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REGULAR ARTICLE

Incorporating positions into asset pricing models with order-based strategies

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Abstract The paper addresses the problem of agent-based asset pricing models with order-based strategies that the implied positions of the agents remain indeterminate. To overcome this inconsistency, two easily applicable risk aversion mechanisms are proposed which modify the original actions of a market maker and the speculative agents, respectively. Here the concepts are incorporated into the classical Beja–Goldman model. For the deterministic version of the thus enhanced model a four-dimensional mathematical stability analysis is provided. In a stochastic version it is demonstrated that jointly the mechanisms are indeed able to keep the agents' positions within bounds, provided the corresponding risk aversion coefficients are neither too low nor too high. A similar result holds for the misalignment of the market price.

Keywords Beja–Goldman model · Financial positions of traders · Four-dimensional stability analysis · Stability reswitching · Misalignment

JEL Classification C 15 · D 84 · G 12

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1 Introduction

In theoretical financial economics it is now widely acknowledged that a paradigm shift has taken place from the representative agent with his or her rational expectations towards a behavioural approach, in which markets are populated by heterogeneous and boundedly rational agents who use rule-of-thumb strategies.[1] If we limit our interest to agent-based asset pricing models with in general non-zero excess demand, then in the vast majority of these cases the strategies of the speculative traders are (explicitly or implicitly) expressed in terms of their orders on the market. This, however, is a dubious feature since it is well-known that the agents' implied inventories are not anchored and so will be indeterminate. In a stochastic setting they may even easily diverge, which is certainly not compatible with the risk constraints of real traders.[2]

There are a few models in the recent literature that account for this problem by letting the traders' strategies directly focus on the desired positions. This alternative approach has been basically initiated by Farmer (2001) and Farmer and Joshi (2002).[3] Nonetheless, given the experience that in the meantime has been gained in all the models working with order-based strategies, it would not be expedient to discard them altogether in order to ensure consistency. It is instead a straightforward idea to explicitly introduce the agents' positions as an additional variable and put forward a simple rule that prevents them from diverging. Specifically it may be supposed that the speculative agents revise their original demand downward (upward) in proportion to the current positive (negative) deviations of their positions from some long-run target level. The proportionality factor can then be interpreted as a risk aversion parameter for these agents.

Regarding a market maker who changes the market price in response to the excess demand for the asset, one can conveniently resort to an idea by Farmer (2001). It is here proposed that if the market maker has accumulated a negative (positive) position, he seeks to encourage selling (buying) by proportionately raising (lowering) the price more than usual. Likewise, this proportionality factor can be said to measure the market maker's risk aversion.

The two concepts can be easily incorporated into any of the models with order-based strategies. An analytical treatment could still be possible if such a model is sufficiently simple. We employ the (deterministic) Beja–Goldman model with its market maker and the two archetypal groups of fundamentalists and chartists for this purpose. The elementary setup and the intuitive outcome that fundamentalists tend to stabilize and chartists tend to destabilize the market had a great influence on the agent-based modelling that has begun in the 1990s. The Beja–Goldman model is therefore a most suitable

1 Recent surveys on the by now voluminous literature are Hommes (2006), LeBaron (2006), Chen et al. (2008), Lux (2008) and Westerhoff (2009).

2 Disequilibrium models with a market maker that refer to mean–variance optimization or, what amounts to the same, to the maximization of expected wealth under a CARA utility function, are not exempt from the indeterminacy problem—unless they reinterpret their price adjustment equation and replace the agents' excess demand with the market maker's excess inventory in it. See the conceptual discussion in Franke (2008a) for a clarification of this statement.

3 Subsequent developments in this framework are Carvalho (2001) and Pape (2007). Franke (2007) takes a step back and sets up a prototype model with position-based strategies.

framework to test the validity of the new concepts, and a study of the thus enhanced model will be the main concern of this paper.

Assuming a conservative law for the market, the introduction of the positions of the agents and their feedback on the formulation of demand adds two other dimensions to the original (continuous-time) model. Our augmented Beja–Goldman model is thus four-dimensional. Fortunately the entries in the Jacobian matrix are so favourable that a number of meaningful stability conditions can be derived, in which the structural parameters of the Beja–Goldman model largely maintain their stabilizing or destabilizing role. Regarding the two new parameters, it will turn out that a high risk aversion of the speculative agents can always stabilize the market, whereas the market maker's risk aversion can, but need not, be destabilizing.

These qualitative insights are to be complemented by a study of a stochastic version of the model (in discrete time), where following a usual practice the price is randomly shocked every period and the fundamental value follows a random walk. The main result, in short, will be that for a moderate misalignment of the market price and moderate fluctuations of the positions of the agents, both risk aversion parameters must neither be too high nor too low. This finding underlines the significance of the simple correction mechanisms here proposed with which the agents seek to keep their positions within bounds.

The remainder of the paper is organized as follows. Section 2 deals with the deterministic continuous-time model. After a brief recapitulation of the Beja–Goldman model it introduces positions and the corresponding correction mechanisms. It then presents the results of the mathematical stability analysis. A grid search over the parameter space reveals that upon a *ceteris paribus* increase in some of the parameters also a reswitching of stability (but not instability) may occur. The stochastic dynamics are investigated in Sect. 3. For a detailed study of the time series characteristics of the variables in the model, two benchmark scenarios are set up; one where convergence in the deterministic counterpart is cyclical and one where it is monotonic. Subsequently we examine the effects of variations in the two risk aversion parameters on price misalignment, the inventory of the market maker, and also the profits of fundamentalists and chartists. Section 4 concludes, and the proofs of the mathematical propositions are collected in an appendix.

2 The deterministic model

2.1 A recapitulation of the Beja–Goldman model

Beja and Goldman (1980) distinguish three groups of participants in an asset market: two groups of speculators—fundamentalists and chartists—and a market maker. Fundamentalists have long time horizons and base their demand on the differences between the current price and the fundamental value. Even though they might expect the gap between the two prices to widen in the immediate future, they do not trade on these short-run expectations, choosing instead to place their bets on an eventual rapprochement. Chartists, on the other hand, neglect deviations of the price from the fundamental value and concentrate on short-run changes. They use past movements of prices as indicators of market sentiments and extrapolate these into the next periods. In general, demand and supply of fundamentalists and chartists will not match the (fixed) supply of the asset. It is the primary function of the market maker to mediate transactions out of equilibrium. He executes all orders at the present price and accordingly decumulates (or accumulates) inventories of the financial asset. His expectations and possible feedback effects from undesired inventories are ignored. So the demand of the market maker is treated as a residual and the only reaction considered here is the setting of a new price for the next trading round.

Let p be the log price and v the log of the fundamental value, on which the agents unanimously agree. As long as we are in a deterministic setting, here and in the following models, v is supposed to remain constant. The fundamentalists specified by Beja and Goldman buy (sell) when p is below (above) v. Chartist demand is characterized by a term π that represents their perceptions of the current trend of prices. Both types of demand, denoted by d^f and d^c, are specified in a linear manner. Introducing two positive coefficients ϕ and χ, which may be taken as indicative of the aggressiveness of fundamentalists and chartists, respectively, or of their share of wealth in total speculative capital, we have

$$d^f = \phi\,(v - p) \tag{1}$$

$$d^c = \chi\,\pi \tag{2}$$

To save an extra symbol for the supply of the asset, d^f and d^c may be interpreted as excess demand (in the verbal discussion we may also economize on the expression 'excess'). The dynamic relationships in the Beja–Goldman model are formulated in continuous time. Let δ denote the price impact of demand, i.e. δ is the coefficient that measures the extent to which the market maker changes the price in response to the current excess demand.[4] Accordingly, the evolution of prices is governed by the differential equation

$$\dot{p} = \delta\,(d^f + d^c) \tag{3}$$

The perception π of the trend by the chartists is a predetermined variable, for which a simple adaptive updating rule is postulated. As the current 'trend' towards which the chartists seek to adjust their expectations is given by the instantaneous price changes, the rule is specified as

$$\dot{\pi} = \alpha\,(\dot{p} - \pi) \tag{4}$$

where α represents the speed at which these adjustments are carried out. This equation completes the description of the Beja–Goldman model. Certainly, each of the four coefficients ϕ, χ, α and δ will vary over time, but in order to study the fundamental dynamic laws of the market they are treated as constants.

[4] It may be noted that δ has no time dimension and is thus *not* a speed of adjustment, as Beja and Goldman (1980, p. 237), themselves seem to imply; see the derivation of the market impact function in Farmer and Joshi (2002, p. 152f).

Substituting (1), (2) into (3) and the latter into (4), the four equations are reduced to two linear differential equations in the price p and the perceived trend π,

$$\begin{aligned} \dot{p} &= \delta\,[\,\phi\,(v - p) \,+\, \chi\,\pi\,] \\ \dot{\pi} &= \alpha\,[\,\delta\phi\,(v - p) \,+\, (\delta\chi - 1)\,\pi\,] \end{aligned} \qquad \text{(BG)}$$

They possess a unique equilibrium point $p = v$ and $\pi = 0$, which (apart from a fluke) is either attractive or repelling. Which of the two cases prevails depends on just one stability condition, which can be formulated as follows.

