{21} W_2 - c_{12} W_1$. The two steady states are then

$$W_1^* = \frac{c_{21}}{c_{12} + c_{21}} M, \quad W_2^* = \frac{c_{12}}{c_{12} + c_{21}} M.$$

The equilibrium value W^* does not depend on the initial conditions of W but only on the set of marginal propensities to spend and on the total amount of money. Besides, it shows the distributional implications of the model: the wealth amount of one agent is increasing in the propensity to spend of the other agent and decreasing in its own.

The 2-agent case can be generalized to any network of agents by the following statement:

Lemma 1 *The equilibrium value W_i^* is decreasing in agent i's spending coefficients and increasing in those of its partners.*

The general validity of Lemma 1 can be argued by simply considering that the spending coefficients are all smaller than one. Consequently, every additional expenditure flow from i to j will be followed in the next period only by a less-than-proportional flow from j to i. Hence, any initial increment of i's expenditure will never be totally compensated by future income, and the long-run amount of i's wealth will decline (we refer to "Appendix 2" for some proofs).[3]

Before proceeding with the introduction of debts, which are the main concern of the paper, we conclude this section by giving a look at the equilibrium aggregate behavior. The steady-state solution W^* implies, in conjunction with Eq. (4), the existence for each agent of a constant flow of expenditure $E_i^* = c_i W_i^* \leq W_i^*$. As equilibrium wealth stocks are constant, the equality between income and expenditure for any individual agent follows from Eq. (2): $I_i^* = E_i^*$. Hence, we can define the steady-state aggregate income Y as

$$Y = \sum_i I_i^* = \sum_i E_i^*.$$

Since $\sum_i E_i^* = \sum_i c_i W_i^* \leq \sum_i W_i^* = M$, it follows that the equilibrium aggregate income is a fraction of the monetary stock, that is $Y \leq M$. By using a mean-field approximation we can write it as

[3] Moreover, computer simulations confirmed the result for any kind of network we tried: different kinds of random, scale-free (power law) and small-world networks.

$$Y = cM, \tag{15}$$

where c is a weighted average of the c_i's, with weights w_i equal to the relative equilibrium wealth stocks $W_i^* / \sum_j W_j^*$, that is

$$c = c_1 w_1 + \cdots + c_n w_n. \tag{16}$$

Equation (15) leads to conclude that the steady-state income Y can increase only if $c \to 1$ and/or if the money stock increases. So, in both cases Y is *always* limited by M. However, this conclusion must be interpreted carefully: it does not mean that an increase in the monetary stock M automatically delivers a corresponding increase in income Y, because the true behavioral parameter here is c. Money stock M only determines the upper bound of nominal income.

3 Debt dynamics

In this section we are going to introduce debts as additional state variables. We limit to consider the case of financial contracts that leave the aggregate money stock unchanged, such as for example corporate bonds and commercial paper. Our model can also encompass the case of Government bonds sold on the financial markets (but not to the Central Bank). Our modeling choice, therefore, has the limitation of not applying in general to bank loans. We are well aware that generally debts and money follow the evolution of the economy and are endogenous. Nonetheless, by keeping debts exogenous we can capture their direct effect on the economy, abstracting from the feed-backs from the economy to debts.

In order to keep things simple, we assume that borrowers and lenders are two disjoint sets of agents such that who borrows does not lend and vice versa. Let $D(t)$ be the $n \times n$ matrix containing the stocks of debt at time t, where the generic entry D_{ij} stands for the outstanding debt that agent i owes to agent j. During the time interval dt debtors have to pay interests and principal to creditors, so we can define

$$F = (i + a)D \tag{17}$$

as the matrix of the financial flows F_{ij} from agent i to agent j, where we make the simplifications of a uniform interest rate i and of a uniform debt repayment coefficient a. The latter coefficient can be interpreted as the reciprocal of the debt contract length: the bigger a, the faster the debt reimbursement. Finally, we define L as the matrix of current credit flows, whose generic entry L_{ij} stands for the new credits supplied by agent i to agent j.

