Chin. Ann. Math. Ser. B
45(3), 2024, 319–348
DOI: 10.1007/s11401-024-0019-3

Chinese Annals of
Mathematics, Series B

A Dual Yamabe Flow and Related Integral Flows*

Jingang XIONG[1]

(*In memory of my father*)

Abstract The author studies a family of nonlinear integral flows that involve Riesz potentials on Riemannian manifolds. In the Hardy-Littlewood-Sobolev (HLS for short) subcritical regime, he presents a precise blow-up profile exhibited by the flows. In the HLS critical regime, by introducing a dual Q curvature he demonstrates the concentration-compactness phenomenon. If, in addition, the integral kernel matches with the Green's function of a conformally invariant elliptic operator, this critical flow can be considered as a dual Yamabe flow. Convergence is then established on the unit spheres, which is also valid on certain locally conformally flat manifolds.

Keywords Hardy-Littlewood-Sobolev functional, Dual Q curvature, Integral flow
2000 MR Subject Classification 45K05, 35B33

1 Introduction

1.1 Background and motivations

Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and

$$[g_0] = \{\rho g_0 : \rho \in C^\infty(M),\ \rho > 0\}$$

be the conformal class of g_0. The Yamabe problem asks whether there exists a metric $g \in [g_0]$ of constant scalar curvature. Recall that the conformal Laplacian of g_0 on M is given by

$$L_{g_0} := \Delta_{g_0} - c(n)R_{g_0},$$

where Δ_{g_0} denotes the Laplace-Beltrami operator associated with the metric g_0, $c(n) = \frac{n-2}{4(n-1)}$ and R_{g_0} represents the scalar curvature of g_0. It satisfies the conformal transformation law

$$L_{u^{\frac{4}{n-2}} g_0}(\phi) = u^{-\frac{n+2}{n-2}} L_{g_0}(u\phi), \quad \forall\ \phi \in C^2(M).$$

Hence, the Yamabe problem amounts to seeking a solution of

$$-L_{g_0} u = cu^{\frac{n+2}{n-2}} \quad \text{on } M, \quad u > 0$$

Manuscript received December 27, 2023. Revised January 23, 2024.
[1]School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing Normal University, Beijing 100875, China. E-mail: jx@bnu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 12325104, 12271028).

for some constant c. The existence was proved by the variational method through Yamabe [38], Trudinger [37], Aubin [1] and Schoen [33]. In 1980s, Hamilton introduced the Yamabe flow

$$\partial_t g = -(R_g - r_g)g,$$

where $t \geq 0$ and $r_g = \mathrm{Vol}_g(M)^{-1} \int_M R_g \,\mathrm{d}\,\mathrm{vol}_g$ and $\mathrm{Vol}_g(M)$ denotes the volume of M with respect to the metric g. The Yamabe flow is the normalized negative L^2 gradient flow of the Yamabe functional

$$F_2[g] := \frac{\int_M R_g \,\mathrm{d}\,\mathrm{vol}_g}{\left(\int_M \mathrm{d}\,\mathrm{vol}_g\right)^{\frac{n-2}{n}}}$$

and the conformal class is preserved along the flow. Thus it has a scalar form

$$\frac{n-2}{(n-1)(n+2)} \partial_t u^{\frac{n+2}{n-2}} = L_{g_0} u + c(n) r_g u^{\frac{n+2}{n-2}}.$$

The convergence was established by Chow [11], Ye [39], Schwetlick-Struwe [34] and Brendle [5].

Analogously, there has been much interest in the fourth-order Yamabe-type problem. If $n \geq 5$, let

$$P_{g_0} := \Delta_{g_0}^2 - \mathrm{div}_{g_0}(a_n R_{g_0} g + b_n \mathrm{Ric}_{g_0}) d + \frac{n-4}{2} Q_{g_0},$$
$$Q_{g_0} := -\frac{1}{2(n-1)} \Delta_{g_0} R_{g_0} + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_{g_0}^2 - \frac{2}{(n-2)^2} |\mathrm{Ric}_{g_0}|^2$$

be the Paneitz operator (see [31]) and the Q curvature (see [4]), respectively. The Paneitz operator satisfies the conformal transformation law

$$P_{u^{\frac{4}{n-4}} g_0}(\phi) = u^{-\frac{n+4}{n-4}} P_{g_0}(u\phi), \quad \forall\, \phi \in C^4(M).$$

Hence, the Yamabe type problem for Q curvature is equivalent to solving

$$P_{g_0}(u) = c u^{\frac{n+4}{n-4}} \quad \text{on } M, \quad u > 0$$

for some constant c. However, both the variational method and potential fourth-order flow approach encounter difficulties in obtaining a positive object.

Recently, Gursky-Malchiodi [16] established the existence of solutions to the 4th order Yamabe problem under the assumption that there is a conformal metric with nonnegative scalar and Q curvature, and that the Q curvature is strictly positive at some points. Hang and Yang [20] demonstrated existence on manifolds of positive Yamabe type, provided there exists a conformal metric whose Q curvature is nonnegative and positive at certain points. Both assumptions of [16, 20] imply that P_{g_0} is invertible, positive and its Green's function is positive. Gursky-Malchiodi's proof employs the normalized $W^{2,2}$ gradient flow of the total Q curvature functional

$$F_4[g] = \mathrm{Vol}_g(M)^{-\frac{n-4}{n}} \int_M Q_g \,\mathrm{d}\,\mathrm{vol}_g, \tag{1.1}$$

whereas Hang-Yang's proof leverages a dual variation problem (see [30] for a survey). We also refer to [21, 32] for earlier results.