Proposition 1 *The equilibrium point of system (BG) is globally asymptotically stable if, and only if,*

$$\chi \;<\; \phi/\alpha \,+\, 1/\delta$$

and it is repelling if the inequality is reversed. A loss of stability under ceteris paribus *variations of a parameter occurs by way of a Hopf bifurcation.*

Proof The determinant of the Jacobian matrix of (BG) is unambiguously positive, $\det J \;=\; \alpha\delta\phi \;>\; 0$, which rules out saddle point instability and zero eigen-values. The trace is negative if and only if the inequality stated in the proposition is satisfied. Certainly, the stability thus ensured is global by virtue of the linearity of (BG). □

The simple condition in the proposition allows us to characterize the parameters of the model as either stabilizing or destabilizing, meaning that sufficiently high values render a possibly unstable equilibrium stable or, respectively, they turn a possibly stable equilibrium into an unstable one. In this sense, the aggressiveness of fundamentalists ϕ, or their weight in the market, is a stabilizing parameter. Conversely, low values of ϕ lead to instability if $\chi > 1/\delta$. The weight χ of chartists plays an opposite role. Sufficiently high vales of χ are always destabilizing, while sufficiently low values give rise to stability.

Presupposing $\chi > 1/\delta$, chartists can furthermore destabilize the equilibrium by a high adjustment speed α in the updating of their expectations about future price changes. Effectively, α is the rate at which old price changes are discounted, so that the instability feature may complementarily be also summarized by the statement that a short memory of chartists is destabilizing. A long memory (low α), on the other hand, can always ensure stability. Finally, a sufficiently sluggish price impact of demand δ is stabilizing, and a high responsiveness is destabilizing if $\alpha\chi > \phi$.

It may also be noted that since each of the four parameters enters the critical condition in Proposition 1 in a monotonic way, a reswitching of stability or instability as a parameter rises from zero to infinity cannot possibly occur.

Because of the linearity of system (BG), the Hopf bifurcation in Proposition 1 is, of course, degenerate; periodic orbits only come into existence at the bifurcation value itself. We mention the Hopf bifurcation, here and in other propositions to follow, as a convenient short-cut which indicates that the positive and negative feedback mechanisms in the model provide considerable scope for cyclical behaviour. There are thus phases in the dynamics where the destabilizing forces (stemming from the chartists) dominate and drive the price away from the fundamental value, and other phases where the stabilizing forces (of the fundamentalists) gain momentum again and reverse this tendency. This interplay, whether explicitly referred to or not, has made the Beja–Goldman model an attractive basis for stochastic and deterministic extensions that give rise to richer dynamic phenomena.

2.2 Introducing asset positions and risk aversion

Buying and selling of the asset means that the agents build up positions.[5] Generally the positions may be positive or negative, according to whether the agents go long or short. For each group of traders we are here concerned with their deviations a^f, a^c, a^m from some fixed long-run target levels, where the superscripts refer to the fundamentalists, chartists and the market maker, respectively. For a parsimonious model design, we employ four assumptions.

First, the market orders of the speculative agents are always fulfilled by the market maker, so that

$$\dot{a}^f = d^f, \quad \dot{a}^c = d^c \tag{5}$$

Second, receipts from an alternative riskless asset as well as dividends from shares or interest from foreign bonds (in case of a foreign exchange market) are neglected as a source of reinvestment and accumulation of wealth. In other words, the market is supposed to be closed and the total number of assets is conserved: every time an agent buys an asset, another agent loses it.[6] With the interpretation as deviations from a target, the agents' positions thus satisfy the following identity in every point in time,

$$a^f + a^c + a^m = 0 \tag{6}$$

Generally, the target levels of the agents' positions need not be fixed but might be thought of as adjusting one-to-one and immediately to variations in the total number of the assets issued, such that (6) is preserved. Practically, of course, the adjustments will be in both directions, depend on many exogenous factors, and involve longer delays. This is here short-circuited to obtain a convenient equilibrium notion where all positions are on target and can remain there in the absence of shocks.[7]

[5] Within a simple model that, in particular, maintains the (discrete-time counterpart of) the price impact equation (3), Day (1997) discusses the conditions under which the market maker can avoid outages and withdraw a positive dividend indefinitely. Closer to our paper is Sethi (1996), who incorporates the traders' inventory accumulation and cash flow into a Beja–Goldman framework. His assumptions, however, include a rationing device, from which we abstain.

[6] In Farmer (2001) this seems to be tacitly understood; in Farmer and Joshi (2002) it is explicitly stated on p. 154.

[7] Most other agent-based models cannot even discuss these conceptual issues but rest on similar—and implicit—assumptions.

A third point deals with the risk aversion of the market maker concerning his inventory, which takes up the empirical fact that market maker positions on stock markets have (or had) a half-life ranging from a few days to a week (Hansch et al. 1998); and the half-life seems to be even shorter on foreign exchange markets (Lyons 1998). A stylized way to incorporate this feature into the market maker's price quotes has been proposed by Farmer (2001, p. 66). His supposition basically maintains the price impact equation (3) from above but additionally states that if the market maker has accumulated a negative (positive) position, then this prompts him to encourage selling (buying) by raising (lowering) the price more than usual. The risk that in this way feeds back on the market is measured by a correction coefficient $\mu > 0$. Accordingly, our assumption on the price changes reads,[8]

$$\dot{p} = \delta\,(d^f + d^c - \mu\, a^m) \tag{7}$$

Apparently, this rule primarily relies on the behaviour of the fundamentalists, whereas the market maker does not systematically try to trigger the orders of the chartists into the desired direction (for which there are probably good reasons, at least in models with further types of technical traders or contrarians). In addition it is worth mentioning that a term like $-\mu\,(a^m - x^m)$, where x^m is the target level of the market maker's inventory, can be part of an optimal policy in a rigourous dynamic setting which, in particular, takes his uncertainty about the future arrival of tenders into account (Bradfield 1979).[9]

In the fourth assumption the basic idea of risk aversion is carried over to the speculative agents. They do not lose track of their accumulated positions, either, and take account of them by a similar correction term. So their market orders are now made up of two components: the original speculative one of Eqs. (1) and (2), respectively, and a second component that seeks to bring their current position in line with their target position. The latter adjustments take place in a gradual manner, and their intensity is measured by a positive coefficient η, which for simplicity is supposed to be the same for fundamentalists and chartists.[10] In this way, the demand equations (1) and (2) are modified as

$$d^f = \phi\,(v - p) - \eta\, a^f \tag{8}$$

$$d^c = \chi\,\pi - \eta\, a^c \tag{9}$$

[8] Westerhoff (2003, p. 366) has a similar idea but specifies it as a mechanism that, conditionally on the market maker's current inventory, switches between (in our notation) two coefficients δ_1 and δ_2 in Eq. (3). While being straightforward, the discontinuity in this approach prohibits an analytical treatment of stability conditions.

[9] The market structure in Bradfield's model is, however, more detailed than is usual in the present kind of modelling. A special result of his analysis is that, due to overnight costs, the coefficient μ depends on the time of the trading day: the later the time of the day, the less willing is the market maker to permit a^m to diverge from x^m, so that transaction prices tend to be more variable at that time of the day.

[10] Differentiated risk aversions η_f and η_c for fundamentalists and chartists would leave the results essentially unaltered; only the analytical expressions would become a bit more cumbersome. Conditions concerned with the stabilizing effects of the risk aversion mechanism could then be treated by postulating a constant ratio η_f/η_c (which typically may be larger than one).

Table 1 Parameters in the BGA model

ϕ	Weight of fundamentalists in the market
χ	Weight of chartists in the market
α	Adjustment speed of chartist trend revisions
δ	(risk adjusted) price impact of demand
μ	Risk aversion of the market maker
η	Risk aversion of fundamentalists and chartists

Equations (5)–(9) together with (4) constitute our augmented Beja–Goldman model; augmented, that is, by two straightforward mechanisms by which the market maker and the speculative agents seek to prevent their positions from drifting away.

Evidently, the positions feed back on the price and so we have now two of the three position variables as additional state variables; the third one can be expressed in terms of the two other by virtue of the closed market assumption (5). Linearity is preserved and one easily obtains the following four-dimensional set of differential equations in p, π, a^f, a^c:

$$\begin{aligned} \dot{p} &= \delta\,[\,\phi\,(v-p) \;+\; \chi\,\pi \;+\; (\mu-\eta)\,(a^f + a^c)\,] \\ \dot{\pi} &= \alpha\,[\,\delta\phi\,(v-p) \;+\; (\delta\chi - 1)\,\pi \;+\; \delta(\mu-\eta)\,(a^f + a^c)\,] \\ \dot{a}^f &= \phi\,(v-p) \;-\; \eta\,a^f \\ \dot{a}^c &= \chi\,\pi \;-\; \eta\,a^c \end{aligned} \tag{BGA}$$

The acronym BGA stands for Beja–Goldman augmented. Before going on, it may be convenient to list the six parameters of the model with their specific meanings. This is done in Table 1.

An obvious equilibrium point of (BGA) is given by $p = v$, $\pi = 0$, $a^f = a^c = a^m = 0$; the price coincides with the fundamental value and all three groups of agents have their positions on target. If both risk aversion coefficients μ and η are strictly positive, this is also the only equilibrium.[11] If $\mu = 0$ and $\eta > 0$, there is a continuum of equilibria where a^f can attain any value. π and a^c are zero then and a^f determines how much the price will deviate from the fundamental value, as $p = v - \eta\,a^f/\phi$ in such a state. On the other hand, if $\eta = 0$ and $\mu \geq 0$, both positions a^f and a^c can attain arbitrary values and give rise to the equilibrium price $p = v + \mu(a^f + a^c)/\phi$. Trivially, with $\mu = \eta = 0$ we are back in the original Beja–Goldman model with its indeterminate positions. Nevertheless, in the following the ordinary equilibrium notion ensured by $\mu > 0$ and $\eta > 0$ will be presupposed.