In order to fulfill our stock-flow consistency requirements, we make the additional assumption that for every creditor i the condition must hold:

$$\sum_j L_{ij} \leq W_i - E_i \equiv (1 - c_i)W_i. \tag{18}$$

Equation (18) simply states that current lending must not be greater than the amount of wealth left after expenditure. We recall that, by construction, flows F_{ij} and L_{ij} are outflows for i and inflows for j. Consequently, the law of motion for the wealth stocks becomes[4]

$$\dot{W}(t) = \sum I(t) - \sum E(t) + \sum (F' + L' - F - L). \tag{19}$$

The system is completed by the law of motion for debt stocks, which in matrix form looks as

$$\dot{D} = L' - aD. \tag{20}$$

We still need to define the loan matrix L. For simplicity we assume the total flow of credit to be proportional to wealth (richer lenders lend more). So, if agent i is a lender, we have

$$L_i = (L_{i1}, L_{i2}, \ldots, L_{in}) \equiv (r_{i1}, r_{i2}, \ldots, r_{in}) W_i, \tag{21}$$

with $\sum_j r_{ij} \leq (1 - c_i)$ in order to satisfy condition (18). The coefficient r_{ij} can be interpreted as agent i's 'propensity to lend' to agent j.

Equations (19) and (20) represent a system of coupled differential equations, whose steady state is given by the pair (W^{**}, D^*). Notice, however, that the system is block-recursive since both groups of equations $\dot{W}$ and $\dot{D}$ depend on the stock variables W and D and do not simultaneously affect each other. Hence, as we are considering the steady states we are allowed to first resolve the second block of Eq. (20), and then substitute its steady state values in the first block (19). Doing so, the sub-system (19) reduces to

$$\dot{W}(t) = \sum I(t) - \sum E(t) + \frac{i}{a} \sum (L - L'), \tag{22}$$

where we used Eq. (17) together with the steady-state solution $D^* = L'/a$ of Eq. (20). The flows of interest payments are given by iL'/a and enter in Eq. (22) with negative sign for debtors, while interest gains iL/a enter with positive sign for creditors. Debt repayments aD^* and new loans L in the long run offset each other and do not affect wealth distribution.

In order to find the steady-state solutions W^{**} we have to carry out some trivial but boring matrix manipulation as in Sect. 2. Rearranging the expression $\sum(L - L')$, Eq. (22) become

$$\dot{W}(t) = \tilde{C}W + \frac{i}{a} HW, \tag{23}$$

where $\tilde{C}$ is the same as in Eq. (10) and H is a singular square matrix. Rows of matrix H are different for lenders and borrowers (we are assuming that lenders do not borrow and vice versa). Supposing that agent l is a lender, then its row in H is

[4] By $\sum X$ we mean the $n \times 1$ column vector of the sum of the columns of matrix X.

$$H_l = \left(0, \ldots, \sum_j r_{lj}, \ldots, 0\right), \tag{24}$$

where the non-null entry is in position l. The row corresponding to a borrower b is

$$H_b = (-r_{1b}, -r_{2b}, \ldots, 0, \ldots, -r_{nb}) \tag{25}$$

where the null entry is in position b. System (23) is provided with a unique steady-state solution W^{**} since it is a linear transformation of system (11) and its coefficients are chosen in order to satisfy the conservation of money.[5] More interesting is to understand how it differs from its counterpart without debts W^*.

Let's first consider a lender l. From (23) and (24) the law of motion for its wealth is

$$\dot{W}_l = c_{1l}W_1 + c_{2l}W_2 + \cdots - c_l W_l + \cdots + c_{nl}W_n + \frac{i}{a}\sum_j r_{lj}W_l.$$

The presence of the last positive term goes to diminish the spending coefficient of the lender towards the other agents from c_l to $c_l - \frac{i}{a}\sum_j r_{lj}$. Thus, by Lemma 1 the value of W_l in W^{**} must be higher than in W^*. The same arguments lead to conclude that the opposite is true for a borrower b, whose steady-state wealth will be lower than in W^*. In fact, from (23) and (25) we have

$$\dot{W}_b = c_{1b}W_1 + c_{2b}W_2 + \cdots - c_b W_b + \cdots + c_{nb}W_n - \frac{i}{a}\sum_j r_{j\,b}W_j.$$

The presence of the last negative term goes to diminish the spending coefficients of the other agents towards the borrower (say from $c_{j\,b}$ to $c_{j\,b} - \frac{i}{a}r_{j\,b}$, for each $j \neq b$). Hence, again by Lemma 1 we have $W_b^{**} < W_b^*$.