The flow in [16] is given by

$$\partial_t u = -u + \mu(t)(P_{\overline{g}})^{-1}(|u|^{\frac{n+4}{n-2}}) \tag{1.2}$$

and satisfies initial condition $u(\cdot, 0) = 1$, where $\overline{g} \in [g_0]$, $(P_{\overline{g}})^{-1}$ is the inverse of $P_{\overline{g}}$ and $\mu(t)$ is a normalization. They showed that the flow converges to a solution of the fourth-order Yamabe problem, gave that $F_4[\overline{g}]$ falls below a natural threshold. However, exploring the dynamic behavior when $\overline{g}$ is any element within the conformal class $[g_0]$ remains an intriguing problem.

In addition to the conformal Laplacian and Paneitz operator, there are many other important conformally invariant operators of fractional and higher orders. See [7–8, 14, 17–18] and many others. The fractional Yamabe flow has been studied by Jin-Xiong [23], Daskalopoulos-Sire-Vázquez [12] and Chan-Sire-Sun [6].

Finally, we note that differential integral flows of (1.2) type were frequently used to deform level sets of functionals in the critical point theory. In particular, it plays a crucial role in [2] about the Nirenberg problem on the three dimensional unit sphere. In this paper, we would like to conduct a detailed study of this type of flow.

1.2 A general framework and main theorems

Let M be a compact smooth Riemannian manifold of dimension $n \geq 1$, and $K_0 : M \times M \to (0, \infty]$ be a C^1 singular kernel of the Riesz potential type. Namely, for any $X, Y \in M$,

$$K_0(X, Y) = K_0(Y, X), \tag{K-1}$$

$$\frac{1}{\Lambda} d_{g_0}(X, Y)^{2\sigma - n} \leq K_0(Y, X) \leq \Lambda d_{g_0}(X, Y)^{2\sigma - n}, \tag{K-2}$$

$$|\nabla_{g_0} K_0(\cdot, Y)| \leq \Lambda d_{g_0}(\cdot, Y)^{2\sigma - n - 1} \quad \text{on } M \setminus \{Y\}, \tag{K-3}$$

where d_{g_0} is the distance function with respect to metric g_0, $0 < \sigma < \frac{n}{2}$ and $\Lambda \geq 1$ are constants.

Define

$$\mathcal{K}_{g_0}(f)(X) := \int_M K_0(X, Y) f(Y) \, \mathrm{d}\,\mathrm{vol}_{g_0}(Y) \quad \text{for } f \in L^1(M).$$

For $T > 0$, we study the Cauchy problem for the differential-integral equation

$$\begin{cases} \partial_t u^m = \mathcal{K}_{g_0}(u) & \text{on } M \times (0, T), \\ u(\cdot, 0) = u_0 \geq 0, \end{cases} \tag{1.3}$$

where $m > 0$ and $u_0 \in C^0(M)$ is not identical to zero. By Riesz potential estimates and Picard-Lindelöf theorem, (1.3) has a unique solution with $u^m \in C^1([0, T^*); C^0(M))$ for some $T^* > 0$ representing the maximum existence time.

If $m > \frac{n-2\sigma}{n+2\sigma}$, $m \neq 1$, the steady problem

$$\mathcal{K}_{g_0}(S) = S^m \quad \text{on } M, \quad S > 0 \tag{1.4}$$

has a continuous solution. Moreover, the solution is unique if $m > 1$. See Section 2 below.

Theorem 1.1 *Let u be a solution of* (1.3) *with $u^m \in C^1([0, T^*); C^0(M))$ for some $T^* > 0$ representing the maximum existence time. Then we have the following results.*

(i) *If* $m>1$, *then* $T^*=\infty$ *and*

$$\lim_{t\to\infty}\|t^{-\frac{1}{m-1}}u(\cdot,t)-cS(\cdot)\|_{C^0(M)}=0.$$

(ii) *If* $\frac{n-2\sigma}{n+2\sigma}<m<1$, *then* $T^*<\infty$ *and*

$$\lim_{t\to(T^*)^-}\|(T^*-t)^{-\frac{1}{m-1}}u(\cdot,t)-cS(\cdot)\|_{C^0(M)}=0.$$

(iii) *If* $m=\frac{n-2\sigma}{n+2\sigma}$, *then* $T^*<\infty$ *and*

$$\lim_{t\to(T^*)^-}(T^*-t)^{-\frac{1}{m-1}}\|u(\cdot,t)\|_{L^{\frac{2n}{n+2\sigma}}}=c.$$

Here S *is a solution of* (1.4) *and* c *is positive constants.*

One can obtain precise convergence rates for the above items (i)–(ii), as exemplified by Jin-Xiong-Yang [25] in their study of a nonlinear boundary control problem. Section 6 provides essential tools such as the linearized operator and eigenfunctions for further exploration. Our proof is inspired by previous studies on the Cauchy-Dirichlet problem for the porous medium equations (see [25] and references therein).

For the linear case $m=1$, we also have $T^*=\infty$ and

$$\lim_{t\to\infty}\|\mathrm{e}^{-t}u(\cdot,t)-\phi_1(\cdot)\|_{C^0(M)}=0,$$

where $\phi_1>0$ is a positive eigenfunction associated to the largest eigenvalue of $\mathcal{K}_{g_0}$. If $0<m<\frac{n-2\sigma}{n+2\sigma}$, we still have $T^*<\infty$. But the blow-up behavior is unclear.