2.3 Temporal structure through the market maker

In assessing the role of the market maker it is interesting to observe that his or her risk reducing behaviour introduces a certain persistence into the model. To quote Farmer

[11] Without going into any details, the uniqueness of the equilibrium is an immediate consequence of the fact that, as made explicit shortly below, the Jacobian matrix of (BGA) has full rank.

(2001, p. 67): "Once the market maker acquires a position, because of her risk aversion, she has to get rid of it. By selling a fraction β [our coefficient μ] at each time step, she will unload the position a bit at a time. This behaviour causes a trend in prices. Any risk-averse behaviour on the part of the market maker will result in a temporal structure of some sort in prices."

Generally, it may be concluded that sufficiently high values of the market maker's risk aversion μ reinforce a monotonic tendency toward the equilibrium, or they may even cause the price to overshoot its equilibrium value. If the latter does not lead to instability, cyclical trajectories would result. The market maker's potential for generating oscillatory behaviour would, however, be hard to identify in a model where pronounced cyclical tendencies are already present—because of the speculative demand of the chartists.

Therefore, in order to disentangle the two possible sources of cyclical behaviour and study the consequences of a higher risk aversion of the market maker in a pure form, let us in this subsection temporarily discard the chartists from the model. System (BGA) is then reduced to two dimensions, the price and the positions of the fundamentalists,

$$\begin{aligned} \dot{p} &= \delta\,[\,\phi\,(v-p) \,+\, (\mu-\eta)\,a^f\,] \\ \dot{a}^f &= \phi\,(v-p) \,-\, \eta\,a^f \end{aligned}$$

The determinant of the Jacobian of this system is computed as det $J = \delta\phi\,\mu$ and the trace as $-\delta\phi - \eta$, which ensures that this fundamentalist market is unambiguously stable. It is furthermore well-known that the eigen-values of the Jacobian are complex if the discriminant (trace $J)^2 - 4\,\det J$ is negative. Checking this condition and defining the critical value $\mu^c := [\delta\phi \,+\, (2+\eta/\delta\phi)\,\eta]\,/\,4$, it is easily seen that convergence is monotonic if $\mu < \mu^c$, and cyclical if $\mu > \mu^c$. On this basis it may be said that the risk aversion of the market maker can indeed be an additional cycle generating mechanism.

2.4 Mathematical stability analysis

Investigating the stability of a four-dimensional dynamic system is usually a formidable task. The mixed terms in the Routh–Hurwitz stability conditions easily get so complicated that economically meaningful and incisive results are no longer obtainable. In the present case, however, the Jacobian matrix J is relatively benign. Equation (10) gives a first impression that it is not so much the number of zero entries, since there are only three of them, but the structure of the entries that will facilitate the computations:

$$J = \begin{bmatrix} -\delta\phi & \delta\chi & \delta(\mu-\eta) & \delta(\mu-\eta) \\ -\alpha\delta\phi & \alpha(\delta\chi-1) & \alpha\delta(\mu-\eta) & \alpha\delta(\mu-\eta) \\ -\phi & 0 & -\eta & 0 \\ 0 & \chi & 0 & -\eta \end{bmatrix} \tag{10}$$

Indeed, a first ray of hope for the mathematical analysis is the determinant of this matrix, which is always positive and as simple as $\det J = \alpha\,\delta\,\phi\,\mu\,\eta > 0$. Hence stability is not ruled out a priori, as it would be with a negative determinant.

It will, of course, be expected that the stability conditions for (BGA) are not too different from those for (BG) if the risk aversion coefficients μ and η are small. As a matter of fact, without much effort we even get the stronger result that equality of the two coefficients, at what level so ever, is the relevant benchmark case (regarding the effort in establishing this feature, see the proof in the appendix). It goes without saying in the following that because of the linearity of (BGA), 'stability' means global stability.

Proposition 2 *If $\mu = \eta$, the equilibrium of (BGA) is asymptotically stable if and only if the equilibrium of (BG) is asymptotically stable.*

Let us then turn to the general case. Clear-cut instability statements, though they will not be the sharpest possible, are readily obtained by checking that the trace of the Jacobian is positive. As the upper 2×2 submatrix of J is identical to the Jacobian of (BG), instability of (BGA) prevails under quite similar conditions (regardless of the level of μ). A negative trace of J, on the other hand, is only one out of four conditions that have to be satisfied to ensure stability. As a consequence, the stability frontier can no longer be described in an equally simple way to the equation of a zero trace. It is, however, still possible to derive a number of qualitative stability conditions. The next proposition collects the statements for the four parameters that both models (BG) and (BGA) have in common. Again, the proof is given in the appendix.

Proposition 3 (a) *The equilibrium is asymptotically stable if the weight of fundamentalists ϕ is sufficiently large*; *given that $\delta\chi > 1 + 2\eta/\alpha$, the equilibrium is unstable if ϕ is sufficiently small.*

(b) *The equilibrium is asymptotically stable if both the weight of chartists χ and the price impact of demand δ are sufficiently small; instability prevails if χ is sufficiently large or, given that $\alpha\chi > \phi$, if δ is sufficiently large.*

(c) *The equilibrium is asymptotically stable if the speed of chartist trend adjustments α is sufficiently low*; *given that $\chi > 1/\delta$, the equilibrium is unstable if α is sufficiently large.*

(d) *A loss of stability under* ceteris paribus *variations of a parameter occurs by way of a Hopf bifurcation.*

The conditions for stability and instability are the same as in the verbal characterization of the critical condition in Proposition 1 for (BG). The only exception is that in order for low values of ϕ to be destabilizing, Proposition 3(*b*) presupposes $\delta\chi > 1+2\eta/\alpha$ rather than $\delta\chi > 1$. Any extension of the Beja–Goldman model that builds on the stabilizing or destabilizing properties of the parameters ϕ, χ, α, δ should, therefore, produce very similar effects when it is carried over to the BGA framework.

To complete the mathematical stability analysis, Proposition 4 summarizes what can be derived for the two risk aversion coefficients μ and η, which are the constituent parameters of the augmented part of Beja–Goldman.

Proposition 4 (a) *The equilibrium is asymptotically stable if the speculative agents' risk aversion η is sufficiently large.*

(b) *If the equilibrium is asymptotically stable for all sufficiently large value of the maker's risk aversion μ, then it is also asymptotically stable if μ gets sufficiently small.*

(c) *If $\alpha\chi \geq \phi$, the equilibrium is unstable for all large values of μ.*

(d) *Admit $\alpha\chi \geq \phi$ but suppose the price impact of demand δ is so small that $\delta(\alpha\chi - \phi) < \min\{2\eta, 2\alpha + \eta\}$ and $\delta(\eta\chi - \phi) < 2\eta$. Then the equilibrium is asymptotically stable if μ is sufficiently small.*

(e) *If the equilibrium is stable for all μ sufficiently small and unstable for all μ sufficiently large, then there is a unique bifurcation value μ^H such that the equilibrium is asymptotically stable if $0 < \mu < \mu^H$ and unstable if $\mu > \mu^H$.*

In the last part of Proposition 4 we are able to establish that if a loss of stability occurs as μ is rising, then this bifurcation is uniquely determined. Multiple bifurcations were also generally ruled out in the original Beja–Goldman model, that is, a reswitching of stability or instability upon *ceteris paribus* variations of one of the parameters. For the five coefficients ϕ, χ, α, δ, η in (BGA), however, an analytical treatment of the reswitching issue is no longer possible. In fact, to this end the sign of the composite term b in the Routh–Hurwitz conditions has to be evaluated, which changes from positive to negative as the equilibrium changes from stable to unstable.[12] It turns out that b is a quadratic function of μ, but at least a cubic function of the other coefficients. In addition, a bifurcation at a parameter value bringing about $b = 0$ requires that the other four Routh–Hurwitz terms are all still positive (a_1, a_2, a_3, a_4 in Eq. (12) in the Appendix). Hence the polynomial character of the functions $b = b$(parameter) suggests that reswitching may now become possible, but the conditions one has to check are too complicated to be analytically tractable.

2.5 Additional numerical findings

It is sufficiently informative to address the reswitching issue by a straightforward numerical investigation. To this end we set up a five-dimensional grid of five of the coefficients (the 'exogenous' parameters) and, for each grid point, consider b as a function of the remaining sixth coefficient (the 'inner' parameter). Computing this function over a certain interval, where b is set to some negative value if one of the a_k terms just mentioned is negative, we only have to record the number of sign changes. The grid of the exogenous parameters is made up of $21^5 = 4,084,101$ points (equally spaced), and the function $b(\cdot)$ is evaluated at 101 values of the inner parameter (likewise equally spaced). The intervals within which the six parameters vary are given in Table 2. As the intervals are fairly wide, we can have some reasonable confidence in the uniqueness of the bifurcation of an inner parameter if no reswitching was found for it. That is, if still some reswitching events were missed, they may be regarded as rather special since only the density of the grid was not sufficient to detect them.

[12] See Eq. (12) in the Appendix and the remark near the end of the proof of Proposition 4.