In principle nothing can be inferred about those who are neither lenders nor borrowers, because their income depends on the wealth of their partners: if their income relies more on borrowers (lenders), then their wealth should decrease (increase).

Above results lead to simple but meaningful conclusions: in the short-run borrowers (lenders) increase (decrease) their spending by using external finance (lending their savings), but in the long run their wealth and, consequently, their expenditure will be lower (higher). Thus, the ultimate effect of debt (at least of debt leaving the monetary stock unchanged) is a redistribution of wealth and income in favor of lenders. This may straightforwardly explain the empirical evidence: if borrowers, as it is likely to be, have a larger marginal propensity to spend than lenders', Eq. (16) implies that an economy with debts will be characterized in the long run by a smaller average marginal propensity to spend. Consequently, nominal income will decrease (see Eq. 15).

[5] In order to have interior solutions we also need that the ratio i/a is not too high, otherwise the model might show a meaningless behavior. In that case, to keep economic meaning and to have corner solutions we should introduce further constraints such as $W_i \geq 0\ \forall i$, but that would not add much to our knowledge.

As one could expect, the entity of the redistribution is increasing in the interest rate i. Less intuitively, the redistribution magnitude is decreasing in the debt repayment velocity a—or increasing in the debt contract duration $1/a$. This result may appear paradoxical (for given interest rates long debt contracts are usually preferred), but actually it simply proves that persistent debts (low a) in the long run are in fact detrimental for borrowers.

As a bottom line we can deduce a non-trivial policy implication. In fact, if the Government is one of the borrowing agents, then part of debts D^* is public debt. Therefore, an economy characterized by high and persistent (low a) public debt in the long run will dispose in general of lower income and in particular of less public expenditure. This outcome is consistent, at least qualitatively, with the debt crisis currently affecting some European Countries and with debt crises historically experienced by developing Countries.

4 Conclusive remarks

Empirical evidence suggests that both public and private debt may have detrimental effects on the economy. Unfortunately, theoretical work has not produced a unified approach to debt but proposed many alternative mechanisms through which debt can negatively affect the economy. The scope of this paper, therefore, was to investigate within a unified framework how debts influence the economy in the long run. Our approach was guided by two major concerns. The first concern was to keep the model as simple as possible, and the second one was to analyze the effect of debt in isolation, abstracting from possible interactions with other factors such as productive investments, asymmetric information, crowding-out effects, or expectations.

As a first step we outlined a baseline model where an economy without debt and with constant money supply is represented as a network of trading agents. This model predicts that (i) the steady-state aggregate income flow is always limited from the above by the amount of money available to the economy and that (ii) individual wealth stocks are decreasing in one's spending propensity.

In the following step we allowed agents to borrow from each other, while retaining the hypothesis of a fixed amount of money. The latter assumption, although it may appear restrictive, is nonetheless compatible with several typologies of debt. The extended model predicts that in the long run debts determine a redistribution of wealth and income from debtors to creditors. This phenomenon is amplified both by the level of interest rates and by the duration of debts. The negative role (for the borrowers) of debt duration is of special importance, as it suggests that situations of persistent debt should be in general avoided. Finally, in the aggregate debts may lower the average marginal propensity to spend and nominal income.

The results bear relevant policy implications. In fact, as Government may be one of the borrowing agents, the model suggests that high and persistent levels of public debt may be detrimental to public finances and to the economy as a whole. We believe that this conclusion is consistent with the empirical evidence and with debt crises experienced in the past decades by developing Countries and currently by some European Nations.