When $m=\frac{n-2\sigma}{n+2\sigma}$, the flow has an independent interest in conformal geometry as follows. Let

$$[g_0]_0=\{u^{\frac{4}{n+2\sigma}}g_0: u\in C^0(M),\ u>0\}$$

be the C^0 conformal class of g_0. For any $g=u^{\frac{4}{n+2\sigma}}g_0\in[g_0]_0$, we let

$$\mathcal{K}_g(f)(X):=\int_M K_g(X,Y)f(Y)\,\mathrm{d}\,\mathrm{vol}_g(Y)\quad\text{for } f\in L^1(M),$$

where

$$K_g(X,Y)=(u(X)u(Y))^{-\frac{n-2\sigma}{n+2\sigma}}K_0(X,Y).$$

By definition, $\mathcal{K}_g$ is a conformally invariant operator and

$$\mathcal{K}_{u^{\frac{4}{n+2\sigma}}g_0}(\phi)=u^{-\frac{n-2\sigma}{n+2\sigma}}\mathcal{K}_{g_0}(u\phi)\quad\text{for all } \phi\in L^1(M).$$

This motivates us to define

$$Q^g_{K_0}:=\mathcal{K}_g(1)\tag{1.5}$$

and

$$F_{K_0}[g]:=\mathrm{Vol}_g(M)^{-\frac{n+2\sigma}{n}}\int_M Q^g_{K_0}\,\mathrm{d}\,\mathrm{vol}_g.\tag{1.6}$$

We may call $Q^g_{K_0}$ a dual Q curvature, which is related to a quantity introduced by Zhu [40] and Han-Zhu [19] for a specific kernel, although they are distinct. The normalized L^2 gradient flow of $F_{K_0}[\cdot]$ takes the form

$$\partial_t g = (Q^g_{K_0} - a(t))g, \tag{1.7}$$

where $t \geq 0$ and $a(t) = \int_M Q^g_{K_0} \,\mathrm{d}\,\mathrm{vol}_g$. By setting $g = u^{\frac{4}{n+2\sigma}} g_0$, we obtain

$$\partial_t u^{\frac{n-2\sigma}{n+2\sigma}} = \mathcal{K}_{g_0}(u) - a(t) u^{\frac{n-2\sigma}{n+2\sigma}}. \tag{1.8}$$

Suppose that

$$u(\cdot, 0) = u_0(\cdot) \geq 0 \quad \text{on } M, \tag{1.9}$$

and $u_0 \in C^0(M)$ is not identical to zero.

If K_0 matches with the Green's function of some invertible conformally invariant differential operator, we may call (1.7) (or (1.8)) a dual Yamabe flow as it stems from the dual type functional $F_{K_0}[g]$.

On the standard n-sphere $\mathbb{S}^n$ equipped with the induced metric g_0 from $\mathbb{R}^{n+1}$, there defines the intertwining operator $P^{g_0}_\sigma$ which is the pull-back operator of the $\sigma \in \left(0, \frac{n}{2}\right)$ power of the Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection

$$P^{g_0}_\sigma(\phi) \circ F = |J_F|^{-\frac{n+2\sigma}{2n}} (-\Delta)^\sigma (|J_F|^{\frac{n-2\sigma}{2n}} (\phi \circ F)),$$

where F is the inverse of the stereographic projection and $|J_F|$ is the determinant of the Jacobian of F. Moreover,

$$(P^{g_0}_\sigma)^{-1}(f)(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{f(\zeta)}{|\xi - \zeta|^{n-2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0}(\zeta) \quad \text{for } f \in L^1(\mathbb{S}^n), \tag{1.10}$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma}\pi^{\frac{n}{2}}\Gamma(\sigma)}$, $p \geq 1$ and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$. The classical result of [29] asserts that

$$\sup\Big\{ \int_{\mathbb{S}^n} f (P^{g_0}_\sigma)^{-1}(f) \,\mathrm{d}\,\mathrm{vol}_{g_0} : \int_{\mathbb{S}^n} |f|^{\frac{2n}{n+2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0} = 1 \Big\} := S_{n,-\sigma} \tag{1.11}$$

is achieved and the maximizers have to be form

$$\overline{U}_{\xi_0,\lambda}(\xi) = \Big(\frac{2\lambda}{2 + (\lambda^2 - 1)(1 - \cos d_{g_0}(\xi, \xi_0))} \Big)^{\frac{n+2\sigma}{2}} \quad \text{for some } \lambda > 0,\ \xi_0 \in \mathbb{S}^n \tag{1.12}$$

upon a sign. Finally, we remark that by taking $\mathcal{K}_{g_0} = (P^{g_0}_\sigma)^{-1}$ and using the stereographic projection, (1.8) can be transformed into the prototype

$$\partial_t v^{\frac{n-2\sigma}{n+2\sigma}} = (-\Delta)^{-\sigma} v - a(t) v^{\frac{n-2\sigma}{n+2\sigma}} \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

where $v = |J_F|^{\frac{n-2\sigma}{2n}} (u \circ F)$.

Our second theorem is as follows.

Theorem 1.2 *The Cauchy problem* (1.8)–(1.9) *has a unique solution satisfying* $u^{\frac{n-2\sigma}{n+2\sigma}} \in C^1([0,\infty); C^0(M))$. *Moreover, as* $t \to \infty$,

$$a(t) \to a_\infty \quad \textit{for some positive constant } a_\infty$$

and

$$\int_M |Q^g_{K_0} - a(t)|^q \,\mathrm{d}\,\mathrm{vol}_g \to 0 \quad \textit{for each } 1 \le q \le \frac{2n}{n-2\sigma} + \frac{n+2\sigma}{n-2\sigma}.$$

This theorem guarantees that the flow is a Palais-Smale flow-line of the associated critical functional (see Corollary 5.5). Consequently, following [36] one can show bubbling or global compactness if

$$\lim_{X \to Y} d_{g_0}(X,Y)^{n-2\sigma} K_0(X,Y) = c \quad \text{uniformly for } Y \in M, \tag{K-4}$$

where $\frac{1}{\Lambda} \le c \le \Lambda$ is a constant, and without loss of generality we assume $c = c_{n,\sigma}$.