Table 2 Changes in stability upon *ceteris paribus* increases of parameters

Parameter	Switching	Reswitching	Interval in grid search
ϕ	U–S	S–U–S (80)	0.05–2.00
χ	S–U	None	0.05–6.00
α	S–U	S–U–S (19)	0.01–1.00
δ	S–U	S–U–S (5,712)	0.05–2.00
μ	S–U	None	0.01–2.00
η	U–S	None	0.01–2.00

U–S (*S–U*) means the equilibrium changes from being unstable to stable (from stable to unstable) as the parameter increases, if the bifurcation is unique over the interval considered. Figures in parentheses are the number of reswitching cases (across more than 4 million grid points of the remaining five parameters; see description in the text)

The second column in Table 2 shows for each parameter the normal cases of how stability changes (if a regime change occurs at all at a given set of exogenous parameters). Thus, as the inner parameter rises from the lower to the upper bound of its interval, the equilibrium may change from unstable (U) to stable (S) or *vice versa*, where it should be pointed out that for all unique bifurcations of a parameter, the change in stability is always in the same direction. In this sense we can succinctly summarize that stabilizing are: the weight of fundamentalists ϕ and the risk aversion of the speculative agents η; the other four parameters are destabilizing: the weight of chartists χ, their trend adjustment speed α, the price impact of demand δ, and the risk aversion of the market maker μ.

Cases of ambiguity have been observed for ϕ, α and δ, though they are rare exceptions if the number of the reswitching events in the fourth column of the table is related to the roughly four million points of the wide and unconstrained grid of the exogenous parameter constellations. In any case, the price impact δ happens to be the parameter with the by far highest likeliness of reswitching. It may also be noted that the reswitching is always of the order stable–unstable–stable, regardless of whether the normal changes are from stable to unstable (the parameters α and δ) or from unstable to stable (the parameter ϕ). In sum, Table 2 provides a compact characterization of the stability properties of the augmented Beja–Goldman model.

To carry the reswitching issue a little bit further, two parameters may be varied simultaneously, though in an economically meaningful way. Let us thus assume that a higher risk aversion μ of the market maker goes along with a higher risk aversion η of the speculative agents. More specifically, consider distinct risk attitudes (to avoid the conclusion from Proposition 2) and let η be linked to μ as $\eta = \mu$ minus a positive constant c. If $\alpha\chi \geq \phi$, Proposition 4(c) tells us that the equilibrium would be unstable if μ gets large and if η remained fixed. If, however, η increases in step with μ then, according to Proposition 4(a), there should nevertheless be some scope for stability, at least if c is large enough. On the other hand, if $\delta\,(\alpha\chi - \phi) < 2\alpha$, also small values of μ entail stability according to Proposition 4(d). For a suitable choice of ϕ, χ, α, δ this should still hold when μ is slightly larger than c (so that $\eta > 0$). Figure 1, which

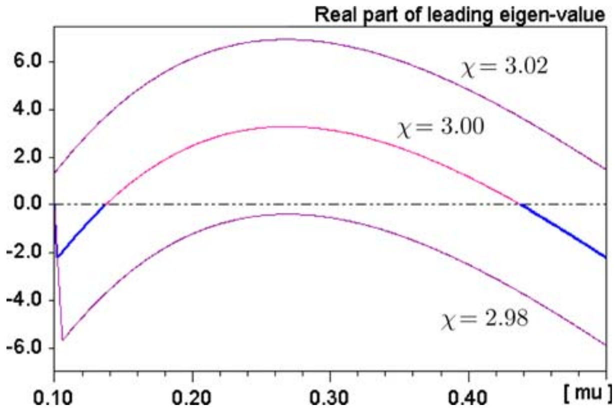

Fig. 1 Real part of the leading eigen-value under variations of μ and $\eta = \mu - 0.10$. *Note*: The values of the real part are multiplied by 10^3. Underlying are the parameters $\phi = 1.00$, $\alpha = 0.75$, $\delta = 0.50$, and χ as indicated

plots the real part of the leading eigen-value (the one with maximum real part) as a function of μ, gives an example of a scenario where all these condition can indeed be met. Again, the reswitching is of the type stable–unstable–stable; see the middle line. The two other functions in the diagram, for χ slightly below and above its scenario value 3.00, illustrate that the reswitching phenomenon is quite sensitive to variations of the other parameters.

3 Fluctuations in a stochastic setting

3.1 Distinguishing a cyclical and a monotonic scenario

With regard to the financial market models with order-based strategies, this paper has started out from the general weakness that, in their present formulation, the positions of the agents are only residually determined from their flow demands. Hence the positions cannot be guaranteed to remain within reasonable bounds if the asset price undergoes persistent oscillatory behaviour. By contrast, the agents in our model are supposed to be fully aware of the boundedness problem and cope with it by counteracting any divergent tendencies in their positions. The speculative agents correspondingly correct their market orders, while the market maker corrects the price he quotes to encourage additional sales and purchases that would reduce his excess inventory.

In order to test whether these mechanisms are really able to fulfill their purpose for the agents, the model must be modified in such a way that it can generate incessant price fluctuations. A straightforward idea to achieve this is to transform the model into discrete time and introduce stochastic shocks in the structural equations. A particularly challenging test for the boundedness of the positions is given by the assumption that the fundamental value is no longer constant but evolves like a random walk. This feature will carry over to the market price, unless it does not disconnect from the fundamental value (which would be another aspect that we will have to have an eye on).

Table 3 Common parameters to both scenarios

χ	α	δ	μ	$\sigma_p \times 100$	$\sigma_v \times 100$
3.00	0.20	1.00	0.50	0.30	0.50

A second type of shocks often employed in the literature are (uncorrelated) random events that induce the market maker to somewhat deviate from the strict price impact rule. In this way the following stochastic version of the BGA model is obtained, which we write down in recursive form and to which we may refer as (BGA–S):

$$
\begin{aligned}
a_t^m &= -a_t^f - a_t^c \\
d_t^f &= \phi\,(v_t - p_t) - \eta\, a_t^f \\
d_t^c &= \chi\, \pi_t - \eta\, a_t^c \\
p_{t+1} &= p_t + \delta\,(d_t^f + d_t^c - \mu\, a_t^m) + \sigma_p\, \varepsilon_{p,t} \qquad \varepsilon_{p,t} \sim N(0,1) \\
v_{t+1} &= v_t + \sigma_v\, \varepsilon_{v,t}\,, \qquad\qquad\qquad\qquad\qquad\quad \varepsilon_{v,t} \sim N(0,1) \\
a_{t+1}^f &= a_t^f + d_t^f \\
a_{t+1}^c &= a_t^c + d_t^c \\
\pi_{t+1} &= \pi_t + \alpha\,(p_t - p_{t-1} - \pi_t)
\end{aligned}
\qquad \text{(BGA–S)}
$$

The random shocks $\varepsilon_{p,t}$ and $\varepsilon_{v,t}$ are drawn from two independent unit normal distributions (with mean zero and variance one); they are scaled by the two standard deviations σ_p and σ_v in the corresponding adjustment equations. Regarding the perceived trend it is supposed that the agents do not yet know the new price when updating π_t. For concreteness, the time unit underlying (BGA–S) will be conceived of as one day.

For the numerical investigation of the stochastic market a base scenario has to be set up. Of course, the model's linearity requires the deterministic skeleton to be asymptotically stable. It should, however, be taken into account that the type of convergence might matter, that is, whether the equilibrium price would be approached in a monotonic or cyclical manner. To check this possibility two such benchmark scenarios will be put forward. They only differ in two coefficients, the risk aversion parameter η and the weight of the fundamentalists ϕ. The values of the other parameters, which are common to both scenarios, are given in Table 3. While the order of magnitude of σ_p for the additive price shocks is obvious, it may be noted that the standard deviation σ_v in the random walk of the fundamental value amounts to an annual volatility of 7.9% (since with 250 market days, $0.079/\sqrt{250} = 0.005$).

Apart from the type of convergence, the two scenarios should be as similar as possible. For simplicity, we use the eigen-values of the deterministic continuous-time system to assess their comparability, positing that the first two distinct eigen-values have identical real parts of -0.20 and -0.30, respectively. This choice entails that convergence takes place at a medium speed.[13] In the scenario with cyclical convergence, or the cyclical scenario (CS) for short, the first two eigen-values are conjugate complex and the third one is real. Conversely, the first eigen-value is real and the next

13 According to the real part of the leading eigen-value in the continuous-time framework, 90 per cent of an initial gap would be closed after 11.5 days.

Table 4 The cyclical and monotonic scenario (CS and MS, respectively)

	ϕ	η	Eigen-values		Log returns	
			First/second	Second/third	SD	AC 1
CS	0.892	0.300	$-0.2000 \pm 0.433i$	$-0.3000 + 0i$	0.764	0.106
MS	1.142	0.200	$-0.2000 + 0i$	$-0.3002 \pm 0.494i$	0.864	−0.121

'SD' and 'AC 1' are the standard deviation and first-order autocorrelation of the daily log returns $r_t = p_t - p_{t-1}$ (25,000 observations, SD in percent)

two are conjugate complex in the scenario with monotonic convergence, which we call the monotonic scenario (MS). The values of ϕ and η bringing this about are given in Table 4, where it can be observed that the real eigen-value coincides with the value assigned to η.