The case with a variable money supply is left for future research. This would require the introduction of some agent (a "bank") generating money at rate $m(t)$ and injecting it in the economy through its network. The hypothesized modification would then turn the system into a forced compartmental system whose steady state grows exactly at rate $m(t)$.

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Appendix 1

In this section we will explicitly solve the 2×2 case with and without debts, showing the correctness of our theoretical predictions.

System (11) becomes

$$\begin{pmatrix} \dot{W}_1 \\ \dot{W}_2 \end{pmatrix} = \begin{pmatrix} -c_1 & c_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix},$$

where we used $c_{12} = c_1$ and $c_{21} = c_2$. Solving the second-degree characteristic polynomial of $\tilde{C}$ we get the two real-valued eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -(c_1 + c_2)$. Hence, as the system is linear and homogeneous its general solution will take the following form:

$$W(t) = k_1 v_1 e^{\lambda_1 t} + k_1 v_2 e^{\lambda_2 t}, \tag{26}$$

where v_1 and v_2 are the eigenvectors associated respectively to λ_1 and λ_2. After obtaining $v_1 = (c_2, c_1)$ and $v_2 = (-1, 1)$ and plugging these values in Eq. (26), we get the general solution

$$W(t) = k_1 \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} e^{0t} + k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-(c_1+c_2)t}.$$

To eliminate the parameters, we need to solve the Cauchy problem. By using initial conditions $W_1(0) = w_1$ and $W_2(0) = w_2$ and the constraint $w_1 + w_2 = M$, we finally find the solution

$$W_1(t) = \frac{c_2}{c_1 + c_2} M + \frac{c_1 w_1 - c_2 w_2}{c_1 + c_2} e^{-(c_1+c_2)t}$$

and

$$W_2(t) = \frac{c_1}{c_1 + c_2} M - \frac{c_1 w_1 - c_2 w_2}{c_1 + c_2} e^{-(c_1+c_2)t}.$$

The solution is clearly stable, as it monotonically converges to the steady state already found in Sect. 2.

We can now consider the introduction of debt, assuming that agent 1 lends to agent 2 at rate r_1. Using directly Eq. (23) we get

$$\dot{W}_1 = -c_1 W_1 + c_2 W_2 + \frac{i}{a} r_1 W_1$$

and

$$\dot{W}_2 = c_1 W_1 - c_2 W_2 - \frac{i}{a} r_1 W_1.$$

Hence, solving for the steady states and using the constraint of constancy of money we find

$$W_1^{**} = \frac{c_2 M}{c_1 + c_2 - \frac{i}{a} r_1}, \quad W_2^{**} = \frac{c_1 - \frac{i}{a} r_1}{c_1 + c_2 - \frac{i}{a} r_1} M,$$

provided that $i/a < (c_1 + c_2)/r_1$. A comparison with the values found in Sect. 2 will show that, as expected, debt increases agent 1's wealth and decreases agent 2's.

In order to assess the stability of the steady states we have to solve systems (19) and (20). In the specific case we will have

$$L(t) = \begin{pmatrix} 0 & L_{12} \\ 0 & 0 \end{pmatrix} \text{ and } D(t) = \begin{pmatrix} 0 & 0 \\ D_{21} & 0 \end{pmatrix}.$$

Then, using Eq. (17) and $L_{12} = r_1 W_1$, the two blocks of equations will look like

$$\begin{pmatrix} \dot{W}_1 \\ \dot{W}_2 \end{pmatrix} = \begin{pmatrix} -c_1 & c_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} + (i + a) \begin{pmatrix} D_{21} \\ -D_{21} \end{pmatrix} + \begin{pmatrix} -L_{21} \\ L_{21} \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{D}_1 \\ \dot{D}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r_1 & 0 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} - a \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}.$$

Considering that $D_1(t) = 0\ \forall\, t$, and using the constraint $W_1 + W_2 = M$, after some manipulations the system to be solved reduces to

$$\begin{pmatrix} \dot{W}_1 \\ \dot{D}_2 \end{pmatrix} = \begin{pmatrix} -(c_1 + c_2 + r_1) & (i + a) \\ r_1 & -a \end{pmatrix} \begin{pmatrix} W_1 \\ D_2 \end{pmatrix} + \begin{pmatrix} c_2 M \\ 0 \end{pmatrix}. \tag{27}$$

The general solution to the non-homogeneous system (27) will be the sum of one of its particular solutions and the general solution of the associated homogeneous system.