Finally, we state a convergence result on $\mathbb{S}^n$.

Theorem 1.3 *Suppose that* $M = \mathbb{S}^n$ *and* $\mathcal{K}_{g_0} = (P^{g_0}_\sigma)^{-1}$ *as in* (1.10). *If* u *is a positive solution of* (1.8) *on* $\mathbb{S}^n \times (0,\infty)$ *and* $u(0)^{\frac{n-2\sigma}{n+2\sigma}} \in C^1(\mathbb{S}^n)$, *then* $u \in C^1(\mathbb{S}^n \times (0,\infty))$ *and*

$$\lim_{t\to\infty} \|u(\cdot,t) - c\overline{U}_{\xi_0,\lambda}(\cdot)\|_{C^0(M)} = 0 \quad \textit{on } \mathbb{S}^n \times [1,\infty),$$

where $c > 0$ *is constant and* $\overline{U}_{\xi_0,\lambda}(\cdot)$ *is the function in* (1.12).

A crucial step in the proof of the above theorem is establishing a differential Harnack inequality, which is inspired by [39]. However, in the current integral framework, we need to borrow ideas from [9] and [27]. The same differential Harnack inequality also holds on the family of locally conformally flat manifolds considered by [32]. To prevent distractions, we exclusively focus on spherical objects.

The limiting case $2\sigma = n$ would involve a kernel having the rate of $\ln d_{g_0}(X,Y)$, which deserves further exploration. In light of Dou-Zhu [13], it is intriguing to investigate the case when $2\sigma > n$. The above differential-integral equations may be viewed as porous medium type equations with Riesz potentials diffusion. In contrast to the (fractional) Laplacian diffusion,

$$\int_M |f|^{p-1} f \mathcal{K}_{g_0}(f) \,\mathrm{d}\,\mathrm{vol}_{g_0} \quad \text{may change signs, if } p > 1. \tag{1.13}$$

We further refer to [3] for the recent studies of weak dual solutions of porous medium type equations, where Riesz potential estimates also play an important role.

The paper is structured as follows. In Section 2, we introduce a class of separable solutions that serve as a guiding principle for Theorem 1.1. These solutions are crucial in the proof of Theorem 1.1. In Section 3, we demonstrate the existence of local solutions to (1.3) with $u(t) \in C(M)$ and provide a blow-up criterion that holds for all $m > 0$. For future applications, we establish a regularity theorem for mild solutions, inspired by some idea of [27]. In Section 4, we establish crucial lower and upper bounds. Section 5 is dedicated to the critical case. In the first subsection, we prove Theorem 1.2. In the second one, we establish a differential Harnack inequality. In Section 6, we prove a convergence theorem for all $m > 0$ if the solutions are bounded between positive constants. The proofs of Theorems 1.1 and 1.3 are completed there.

Notations Letters x, y, z represent points in $\mathbb{R}^n$, and capital letters X, Y, Z represent points on Riemannian manifolds. Denote by $B_r(x) \subset \mathbb{R}^n$ the ball centered at x with radius $r > 0$. We may write B_r in replace of $B_r(0)$ for brevity. For $X \in M$, $\mathcal{B}_\delta(X)$ denotes the geodesic ball centered at X with radius δ. When it is clear from the context, we may use $u(t) = u(\cdot, t)$ for instance. Throughout the paper, constants $C > 0$ in inequalities may vary from line to line and are universal, which means that they depend on given quantities but not on solutions.

2 Separable Solutions

Let $K_0 : M \times M \to (0, \infty]$ satisfy (K-1), (K-2) and (K-3). We consider the functional

$$J_m(f) = \frac{\int_M f \mathcal{K}_{g_0}(f) \,\mathrm{d}\,\mathrm{vol}_{g_0}}{\left(\int_M |f|^{m+1} \,\mathrm{d}\,\mathrm{vol}_{g_0}\right)^{\frac{2}{m+1}}} \tag{2.1}$$

and the variational problem

$$\overline{J}_m = \sup_{f \in L^{m+1}(M) \setminus \{0\}} J_m(f) > 0.$$

By the Hardy-Littlewood-Sobolev inequality, $\overline{J}_m < \infty$ if $m \geq \frac{n-2\sigma}{n+2\sigma}$.

Proposition 2.1 *If $m > \frac{n-2\sigma}{n+2\sigma}$, then $\overline{J}_m$ is achieved by a positive Hölder continuous function.*

Proof Let $\{f_j\}_{j=1}^\infty$ be a maximizing sequence with $\|f_j\|_{L^{m+1}(M)} = 1$ and

$$\lim_{j \to \infty} \int_M f_j \mathcal{K}_{g_0}(f_j) \,\mathrm{d}\,\mathrm{vol}_{g_0} = \overline{J}_m. \tag{2.2}$$

We may assume that f_j are nonnegative and

$$f_j \rightharpoonup f \quad \text{in } L^{m+1}(M).$$

After passing to a subsequence (still denoted by $\{f_j\}$), we have

$$\mathcal{K}_{g_0}(f_j) \to \mathcal{K}_{g_0}(f) \quad \text{in } L^q(M)$$

for any $1 \leq q < \frac{n(m+1)}{(n-2\sigma(m+1))_+}$. See for instance the proof of [19, Proposition 1.1] or [24, Proposition 5.1]. Since $m > \frac{n-2\sigma}{n+2\sigma}$, we have $\frac{n(m+1)}{(n-2\sigma(m+1))_+} > \frac{m+1}{m}$ with noting that $\frac{m+1}{m}$ is the Hölder conjugate exponent of $m+1$. Thus, by (2.2),

$$\int_M f_j \mathcal{K}_{g_0}(f_j) \,\mathrm{d}\,\mathrm{vol}_{g_0} \to \int_M f \mathcal{K}_{g_0}(f) \,\mathrm{d}\,\mathrm{vol}_{g_0} = \overline{J}_m.$$