On the basis of a sufficiently long sample run of 25,001 days for each of the two scenarios, Table 4 also shows that the scaling of the parameters is quite acceptable in the sense that the standard deviation of the daily log returns r_t is within a reasonable range, at least as far as the period before the 1987 stock market crash is concerned (afterwards this standard deviation is somewhat higher).[14] A low positive first-order autocorrelation of r_t is also compatible with that period, whereas after the crash it tends to become insignificant.[15] As might have been expected, the cyclical scenario gives rise to a positive autocorrelation of the returns and the monotonic scenario to a negative autocorrelation. If desired, these coefficients could be driven down to zero by sufficiently increasing the noise in prices relative to the noise in the random walk of the fundamental value.

Nevertheless, the numerical parameters presented in Tables 3 and 4 cannot be taken as an attempt to fit the stylized facts of the daily returns. The model is still much too simple to match the empirical higher order autocorrelations, let alone the slow decay in the autocorrelation function of the absolute or squared returns.[16]

3.2 Prices, positions and profits in the two scenarios

In this subsection we concentrate on the two scenarios just set up and study their dynamic properties in greater detail. To this end, as already mentioned, CS and MS

[14] For example, over the periods January 1970 until March 1987 and January 1960 until December 1980, respectively, the S&P 500 stock market index yields standard deviations of 0.878 and 0.773 for the returns r_t.

[15] Over the sample periods of the previous footnote, the first-order autocorrelation of r_t for the S&P 500 is still 0.181 and 0.207, respectively. For the period 1988–1999 it is almost zero.

[16] In this respect one might think of the model by He and Li (2007), which could be viewed as a Beja–Goldman model with a more flexible formulation of demand and which does generate autocorrelation patterns with the desired features. In Franke (2008b) it is, however, shown that this goes at the price of a questionable specification of the model's random shocks. It is furthermore argued that a matching of the stylized facts is still a severe challenge to this (and the Brock–Hommes) type of structural models. We conjecture that if in these purely order-based models some progress is made, this will also carry over if they are augmented by our mechanisms in Eqs. (5)–(9).

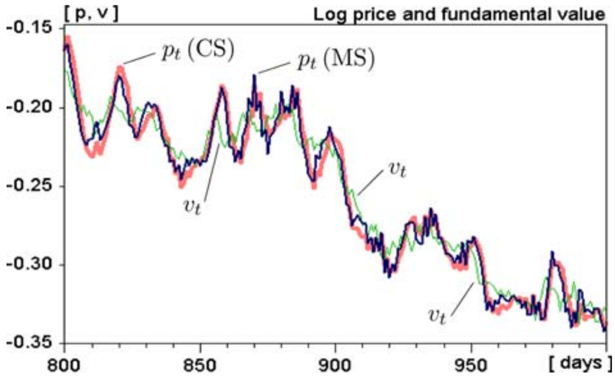

Fig. 2 Fundamental value v_t and price p_t in the cyclical (*CS*) and monotonic scenario (*MS*)

were simulated over 100 years = 25,001 days, where the prices and fundamental values were subjected to identical random shocks. Let us first consider the model's central variable, the (log) market price p_t and its relationship to the (log) fundamental value v_t.

Despite the differences in the two summary statistics of the log returns in Table 4, the evolution of the price appears to be quite similar in the two scenarios. This is illustrated in Fig. 2 for a time span of 201 days. The price series for both the cyclical scenario: the bold (red) line, and the monotonic scenario: the solid (blue) line, remain fairly close to the fundamental value, which is depicted as the thin (green) line. Common to the two series is also that both of them overshoot the fundamental value when it takes a turn after a temporary rise or decline; or already when such a motion slows down a little (at around $t = 880$ or $t = 900$, for example). It cannot even be said that CS produces a systematically stronger overshooting: sometimes it is the turning point of CS and sometimes the turning point of MS that deviates more from the fundamental value.

Only a numerical calculation discloses that, on average and in accordance with what one will have been expected, the cyclical scenario leads to a stronger misalignment. That is, writing $\sigma(x_t)$ for the standard deviation of a time series x_t, we get $\sigma(p_t - v_t) = 1.341\%$ for the cyclical scenario in Fig. 2, and $\sigma(p_t - v_t) = 1.184\%$ for the monotonic scenario. In the complete simulation runs the differences are somewhat more pronounced, though they are still quite moderate; see the second column in Table 5.

Figure 3 illustrates the answer to the boundedness problem of the agents' positions. It shows that the present choice of the risk aversion parameters η is for all three groups of the agents sufficient to keep their positions within a limited range. This basic feature holds for the monotonic and the cyclical scenario alike, the time series of the positions are indeed quite congruent. The bandwith obtained in Fig. 3 is fairly representative for the entire simulation runs.

The positions of the market maker are moreover distinctly centred, already in the short run. On the other hand, besides their larger amplitude, the positions of the fundamentalists and chartists fluctuate at a lower frequency, with an average period between 15 and 20 days. So, over the 200 days shown in Fig. 3, especially the positions of chartists do not exhibit a similarly clear pattern of centring. There is nevertheless an

Table 5 Summary statistics of the two scenarios

		RMSD of positions			Profits		
	misal.	F	C	MM	F	C	MM
CS	1.368	2.56	2.29	1.01	6.24	−2.13	−4.11
					(14.62)	(11.90)	(6.69)
MS	1.173	2.87	2.53	1.02	7.30	−2.61	−4.69
					(18.35)	(14.91)	(7.67)

'misal.' stands for misalignment, specified as $100 \cdot \sigma(p_t - v_t)$. RMSD are the root mean square deviations from zero (times 100). The last three columns show the average profits (times 10^5) per day with their standard deviations in parentheses; see Eq. (11) below. F, C, MM denote fundamentalists, chartists, and the market maker, respectively

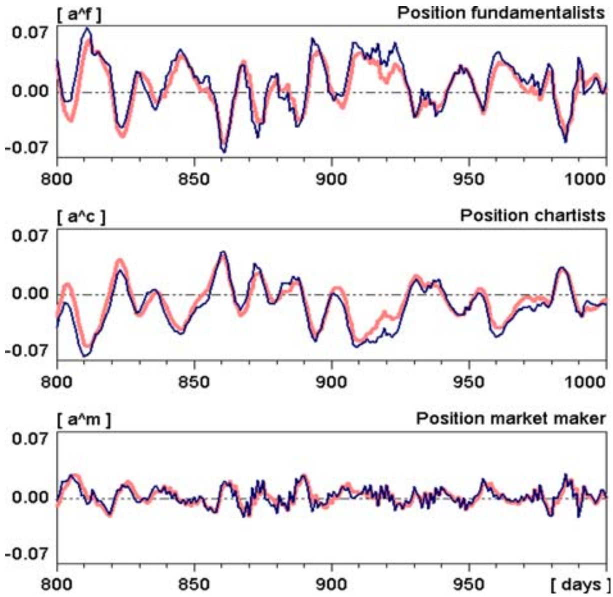

Fig. 3 Positions in CS (*bold line*) and MS (*thin line*)

obvious tendency for the positions to return to their target levels within a rather short period of time, even after a strong decline of the fundamental value (between $t = 895$ and $t = 920$ in Fig. 2, for example). At least over a longer time horizon of 2 or 4 years we would not hesitate to characterize the positions a_t^f and a_t^c as being well centred, too.

The 'amplitude' of the fluctuations of the agents' positions is conveniently measured by their root mean square deviation from the target (RMSD). Table 5 reports these statistics for the full sample period (the values for the period covered by Fig. 3 are not very different). While it confirms the considerably lower fluctuations of the market maker, there are only minor differences in the RMSD of a_t^f and a_t^c. Note, however, that these are the positions of all members of the group of fundamentalists and chartists. So without a more detailed specification of the individual trading capital invested, nothing can be said about the relative variability that originates with a fundamentalist versus a chartist strategy as such.

What can be directly compared is only the RMSD across different parameter sets. In this respect the market maker's RMSD is practically the same in both scenarios, whereas, perhaps somewhat surprisingly and in contrast to the misalignment, the variability of the fundamentalist as well as of the chartist positions is slightly higher in the monotonic scenario rather than in the cyclical scenario. This is also true for the time series shown in Fig. 3.

We may also have a look at the profits to be earned with trading. For group h of the agents ($h = f, c, m$) we consider the daily capital gains g_t^h, which under the simplifying assumptions on dividend and interest rates are exactly given by the change in the value of the current positions,[17]

$$g_t^h = [\exp(p_t) - \exp(p_{t-1})]\, a_t^h \tag{11}$$

As an immediate consequence of the conservation law (6) for positions, the positive gains of one group always go at the expense of at least one of the other groups: $g_t^f + g_t^c + g_t^m = 0$ in each period t.

The long-run time averages of the daily profits (scaled by 10^5) are reported in the last three columns of Table 5. Quite in line with general (academic) wisdom, fundamentalists win and chartists lose in the long-run. Nevertheless, as the large standard deviations in parentheses show, the fundamentalists do not earn their profits without risk, which would be especially true for individual traders with limited capital. On the other hand, even for the straightforward chartist strategy specified by Beja–Goldman there are still some prospects for temporary capital gains. Fundamentalists fare a bit better and chartists fare a bit worse in the monotonic scenario relative to the cyclical scenario, but note that the standard deviations of the daily profits increase, too (even more than proportionately for the fundamentalists).