As a particular solution we try a constant one:

$$W_1(t) = \bar{W}_1 \text{ and } D_2(t) = \bar{D}_2.$$

These values must satisfy system (27), which therefore becomes

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -(c_1 + c_2 + r_1) & (i + a) \\ r_1 & -a \end{pmatrix} \begin{pmatrix} \bar{W}_1 \\ \bar{D}_2 \end{pmatrix} + \begin{pmatrix} c_2 M \\ 0 \end{pmatrix}.$$

Solving for the two constants we get

$$\bar{W}_1 = \frac{c_2}{c_1 + c_2 - \frac{i}{a} r_1} M$$

and

$$\bar{D}_2 = \frac{c_2}{(c_1 + c_2)\frac{a}{r_1} - 1} M,$$

provided again that $i/a < (c_1 + c_2)/r_1$. From the monetary constraint $\bar{W}_1 + \bar{W}_2 = M$ we also obtain

$$\bar{W}_2 = \frac{c_1 - \frac{i}{a} r_1}{c_1 + c_2 - \frac{i}{a} r_1} M.$$

Hence, $\bar{W}_1$ and $\bar{W}_2$ are exactly the steady-state solutions W_1^{**} and W_2^{**} we have previously found by directly using Eq. (23).

As for the general solution of the homogeneous system, we need to find the eigenvalues of the coefficient matrix in (27). These will be

$$\lambda_{1,2} = \frac{-(c_1 + c_2 + r_1 + a) \pm \sqrt{\Delta}}{2} \equiv \frac{-B \pm \sqrt{\Delta}}{2},$$

where

$$\Delta = (c_1 + c_2 + r_1 + a)^2 - 4[a(c_1 + c_2) - i r_1] \equiv B^2 - 4C.$$

As we are assuming $i/a < (c_1 + c_2)/r_1$, then $C > 0$.

Now, there are three possible cases (e.g. Gandolfo 1997, p. 240). Case 1: if $\Delta > 0$, then we have two distinct real eigenvalues. Moreover,

$$B^2 - 4C < B^2,$$

so

$$|\sqrt{\Delta}| < |B|,$$

which implies that λ_1 and λ_2 are both negative. As a consequence, the system will converge to the particular solutions $\bar{W}_1$, $\bar{D}_2$ and $\bar{W}_2$, implying that the steady states are stable. Case 2: if $\Delta = 0$, then we have two identical real eigenvalues $\lambda_{1,2} = -B/2 < 0$, which again imply the stability of the steady states. Case 3: if $\Delta < 0$, then we have two complex eigenvalues. However, their real part is $-B/2 < 0$, which assures that the system will show damped oscillations converging to the steady states.

Appendix 2

In this section we will prove the stability of the steady-state solutions of system (11).

First, notice that the system is closed ($\tilde{C}$ is singular), which implies that the n solutions $W_i(t)$ are not linearly independent. In other words, when we know $n-1$ solutions we also know the last one. Hence, we can limit ourselves to analyze the solution of the reduced $n-1 \times n-1$ sub-system

$$\begin{pmatrix} \dot{W}_1 \\ \dot{W}_2 \\ \cdots \\ \dot{W}_{n-1} \end{pmatrix} = \begin{pmatrix} -c_1 & c_{21} & \ldots & c_{n-1\,1} \\ c_{12} & -c_2 & \ldots & c_{n-1\,2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n-1} & c_{2n-1} & \ldots & -c_{n-1} \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ \cdots \\ W_{n-1} \end{pmatrix}, \tag{28}$$

which is nothing but the original one without agent n. In the new coefficient matrix all the diagonal elements are negative and dominate their respective column, that is

$$|-c_i| \geq \sum_{j \neq i} |c_{ij}|, \;\; \forall i \neq n,$$

with strict inequality for some agents (namely, those who are connected to the missing agent n). The matrix, therefore, satisfies the sufficient stability conditions of the 'quasi-dominant negative diagonal' with weights $h_i = 1$ and $h_j = 1, \forall\, i, j$ (Gandolfo 1997, p. 253: method V). Hence, the steady states $(W_1^*, \ldots, W_{n-1}^*)$ of sub-system (28) are stable. But because of its linear dependence, also the steady state W_n^* relative to agent n is stable.