By the lower semi-continuity of $\|\cdot\|_{L^{m+1}(M)}$, we have $\|f\|_{L^{m+1}(M)} \leq 1$ and thus

$$J(f) \geq \overline{J}_m.$$

Hence, f is a maximizer and satisfies the Euler-Lagrange equation

$$\mathcal{K}_{g_0}(f) = \overline{J}_m f^m \quad \text{on } M. \tag{2.3}$$

By the standard potential estimates, f^m is Hölder continuous. Since $f \geq 0$ but not identical to zero, $\mathcal{K}_{g_0}(f)$ must be positive everywhere on M. The proposition is proved.

Proposition 2.2 *If $m > 1$, then* (2.3) *has a unique positive continuous solution.*

Proof Suppose that we have two positive continuous solutions S_1 and S_2. Let

$$\overline{\alpha} = \inf\{\alpha > 0 : S_1 \leq \alpha S_2 \text{ on } M\}.$$

Then $S_1 \leq \overline{\alpha} S_2$ and S_1 is equal to $\overline{\alpha} S_2$ at some points of M. Since the kernel $K_0(\cdot,\cdot)$ is positive, it follows that either

$$\mathcal{K}_{g_0}(\overline{\alpha} S_2 - S_1) > 0 \quad \text{on } M$$

or

$$\overline{\alpha} S_2 - S_1 \equiv 0.$$

If the former happens,

$$(\overline{\alpha} S_2)^m - S_1^m = \overline{\alpha}^{m-1} \mathcal{K}_{g_0}(\overline{\alpha} S_2) - \mathcal{K}_{g_0}(S_1) \geq \mathcal{K}_{g_0}(\overline{\alpha} S_2 - S_1) > 0 \quad \text{everywhere on } M.$$

We obtain a contradiction. Hence, $\overline{\alpha} S_2 - S_1 \equiv 0$. By (2.3), we must have $\overline{\alpha} = 1$. The proposition is proved.

The following proposition extends a criterion of Aubin.

Proposition 2.3 *Suppose that K_0 additionally satisfies* (K-4). *If*

$$\overline{J}_{\frac{n-2\sigma}{n+2\sigma}} > S_{n,-\sigma},$$

where $S_{n,-\sigma}$ is the best constant of Hardy-Littlewood-Sobolev inequality in (1.11), *then $\overline{J}_{\frac{n-2\sigma}{n+2\sigma}}$ is achieved by a positive Hölder continuous function.*

A related result has been obtained by [19]. Since (K-4) holds, one can show that $\overline{J}_{\frac{n-2\sigma}{n+2\sigma}} \geq S_{n,-\sigma}$, but the strict inequality needs some further conditions, such as "positive mass" type condition

$$K_0(X,Y) = c_{n,\sigma} d_{g_0}(X,Y) + A(X,Y),$$

where $A > 0$ on M.

Proof of Proposition 2.3 We shall adapt the blow-up analysis method in [24] (see Proposition 5.3). Let $m_i > \frac{n-2\sigma}{n+2\sigma}$ and $\lim\limits_{i\to\infty} m_i = \frac{n-2\sigma}{n+2\sigma}$.

First, we claim that

$$\liminf_{i\to\infty} \overline{J}_{m_i} \geq \overline{J}_{\frac{n-2\sigma}{n+2\sigma}}. \tag{2.4}$$

Indeed, for any $\varepsilon > 0$, we choose $\phi \geq 0$ such that

$$\|\phi\|_{L^{\frac{2n}{n+2\sigma}}(M)} = 1, \quad \int_M \phi \mathcal{K}_{g_0}(\phi)\, \mathrm{d}\,\mathrm{vol}_{g_0} \geq \overline{J}_{\frac{n-2\sigma}{n+2\sigma}} - \varepsilon,$$

and set

$$\phi_i = \frac{\phi}{\|\phi\|_{L^{m_i+1}(M)}}.$$

Then we have

$$\liminf_{i\to\infty} \overline{J}_{m_i} \geq \liminf_{i\to\infty} J_{m_i}(\phi_i) \geq \liminf_{i\to\infty} \|\phi\|^{-2}_{L^{m_i+1}(M)}(\overline{J}_{\frac{n-2\sigma}{n+2\sigma}} - \varepsilon).$$

Sending $\varepsilon \to 0$, (2.4) follows immediately.

Upon passing to a subsequence, we may assume that $\lim_{i\to\infty} J_{m_i} = \Lambda \geq \overline{J}_{\frac{n-2\sigma}{n+2\sigma}}$.

Next, let f_i be the positive maximizers obtained in Proposition 2.1 for J_{m_i}, satisfying $\|f_i\|_{L^{m_i+1}} = 1$ and (2.3) with $m = m_i$.

We claim that $\{f_i\}$ are uniformly bounded. By Riesz potential estimates, they will be uniformly bounded in some Hölder space. The proposition will be proved by extracting a subsequence.