Insofar as he passively absorbs the excess demand and supply on the market, the market maker is systematically losing money. In principle, he could make good for this by specific active trading on his own account or by charging transaction fees (directly or indirectly through bid-ask spreads), which in turn would reduce the net profits of the speculative traders. However, the many small-scale models in the literature with only a few groups of agents usually do not go into these details but discuss them at most informally.

3.3 Variations of the risk aversion parameters

After getting some basic insights into the properties of the stochastic dynamics (BGA–S), we can now turn to the behavioural parameters and study their effects

[17] To see this, let $P_t = \exp(p_t)$ be the price of the asset, c_t the cash holdings of a group and W_t their wealth. Then with $c_t = c_{t-1} - P_{t-1}d_{t-1} = c_{t-1} - P_{t-1}(a_t - a_{t-1})$ one has $g_t = W_t - W_{t-1} = P_t a_t + c_t - P_{t-1}a_{t-1} - c_{t-1} = P_t a_t - P_{t-1}a_{t-1} - P_{t-1}(a_t - a_{t-1}) = (P_t - P_{t-1})a_t$.

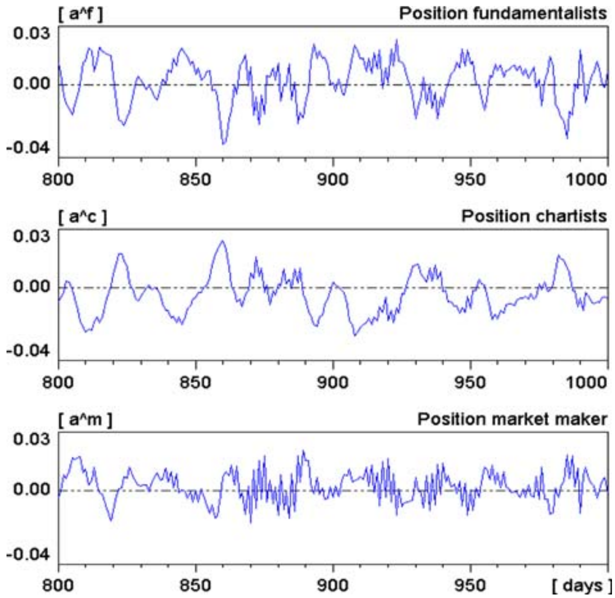

Fig. 4 Positions in the cyclical scenario with $\eta = 0.60$

on the misalignment of prices and the other summary statistics discussed above. We limit ourselves to the *ceteris paribus* variations of the model's innovative parameters, the risk aversion coefficients η and μ. Let us begin with the risk aversion of fundamentalists and chartists, which was identified as a stabilizing parameter. While in the continuous-time model η can be arbitrarily large, it has a natural upper bound of unity in the discrete-time setting. Already values $\eta < 1$ may give rise to unreasonable behaviour in the evolution of the agents' positions, which would induce them to change their η or another parameter. Figure 4, where in the cyclical scenario η is increased from 0.30 to $\eta = 0.60$, is an example of the problems that may typically arise.

If Fig. 4 is compared to the time series over the same sample period in Fig. 3, first the lower scale of the three panels may be noticed. Hence higher values of the risk aversion η fulfill their purpose of narrowing down the range within which the positions of the speculative agents are fluctuating. On the other hand, the main cyclical pattern of a_t^f and a_t^c from the base scenario in Fig. 3 is still clearly visible; there is only more noise in the very short term. This weak trembling in a_t^f and a_t^c, however, adds up and leads to a more pronounced raggedness in the positions of the market maker. In Fig. 4 these features are still in their infancy, but they become stronger as η is further increased. Eventually they carry over to the price itself. It then exhibits extreme fickleness and alternates in moving up and down from one day to another. As a result, along with the RMSD of a_t^m also the standard deviation of of returns and the price misalignment $\sigma(p_t - v_t)$ will rise. We also note that the same phenomenon is observed if η is increased in the monotonic scenario.

The two top panels in Fig. 5 show that the misalignment and the fluctuations in a_t^m begin to worsen at around $\eta = 0.80$. From then on small increments in η have

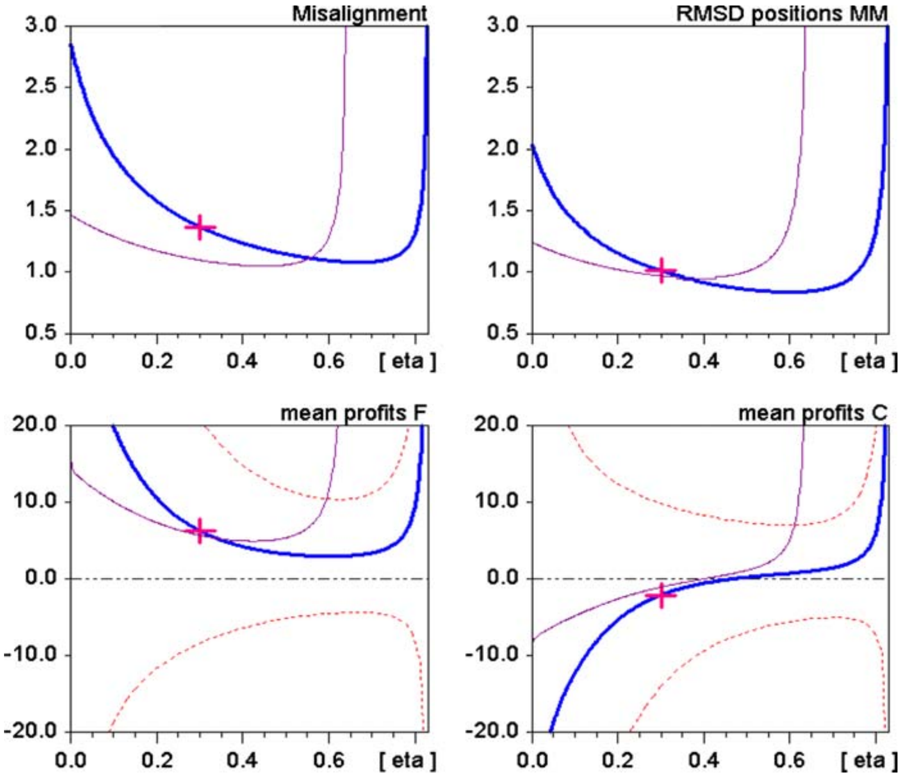

Fig. 5 Variations of the speculative agents' risk aversion η *Note*: *Bold (thin) lines* refer to the parameter values from CS (MS). Misalignment $\sigma(p_t - v_t)$ and RMSD multiplied by 100. The *dotted lines* indicate a one standard deviation band around the mean profits in CS ($\times 10^5$). Underlying is a sample period of 25,000 days. The *crosses mark* CS with $\eta = 0.30$

dramatic effects on the summary statistics (see the bold lines). At the upper end of the parameter interval, $\eta = 0.83$, misalignment rises to 4.40% and the RMSD of the market maker's positions (multiplied by 100) to 5.79. Increasing η only slightly above 0.83 causes the amplitude of the fluctuations to explode. The thin lines in the diagrams indicate that in the monotonic scenario the upper limit of the risk aversion coefficient for bounded price dynamics is even lower (around $\eta = 0.65$).

If conversely in the cyclical scenario η decreases from $\eta = 0.30$, the real part of the leading eigen-value of (BGA) declines, too, and so weakens the stabilizing forces in the system. At $\eta = 0.140$ the dynamics become monotonic, in the sense that from then on the leading eigen-value is real and equal to $-\eta$. It follows that the decrease in η increases the misalignment as well as the RMSD of the market maker's positions, though these statistics tend to finite and quite moderate values as η approaches zero.

It should be added that as η gets (very) small, the positions of the speculative agents begin to wander around (while the positions of the market maker remain centered around zero). As the bottom panels of Fig. 5 show, this indeterminacy is associated with a strong increase in the profits of fundamentalists and the losses of chartists; with an even larger increase in the corresponding standard deviations. Hence at least the chartists would not maintain their behaviour. Besides their technical rule they will certainly design a rule that intents to prevent their positions from deviating so persistently from target. Most straightforwardly, they will considerably increase η in the original formulation of chartist demand, Eq. (9). The observation that the implications of the indeterminacy of the positions a_t^f and a_t^c at $\eta \approx 0$ can indeed be quantitatively significant underlines the general and qualitative criticism that the purely order-based strategies provide no anchor for the agents' positions.

Regarding rising values of η it is interesting to note that they mildly increase the average profits of the chartists relative to the fundamentalists. This holds in the cyclical as well as in the monotonic scenario. Because of the thus increasing short-term noise in positions and prices that was pointed out above, these higher values of η cannot be sold as "more plausible". However, they can call attention to additional aspects that more ambitious models may take into account. The agents may not only try alternative mechanisms to predict the market price, they may also vary the intensity of the control mechanism with which they seek to keep the positions within bounds.

If for the sake of the argument we imagine a situation where an increase of η affects the market maker's inventory in undesired ways, he may react by a suitable change in his control parameter. This brings us to the effects of *ceteris paribus* variations in the second risk aversion coefficient, μ, which are studied in Fig. 6.