The same stability conditions can be applied also to the system with debts, although we have first to rewrite it. Basically, we will augment the system by introducing a new set of state variables A_i. So, writing explicitly Eq. (19) for the generic agent i, we get

$$\begin{aligned} \dot{W}_i = &(c_{1i} + r_{1i})W_1 + \cdots - (c_i + r_i)W_i + \cdots \\ &+(c_{ni} + r_{ni})W_n - (i + a)D_i + (i + a)A_i, \end{aligned}$$

where $A_i = \sum_j D_{ji}$ is agent i's credits. Consequently, as lenders and borrowers form two disjoint sets, D_i and A_i cannot be both positive. In the case i is a borrower, from Eq. (20) we have

$$\dot{D}_i = \sum_j r_{ji} W_j - a D_i. \tag{29}$$

In the case i is a lender, using Eq. (29) and considering that $\dot{A}_i = \sum_j \dot{D}_{ji}$ we get

$$\dot{A}_i = r_i W_i - aA_i.$$

Hence, we build a new block-recursive system with $3n$ equations. Now, as building the coefficient matrix for a generic number of agents is extremely complicated (letting alone checking the stability conditions), for expositional convenience we will consider the case with three agents. Supposing that both agent 1 and agent 2 lend to agent 3, the system will look like

$$\dot{Y} = \Phi Y,$$

where $Y' = (W_1, W_2, W_3, D_1, D_2, D_3, A_1, A_2, A_3)$, and

$$\Phi = \begin{pmatrix} -(c_1+r_1) & c_{21} & c_{31} & 0 & 0 & 0 & i+a & 0 & 0 \\ c_{12} & -(c_2+r_2) & c_{32} & 0 & 0 & 0 & 0 & i+a & 0 \\ c_{13}+r_1 & c_{23}+r_2 & -c_3 & 0 & 0 & -(i+a) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_1 & r_2 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\ r_1 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice again that the system is closed, so we can get rid of agent 3. Thus, upon eliminating W_3, D_3, A_3 and their respective derivatives, the coefficient matrix reduces to

$$\Phi' = \begin{pmatrix} -(c_1+r_1) & c_{21} & 0 & 0 & i+a & 0 \\ c_{12} & -(c_2+r_2) & 0 & 0 & 0 & i+a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & 0 & -a & 0 \\ 0 & r_2 & 0 & 0 & 0 & -a \end{pmatrix}.$$

We can prove that matrix Φ' satisfies above stability conditions as its diagonal elements are negative and dominate their respective columns for a suitable choice of the weights h_i.[6] For example, after setting $h_1 = h_2 = 1$, from the first column we get

$$c_1 + r_1 \geq c_{12} + h_5 r_1 \rightarrow h_5 \leq 1 + \frac{c_{13}}{r_1},$$

[6] Strictly speaking, the conditions are not satisfied as some diagonal element is not negative but zero. However, when a diagonal element is zero its corresponding variable is identically zero (e.g. D_1), which means that the variable plays no role in the system and consequently can be neglected.

whereas from column 5 we get

$$h_5 a \geq i + a \ \rightarrow h_5 \geq 1 + \frac{i}{a}.$$

Doing the same for column 2 and 6, we finally find two sufficient conditions:

$$\frac{i}{a} \leq h_5 \leq \frac{c_{13}}{r_1}$$

and

$$\frac{i}{a} \leq h_6 \leq \frac{c_{23}}{r_2},$$

which can be satisfied if the ratio i/a is not "too high" (see also footnote 5).