If not, there is a subsequence, still denoted by $\{f_i\}$, which satisfies

$$f(X_i) := \max_M f_i \to \infty \quad \text{as } i \to \infty,$$

where $X_i \in M$. Choose a geodesic normal coordinates system $\{x^1, \cdots, x^n\}$ centered at X_i, and write the integral equation as

$$\overline{J}_{m_i} f_i(\exp_{X_i} x)^{m_i} = \int_{B_\delta} K_0(\exp_{X_i} x, \exp_{X_i} y)\sqrt{\det g_0} f_i(\exp_{X_i} y)\,\mathrm{d}y + h_i(x),$$

where $h_i(x) = \int_{M\setminus\mathcal{B}_\delta(X_i)} K_0(\exp_{X_i} x, \zeta) f_i(\zeta)\,\mathrm{d}\,\mathrm{vol}_{g_0}(\zeta)$ and $\delta > 0$ is a small constant. Since K_0 additionally satisfies (K-4), following the blow up analysis procedure in the proof of [22, Proposition 2.11] or [28, Proposition 4.1], we have, after passing to a subsequence, as $i \to \infty$,

$$v_i(x) := \frac{1}{f_i(X_i)} f_i(\exp_{X_i} f_i(X_i)^{-\frac{1-m_i}{2\sigma}} x) \to v(x) \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^n)$$

for some $v > 0$ satisfying

$$\Lambda v(x)^{\frac{n-2\sigma}{n+2\sigma}} = c_{n,\sigma} \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^{n-2\sigma}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

In fact, v is classified in [9]. It follows that

$$\Lambda \int_{\mathbb{R}^n} v^{\frac{2n}{n+2\sigma}} = c_{n,\sigma} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{v(x)v(y)}{|x-y|^{n-2\sigma}}\,\mathrm{d}y\mathrm{d}x \leq S_{n,-\sigma}\Big(\int_{\mathbb{R}^n} v^{\frac{2n}{n+2\sigma}}\,\mathrm{d}x\Big)^{\frac{n+2\sigma}{n}}.$$

Since $\|v_i\|_{L^{m_i+1}(B_{\delta\cdot f_i(X_i)^{\frac{1-m_i}{2\sigma}}})} \leq \|f_i\|_{L^{m_i+1}(M)} = 1$, $\int_{\mathbb{R}^n} v^{\frac{2n}{n+2\sigma}} \leq 1$. It follows that

$$\overline{J}_{\frac{n-2\sigma}{n+2\sigma}} \leq \Lambda \leq S_{n,-\sigma},$$

which contradicts to the assumption of the proposition. Hence, the claim is verified and the proposition is proved.

If $m \neq 1$, then $S = \overline{J}_m^{\frac{1}{m-1}} f$ is a solution of (1.4).

Suppose that S is a continuous positive solution of (1.4) and $h(t)S$ is a positive separable solution of

$$\partial_t u^m = \mathcal{K}_{g_0}(u) \quad \text{on } M \times (0,T). \tag{2.5}$$

Then h must satisfy the ODE

$$\frac{\mathrm{d}h^m}{\mathrm{d}t} = h \quad \text{on } (0,T).$$

Integrating the above equation yields

$$h = h_c := \Big(c + \frac{m-1}{m} t\Big)^{\frac{1}{m-1}}, \quad c \geq 0,$$

where $c > 0$ if $m < 1$. Hence, we obtain the separable solutions

$$U_c(X,t) = \Big(c + \frac{m-1}{m} t\Big)^{\frac{1}{m-1}} S(X). \tag{2.6}$$

It is worth noting that if $m > 1$, $U_0(0) \equiv 0$ and hence U_0 corresponds to the "friendly giant" of porous medium equations in bounded domains. This solution originates from the loss of uniqueness of the ODE.

3 Existence and Regularity

By the standard estimates for Riesz potentials, we have

$$\|\mathcal{K}_{g_0}(f)\|_{L^{\frac{np}{n-2\sigma p}}(M)} \leq C_1 \|f\|_{L^p(M)}, \quad \forall\, f \in L^p(M), \quad 1 < p < \frac{n}{2\sigma}, \tag{3.1}$$

$$|\mathcal{K}_{g_0}(f)\|_{C^\alpha(M)} \leq C_1 \|f\|_{L^p(M)}, \quad \forall\, f \in L^p(M), \quad \frac{n}{2\sigma} < p \leq \infty, \tag{3.2}$$

where $0 < \alpha < \min\{2\sigma, 1\}$ and $C_1 > 0$ depends only on M, g_0, p, Λ and σ.

Lemma 3.1 *Suppose that $u_0 \in C^0(M)$. Then (1.3) has a unique solution u satisfying $u^m \in C^1([0,T^*); C^0(M))$ for some $T^* > 0$ representing the maximum existence time. If $T^* < \infty$, then*

$$\lim_{t \to T^*} \|u(t)\|_{C^0(M)} = \infty.$$

Proof Integrating (1.3) in t, we obtain

$$u(X,t)^m = u_0(X)^m + \int_0^t \mathcal{K}_{g_0}(u)(X,s)\,\mathrm{d}s. \tag{3.3}$$

The existence follows from Picard-Lindelöf theorem, using (3.2). Noting that

$$\partial_t u(X,t)^m|_{t=0} = \mathcal{K}_{g_0}(u_0) > 0 \quad \text{on } \mathbb{S}^n,$$

we conclude that u^m is positive for $t > 0$ and increasing in t. If $\lim\limits_{t \to T^*} \|u(t)\|_{C^0(M)}$ is finite, we can extend the solution further, which contradicts to the definition of T^*. The lemma is proved.

We may say that a nonnegative function $u \in C((0,T); L^p(M))$, $p \geq \max\{m, 1\}$, is a mild solution of (1.3) if the integral identity (3.3) holds for almost every $X \in M$ and $t \in [0,T)$.