Although generally μ was found to be destabilizing, in the present two scenarios, the cyclical and the monotonic one, the continuous-time equilibrium is stable for all values of this parameter. However, similarly to what was obtained for the speculative agents in the stochastic setting when their risk aversion η increases, a moderately higher risk aversion μ of the market maker diminishes the RMSD of his positions, whereas a strong risk aversion introduces too much short-term noise. At the upper bound of the interval considered in Fig. 6, $\mu = 2.23$, the market maker's inventory as well as the price he quotes typically go up for two consecutive days, and then again decrease for the next 2 days. The price does not disconnect from the fundamental value v_t, but its short-term fluctuations around v_t become excessively large. The quantitative evidence for these features is given in the top panels of Fig. 6.

At the other end of the parameter interval, as μ tends to zero, the marker maker's inventory loses contact with its target level. Note that owing to the conservation law (6) this would not happen if the positions of the speculative agents remained largely on target. In fact, the positions of chartists do; their RMSD is even lower than in CS and MS with their positive value of μ. Responsible for the persistent one-sided deviations of a_t^m from zero are the opposite and equally persistent deviations of a_t^f from zero.

This phenomenon corresponds to the observation made at the end of Sect. 2.2 that $\mu = 0$ and $\eta > 0$ give rise to a continuum of equilibrium points, where $a^m = -a^f$ can attain any value and induce price deviations at the order of $-\eta a^f/\phi$ from the fundamental value. Accordingly, the random forces in (BGA–S) cause the price to

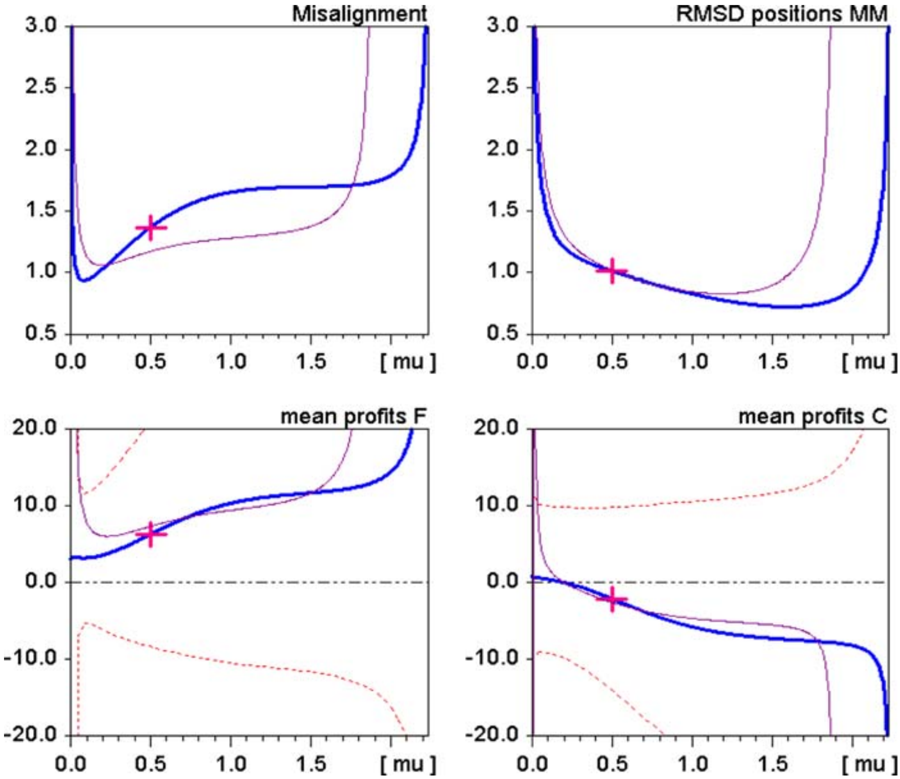

Fig. 6 Variations of the market maker's risk aversion μ

disconnect from the fundamental value after a while, and it then shows no tendency to return to it. The reference to the deterministic equilibrium concept makes it also clear that a higher risk aversion on the part of the fundamentalists would not be sufficient to eliminate this systematic misalignment. It follows that in the deterministic and stochastic framework both risk aversion coefficients μ and η are needed to prevent any indeterminacy in the agents' positions and its implication for the market price.

It is also interesting to note that the value of the market maker's risk aversion that minimizes the RMSD for his position is quite distinct from the one that entails a minimal misalignment. This observation seems to be of more general importance since a similar result holds in the model by Westerhoff (2003, p. 367), despite its different specification of the market maker's price quotes.

Lastly, we briefly mention the two bottom panels in Fig. 6, which show the impact of the variations of the market maker's risk aversion on the profits of the other traders. The positive effects of an increase in μ for the fundamentalists and the negative effects for the chartists are, however, quite limited—as long as μ is sufficiently bounded away from the end-points of the parameter interval with their unreasonable behaviour.

4 Conclusion

The paper has advanced two elementary mechanisms for speculative agents and the market maker which, jointly, can overcome the fundamental inconsistency in the many asset pricing models with order-based strategies, according to which the agents' implied positions are indeterminate and so may easily diverge. To begin with, the modelling device keeps track of the positions. For the speculative agents it is then supposed that they change their original demand from the order-based strategies inversely to the deviations of their positions from a long-run target level. Concerning the market maker it is supposed that he quotes a lower (higher) price than usual in proportion to the current positive (negative) deviations of his inventory from normal. The two proportionality factors measure the agents' risk aversion in this respect.

Incorporating these concepts into the (deterministic and continuous-time) Beja–Goldman model increased its dimension from two to four but still allowed an analytical treatment of stability. As expected, the formal stability conditions for the structural parameters of the Beja–Goldman model were not dramatically affected. A grid search over the parameter space, however, revealed that now some of these parameters can be ambiguous; in the sense that the equilibrium is stable at low values of such a parameter, then turns to unstable as the parameter rises, and eventually becomes stable again as it rises further (though occurrence of this reswitching phenomenon appeared to be quite rare). Regarding the new risk aversion coefficients the mathematical analysis established that the one characterizing the speculative agents is stabilizing, and the one characterizing the market maker is, if anything, destabilizing.

In the (discrete-time) stochastic version of the model where a random walk of the fundamental value brings in a considerable degree of exogenous price variability, it was demonstrated that the two correction mechanisms are indeed able to keep the agents' positions within bounds, and also to check the misalignment of the market price. In order for the corresponding summary statistics to be limited, we observed that the risk aversion of the market maker as well as of the speculative agents should neither be too low nor too high.

The mechanisms proposed in this paper can be easily employed in other models with order-based strategies. Since in many of these models the original properties of the Beja–Goldman model are still shining through, even under endogenous population shares of fundamentalists and technical traders, it can be expected that the two risk aversion coefficients of the speculative agents and the market maker (if he is present) will produce similar effects in other applications. In any case, our risk aversion mechanisms are a comfortable option to ensure consistency in the evolution of the agents' positions, especially since the basic structure of the original model will be left intact.

Whether in the Beja–Goldman model or another framework, we may finally mention a specific issue that has already been touched upon in the discussion of the profits to be earned on the market. It has there been indicated that the agents may not only change their forecasting strategies for the market price but also their risk aversion parameters, i.e. the intensity of their revision mechanism. This also applies to the market maker. As it has been found that excessively low or large parameter values will not be in the agents' interest, one could think of designing evolutionary rules that also include alternative choices for these coefficients. In sum, the two risk aversion concepts put forward in this paper in general, and the last idea in particular, may somewhat widen the perspective of the agent-based modelling of financial markets.

Appendix

Proof of Proposition 2 We first point out that two eigen-values of the Jacobian J are given by $\lambda_1 = \lambda_2 = -\eta < 0$. In fact, it is immediately seen that $x^1 = (0, 0, 1, 0)'$ and $x^2 = (0, 0, 0, 1)'$ bring about the eigen-value equations $Jx^k = -\eta x^k$ for $k = 1, 2$.

The other two eigen-values of J coincide with the eigen-values of its upper-left 2×2 submatrix, which we denote by $J^{(2)}$. We verify this by indicating an eigen-vector $x \in \mathbb{R}^4$ that, with respect to an eigen-value λ of $J^{(2)}$, satisfies $Jx = \lambda x$. Two cases have here to be distinguished. Suppose first that λ equals $-\eta$. Then, by virtue of $j_{13} = j_{14} = j_{23} = j_{24} = 0$, a corresponding eigen-vector is given by $x = (0, 0, 1, 1)'$. For the second case $\lambda \neq -\eta$ let $(v_1, v_2)'$ be an eigen-vector of the submatrix $J^{(2)}$, and let x_3 and x_4 solve the equations $-\phi v_1 - \eta x_3 = \lambda x_3$ and $\chi v_2 - \eta x_4 = \lambda x_4$, respectively. The eigen-vector of J associated with λ is then $x = (v_1, v_2, x_3, x_4)'$. (Incidentally, the present Jacobian is an example that the eigen-vectors need not change continuously with the entries of the matrix.) □

Proof of Proposition 3 The Hopf bifurcation part (d) of the proposition is an immediate consequence of the fact that the determinant of the Jacobian J is always different from zero, so that zero eigen-values are ruled out and eigen-values can cross the imaginary axis in the complex plane only in (non-degenerate) conjugate complex pairs.