Appendix 3

In this section we will first prove Lemma 1 for two particular cases: (i) a generic network with three agents and (ii) a regular ring lattice network with a generic number n of agents. Then, we will also provide a general but more intuitive proof. However, first of all we have to point out that, after a shock to c_{ij}, W_i^* and W_j^* must always move in opposite directions. In fact, if both increase (decrease), then also the other equilibrium wealth stocks must increase (decrease), contradicting the assumption of a constant money supply.

Case (i). Suppose c_{12} (and so also c_1) increases: according to the lemma W_1^* must decrease and W_2^* must increase, but suppose the opposite is true. Equation (9) is

$$\dot{W}_1 = c_{21} W_2 + c_{31} W_3 - c_1 W_1,$$

so that at equilibrium ($\dot{W}_1 = 0$) we have

$$W_1^* = \frac{c_{21} W_2^* + c_{31} W_3^*}{c_1}.$$

As c_1 increased and W_2^* decreased, W_1^* can increase if and only if W_3^* increases. But then also W_2^* should increase as

$$W_2^* = \frac{c_{12} W_1^* + c_{32} W_3^*}{c_2},$$

and this contradicts the assumption that W_2^* decreased.

Case (ii). As agents form a closed ring, agent i gives money to agent $i+1$ at rate $c_{i\,i+1}$ and receives money from agent $i-1$ at rate $c_{i-1\,i}$. Suppose $c_{i\,i+1}$ increases and that, contradicting the lemma, W_i^* increases and W_{i+1}^* decreases. From Eq. (9), at equilibrium we have

$$W_i^* = \frac{c_{i-1\,i} W_{i-1}^*}{c_i},$$

which implies that also W_{i-1}^* increases. On its turn, this implies that also W_{i-2}^* increases as

$$W_{i-1}^* = \frac{c_{i-2\,i-1} W_{i-2}^*}{c_{i-1}}.$$

Proceeding backwards, and considering that agent 1 is also agent $n+1$, we will discover that also W_{i+1}^* must increase, contradicting our assumption.

General case. Suppose the system is in equilibrium ($W_i = W^*, \ \forall i$) and that at time t^* c_{ij} increases to c'_{ij}. This will change the steady states to $W^\#$ and, therefore, the system will no longer be in equilibrium. However, we know that the system is stable (see "Appendix 2") and that it is bound to converge to $W^\#$. Now, there exist only two possible converging paths: a monotonic path (if all the eigenvalues of matrix $\tilde{C}$ are real) or an oscillatory one (when some of the eigenvalues is complex). In both cases, the initial direction of the convergence process will tell if the steady state has increased or decreased: if $W_i(t)$ starts decreasing means that $W_i^\# < W_i^*$, whereas if $W_i(t)$ starts increasing means that $W_i^\# > W_i^*$.[7] So, we compute the derivative of $W_i(t)$ at time t^* when c_i becomes c'_i:

$$\dot{W}_i(t^*) = c_{1i} W_1^* + \cdots + c_{ni} W_n^* - c'_i W_i^*.$$

As $c'_i = c_i + c'_{ij} - c_{ij}$, above expression becomes

$$\dot{W}_i(t^*) = c_{1i} W_1^* + \cdots + c_{ni} W_n^* - c_i W_i^* - (c'_{ij} - c_{ij}) W_i^* = -(c'_{ij} - c_{ij}) W_i^* < 0$$

because by definition of the steady state W^* we have $c_{1i} W_1^* + \cdots + c_{ni} W_n^* - c_i W_i^* = 0$. Thus, after the shock $W_i(t)$ starts decreasing, meaning that $W_i^\# < W_i^*$ as predicted by Lemma 1. With the same procedure we can prove that $\dot{W}_j(t^*)$ is positive, which means that $W_j^\# > W_j^*$.

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[7] This is true not only for the (obvious) monotonic case, but also for the oscillatory one. This is because, after starting its convergence process, $W_i(t)$ can never revert to the original level as fluctuations are damped.

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