Let us write the equation in local coordinates. For an arbitrary point $\overline{X} \in M$, choose a geodesic normal coordinates system $\{x^1, \cdots, x^n\}$ centered at $\overline{X}$, and write the integral equation (3.3) as

$$\widehat{u}(x,t)^m = \widehat{u}(x,0)^m + \int_0^t \int_{B_\delta} \widehat{K}_0(x,y)\widehat{u}(y,s)\,\mathrm{d}y\mathrm{d}s + h(x,t), \tag{3.4}$$

where $\delta > 0$ is a constant,

$$\widehat{u}(x,t) = u(\exp_{\overline{X}} x, t), \quad \widehat{K}_0(x,y) = K_0(\exp_{\overline{X}} x, \exp_{\overline{X}} y)\sqrt{\det g_0(y)}$$

and

$$h(x,t) = \int_0^t \int_{M\setminus \mathcal{B}_\delta(\overline{X})} K_0(\exp_{\overline{X}} x, \zeta)u(\zeta,s)\,\mathrm{d}\,\mathrm{vol}_{g_0}\mathrm{d}s.$$

In the next lemma, we show that h is a good term.

Lemma 3.2 *Assume as above. Then the nonnegative function h satisfies that, for every $0 < t < T^*$,*

$$\sup_{B_{\frac{\delta}{2}}} h(\cdot,t) \le C \inf_{B_{\frac{\delta}{2}}} h(\cdot,t), \tag{3.5}$$

$$\sup_{B_{\frac{\delta}{2}}} h(\cdot,t) \le C \fint_{\mathcal{B}_{\frac{\delta}{2}}(\overline{X}))} u^m\,\mathrm{d}\,\mathrm{vol}_{g_0} \tag{3.6}$$

and

$$|\nabla h(x,t)| \le \frac{C}{\delta} h(x,t), \quad \forall\, x \in B_{\frac{\delta}{2}}, \tag{3.7}$$

where $\fint_{\mathcal{B}_{\frac{\delta}{2}}(\overline{X})} = \frac{1}{\mathrm{vol}_{g_0}(\mathcal{B}_{\frac{\delta}{2}}(\overline{X}))}\int_{\mathcal{B}_{\frac{\delta}{2}}(\overline{X})}$, and $C > 0$ depends only on n, σ, λ and $\|g_0\|_{C(M)}$.

Proof Note that for any $x_1, x_2 \in B_{\frac{\delta}{2}}$, using (K-2),

$$\begin{aligned} h(x_1,t) &= \int_0^t \int_{M\setminus\mathcal{B}_\delta(\overline{X})} \frac{K_0(\exp_{\overline{X}} x_1,\zeta)}{K_0(\exp_{\overline{X}} x_2,\zeta)} K_0(\exp_{\overline{X}} x_2,\zeta)u(\zeta,s)\,\mathrm{d}\,\mathrm{vol}_{g_0}\mathrm{d}s \\ &\le \Lambda^2 \int_0^t \int_{M\setminus\mathcal{B}_\delta(\overline{X})} \Big(\frac{d_{g_0}(x_1,\zeta)}{d_{g_0}(x_2,\zeta)}\Big)^{2\sigma-n} K_0(\exp_{\overline{X}} x_2,\zeta)u(\zeta,s)\,\mathrm{d}\,\mathrm{vol}_{g_0}\mathrm{d}s \\ &\le Ch(x_2,t). \end{aligned}$$

Hence, (3.5) is verified. Since $h(x,t) \le \widehat{u}(x,t)^m$, (3.6) follows immediately from (3.5).

Using (K-3) and (K-2), we have

$$|\nabla h(x,t)| \le \frac{\Lambda^2 C}{\delta} \int_0^t \int_{M\setminus\mathcal{B}_\delta(\overline{X})} K_0(\exp_{\overline{X}} x,\zeta)u(\zeta,s)\,\mathrm{d}\,\mathrm{vol}_{g_0}\mathrm{d}s.$$

Hence, (3.7) is verified. The lemma is proved.

We shall establish regularity results for mild solutions, which are inspired by [27, Theorem 1.3]. For $T > 0$, suppose that $V \in L^\infty([0,T]; L^{\frac{n}{2\sigma}}(B_3))$ and $h \in L^\infty([0,T]; L^q(B_2))$, $q > \frac{n}{n-2\sigma}$

are nonnegative functions. We study the integrability improvement for nonnegative solutions of the integral inequality

$$w(x,t) \le \int_0^t \mathrm{e}^{s-t} \int_{B_3} \frac{V(y,s)w(y,s)}{|x-y|^{n-2\sigma}} \,\mathrm{d}y\mathrm{d}s + h(x,t), \quad a.e.\ x \in B_2,\ 0 \le t \le T. \tag{3.8}$$

Suppose that $w \in L^\infty((0,T); L^p(B_3))$ for some $\frac{n}{n-2\sigma} < p < q$.

The factor e^{s-t} will serve to establish estimates independent of T. In subsequent applications, it will be substituted with $\mathrm{e}^{\alpha(s-t)}$ for a positive constant α. For brevity, we set $\alpha = 1$ in this context.

Theorem 3.1 *Assume as above. Suppose additionally that* $V \in C([0,T]; L^{\frac{n}{2\sigma}}(B_3))$. *Then* $w \in L^\infty((0,T); L^q(B_1))$.

First, we prove the following proposition.