All of the instability statements of the proposition derive from the condition of a positive trace of J. Stability is proved by checking the Routh–Hurwitz conditions for a 4×4 matrix or, more precisely, the equivalent Liénart–Chipart conditions; see (Gandolfo, 1997, p. 223, Eqs. (16.37a) and (16.37c)). Entering them are the following terms:

$$
\begin{aligned}
a_1 &= -\text{trace}\, J \;=\; \delta\phi + \alpha(1-\delta\chi) + 2\eta \\
a_2 &= \text{sum of the principal second-order minors of } J \\
&= \begin{vmatrix} -\delta\phi & \delta\chi \\ -\alpha\delta\phi & \alpha(\delta\chi-1) \end{vmatrix} + \begin{vmatrix} -\delta\phi & \delta(\mu-\eta) \\ -\phi & -\eta \end{vmatrix} + \begin{vmatrix} -\delta\phi & \delta(\mu-\eta) \\ 0 & -\eta \end{vmatrix} \\
&\quad + \begin{vmatrix} \alpha(\delta\chi-1) & \alpha\delta(\mu-\eta) \\ 0 & -\eta \end{vmatrix} + \begin{vmatrix} \alpha(\delta\chi-1) & \alpha\delta(\mu-\eta) \\ \chi & -\eta \end{vmatrix} + \begin{vmatrix} -\eta & 0 \\ 0 & -\eta \end{vmatrix} \\
&= \alpha\delta\phi + \delta\phi\mu + \delta\phi\eta + \alpha(1-\delta\chi)\eta + \alpha(1-\delta\chi)\eta + \alpha\delta\chi(\eta-\mu) + \eta^2 \\
a_3 &= -\;(\text{sum of the principal third-order minors of } J) \\
&= -\begin{vmatrix} -\delta\phi & \delta\chi & \delta(\mu-\eta) \\ -\alpha\delta\phi & \alpha(\delta\chi-1) & \alpha\delta(\mu-\eta) \\ -\phi & 0 & -\eta \end{vmatrix} - \begin{vmatrix} -\delta\phi & \delta\chi & \delta(\mu-\eta) \\ -\alpha\delta\phi & \alpha(\delta\chi-1) & \alpha\delta(\mu-\eta) \\ 0 & \chi & -\eta \end{vmatrix}
\end{aligned}
$$

$$- \begin{vmatrix} -\delta\phi & \delta(\mu-\eta) & \delta(\mu-\eta) \\ -\phi & -\eta & 0 \\ 0 & 0 & -\eta \end{vmatrix} - \begin{vmatrix} -\alpha(\delta\chi-1) & \alpha\delta(\mu-\eta) & \alpha\delta(\mu-\eta) \\ 0 & -\eta & 0 \\ \chi & 0 & -\eta \end{vmatrix}$$

$$= \alpha\delta\phi\mu \; + \; \alpha\delta\phi\eta \; + \; \delta\phi\mu\eta \; + \; \alpha\eta(\eta - \delta\chi\mu)$$

$$a_4 = \det J \;=\; \alpha\delta\phi\mu\eta$$

The eigen-values of J have negative real parts if, and only if,

$$a_k > 0 \quad \text{for } k = 1, 2, 3, 4\,, \quad b = a_1\,a_2\,a_3 \;-\; a_1^2\,a_4 \;-\; a_3^2 > 0 \tag{12}$$

($a_2 > 0$, or alternatively $a_3 > 0$, is implied by the other inequalities.)

To demonstrate the stability part of Proposition 3(a), note first that a_1, a_2, a_3, a_4 are all positive for large values of ϕ. Write these terms as linear functions of ϕ, $a_k = c_k + d_k\phi$ ($k = 1, 2, 3, 4$), with $c_4 = 0$ and $d_1 = \delta$, $d_2 = \delta(\alpha + \mu + \eta)$, $d_3 = \delta(\alpha\mu + \alpha\eta + \mu\eta)$, $d_4 = \alpha\delta\mu\eta$. The composite term b is a cubic function of ϕ, where the coefficient on ϕ^3 is given by

$$d_1\,d_2\,d_3 - (d_1)^2 d_4 = \delta^3\,[(\alpha + \mu + \eta)(\mu\eta + \alpha\mu + \alpha\eta) \;-\; \alpha\mu\eta)] \;>\; 0$$

Thus $b > 0$ if ϕ is chosen sufficiently large.

Regarding the stability part of part (b), consider the limiting case $\chi = \delta = 0$. This gives $a_1 = \alpha + 2\eta > 0$, $a_2 = \eta^2 + 2\alpha\eta > 0$, $\alpha_3 = \alpha\eta^2$, $a_4 = 0$. Furthermore,

$$b \;=\; (a_1\,a_2 - a_3)\,a_3 = [(\alpha + 2\eta)(\eta^2 + 2\alpha\eta) - \alpha\eta^2]\,a_3 \;>\; 0$$

The inequalities are preserved if χ and δ are slightly increased, while a_4 turns positive in this case.

The proof of the stability part of Proposition 3(c) proceeds in a similar way. Putting $\alpha = 0$ gives $a_1 = \delta\phi + 2\eta > 0$, $a_2 = \delta\phi(\mu+\eta) + \eta^2 > 0$, $a_3 = \delta\phi\mu\eta > 0$, $a_4 = 0$. The term b is positive if

$$a_1\,a_2 - a_3 = (2\eta + \delta\phi)(\delta\phi\mu + \delta\phi\eta + \eta^2) \;-\; \eta\delta\phi\mu \;>\; 0$$

which clearly is the case. Sufficiently small values of $\alpha > 0$ maintain these inequalities and achieve $a_4 > 0$. □

Proof of Proposition 4 To begin with part (a), write the a_k in the Routh–Hurwitz conditions as functions of the parameter η: $a_1 = 2\eta + c_1$, $a_2 = \eta^2 + d_2\eta + c_2$, $a_3 = \alpha\eta^2 + d_3\eta + c_3$, $a_4 = \alpha\delta\phi\mu\,\eta$, where c_1, c_2, c_3, d_2, d_3 are some constants. Certainly, all a_k are positive if η is large enough. Furthermore, the term b is a fifth-order polynomial of η, with a positive coefficient on η^5 (from the product $a_1a_2a_3$). Hence b, too, is positive for large values of η.

Regarding the other parts of the proposition, write the a_k as functions of μ, $a_k = c_k + d_k\,\mu$, where $d_1 = c_4 = 0$. The nonzero slope coefficients are $d_2 = \delta(\phi - \alpha\chi)$, $d_3 = \alpha\delta\phi + \eta d_2$, $d_4 = \alpha\delta\phi\eta$, and the intercepts $c_1 = \alpha + 2\eta + d_2$, $c_2 = \alpha\delta\phi + (2\alpha + \eta + d_2)\eta$, $c_3 = \alpha(\eta + \delta\phi)\eta$.

To prove part (c) note that its assumption means $d_2 \leq 0$. If the strict inequality prevails, a_2 will eventually become negative for large μ. If $d_2 = 0$, consider the coefficient on μ^2 in the composite term b, which is given by $(c_1 d_2 - d_3)\, d_3 = -(d_3)^2 < 0$. Hence eventually b becomes negative if μ is large enough.

Regarding part (d) it suffices to check the stability conditions for $\mu = 0$. The first inequality in the additional assumption is tantamount to $d_2 + 2\eta > 0$ and $2\alpha + \eta + d_2 > 0$. Hence $a_k = c_k > 0$ for $k = 1, 2, 3$. As the composite term reduces to $b = (c_1 c_2 - c_3)c_3 - 0$, we only need to verify $c_1 c_2 - c_3 > 0$. The first product is given by $c_1 c_2 = (\alpha + 2\eta + d_2)\,(\eta^2 + \delta\phi\eta + \delta\phi\alpha - \delta\chi\eta\alpha + 2\eta\alpha) = \alpha\eta^2 + \alpha\delta\phi\eta + \alpha[\delta(\phi - \chi\eta) + 2\eta]\,\alpha + (2\eta + d_2)\, c_2$. The first two terms cancel against $-c_3$, the square bracket is positive by the second inequality assumption, and the rest is already known to be positive. Hence $b > 0$, too.

In the proof of part (e), existence of at least one (Hopf) bifurcation value μ^H is obvious, as well as that there must be an odd number of such bifurcations. Then, note that any loss of stability is tantamount to a switching of the composite term b from positive to negative. This follows from the fact that at any such bifurcation we have $a_1 > 0$, $a_3 > 0$, $a_4 > 0$ and $b = 0$ (see Theorem 2(ii) in Asada and Yoshida (2003, p. 527)), while $b > 0$ under these circumstances would imply $a_2 > 0$ and thus stability. So it remains to recall that b is a quadratic function of μ, which implies that it has at most two roots. Hence μ^H is uniquely determined.

To prove part (b) it again suffices to establish the stability conditions for $\mu = 0$. The stability assumption for large μ implies $d_2 > 0$ (see part (c)) and, thus, $c_1 > 0$, $c_2 > 0$, $c_3 > 0$. So, as before, it remains to show $c_1 c_2 > c_3$. Putting $\eta = 0$ we have $c_1 c_2 = (\alpha + d_2)\,\alpha\delta\phi > 0 = c_3$. Differentiation with respect to η gives us $\partial(c_1 c_2)/\partial\eta = 2\,(\alpha\delta\phi + \eta^2 + 2\alpha\eta + d_2\eta) + (\alpha + d_2 + 2\eta)\, 2\eta +$ positive terms $> 2\alpha\delta\phi + 2\alpha\eta > 2\alpha\eta + \alpha\delta\phi = \partial(c_3)/\partial\eta$. Hence $c_1 c_2 > c_3$ for all values of $\eta > 0$, which completes the proof of the proposition. □

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