Proposition 3.1 *For* $q > p > \frac{n}{n-2\sigma}$, *there exist positive constants* $\overline{\delta} < 1$ *and* $C \ge 1$, *depending only on* n, σ, p *and* q, *such that if*

$$\|V\|_{L^\infty((0,T); L^{\frac{n}{2\sigma}}(B_3))} \le \overline{\delta}, \tag{3.9}$$

then $w \in L^\infty((0,T); L^q(B_1))$ *and*

$$\|w\|_{L^\infty((0,T);L^q(B_1))} \le C(\|w\|_{L^\infty((0,T);L^p(B_3))} + \|h\|_{L^\infty((0,T);L^q(B_2))}).$$

Proof We may assume a priori that $w \in L^\infty((0,T); L^q(B_2))$ and only prove the estimate. Otherwise, one can truncate the kernel and take an approximation as implemented by Li [27].

For any open set $\omega \subset B_3$, we let

$$D_\omega(x,s) = \int_\omega \frac{V(y,s)w(y,s)}{|x-y|^{n-2\sigma}} \,\mathrm{d}y, \quad x \in B_3,$$

and let

$$\begin{aligned} I_{1,r}(x,t) &= \int_0^t \mathrm{e}^{s-t} D_{B_r}(x,s)\mathrm{d}s, \\ I_{2,r}(x,t) &= \int_0^t \mathrm{e}^{s-t} D_{B_3\setminus\overline{B}_r}(x,s)\,\mathrm{d}s \end{aligned}$$

with $0 < r < \frac{3}{2}$. For any fixed t, by the Minkowski inequality and estimates of Riesz potential we have, for $0 < \rho < r < \frac{3}{2}$,

$$\begin{aligned} \|I_{1,r}(\cdot,t)\|_{L^q(B_\rho)} &\le \int_0^t \mathrm{e}^{s-t} \Big(\int_{B_\rho} D_{B_r}(x,s)^q \,\mathrm{d}x \Big)^{\frac{1}{q}} \,\mathrm{d}s \\ &\le C \int_0^t \mathrm{e}^{s-t} \|V(s)^\nu w(s)^\nu\|_{L^1(B_r)}^{\frac{1}{\nu}} \,\mathrm{d}s \\ &\le C \int_0^t \mathrm{e}^{s-t} (\|V(s)^\nu\|_{L^{\frac{q}{q-\nu}}(B_r)} \|w(s)^\nu\|_{L^{\frac{q}{\nu}}(B_r)})^{\frac{1}{\nu}} \,\mathrm{d}s \\ &= C \int_0^t \mathrm{e}^{s-t} \|V(s)\|_{L^{\frac{n}{2\sigma}}(B_r)} \|w(s)\|_{L^q(B_r)} \,\mathrm{d}s \end{aligned}$$

$$\begin{aligned}&\le C\overline{\delta}\|w\|_{L^\infty((0,T);L^q(B_r))}\int_0^T \mathrm{e}^{s-T}\,\mathrm{d}s\\&\le C\overline{\delta}\|w\|_{L^\infty((0,T);L^q(B_r))}\\&\le \frac{1}{2}\|w\|_{L^\infty((0,T);L^q(B_r))},\end{aligned}$$

where $\frac{1}{q}=\frac{1}{\nu}-\frac{2\sigma}{n}$, if $C\overline{\delta}\le\frac{1}{2}$. For $x\in B_\rho$, using

$$\begin{aligned}|D_{B_3\backslash\overline{B}_r}(x,s)|&\le\frac{1}{(r-\rho)^{n-2\sigma}}\int_{B_3\backslash\overline{B}_r}V(y,s)w(y,s)\,\mathrm{d}y\\&\le\frac{C}{(r-\rho)^{n-2\sigma}}\|V\|_{L^{\frac{n}{2\sigma}}(B_3)}\|w\|_{L^p(B_3)},\end{aligned}$$

we have

$$\|I_{2,r}(\cdot,t)\|_{L^q(B_\rho)}\le\frac{C}{(r-\rho)^{n-2\sigma}}\|w\|_{L^p(B_3)}.$$

Since $w\ge0$,

$$\begin{aligned}\|w\|_{L^\infty((0,T);L^q(B_\rho))}\le&\frac{1}{2}\|w\|_{L^\infty((0,T);L^q(B_r))}\\&+\frac{C}{(r-\rho)^{n-2\sigma}}\|w\|_{L^p(B_3)}+C\|h\|_{L^\infty((0,T);L^q(B_2))}.\end{aligned}$$

Using Lemma 1.1 of [15], we are able to establish the desired estimate.

Proof of Theorem 3.3 Let $x_0\in B_1$ and $0<\varepsilon<\frac{1}{4}$ be small, we have

$$w_\varepsilon(x,t)=\varepsilon^{\frac{n-2\sigma}{2}}w(x_0+\varepsilon x,t),\quad V_\varepsilon(x,t)=\varepsilon^{2\sigma}V(x_0+\varepsilon x,t),\quad x\in B_3$$

and

$$h_\varepsilon(x,t)=\varepsilon^{\frac{n-2\sigma}{2}}\int_0^t\mathrm{e}^{s-t}\int_{B_3\backslash B_{3\varepsilon}(x_0)}\frac{V(y,s)w(y,s)}{|x_0+\varepsilon x-y|^{n-2\sigma}}\,\mathrm{d}y\mathrm{d}s+\varepsilon^{\frac{n-2\sigma}{2}}h(x_0+\varepsilon x,t).$$

Since $V\in C([0,T];L^{\frac{n}{2\sigma}}(B_3))$ and

$$\|V_\varepsilon(\cdot,t)\|_{L^{\frac{n}{2\sigma}}(B_3)}=\|V(\cdot,t)\|_{L^{\frac{n}{2\sigma}}(B_{3\varepsilon}(x_0))},$$

we can find a small ε such that $\|V_\varepsilon\|_{L^\infty((0,T);L^{\frac{n}{2\sigma}}(B_3))}\le\overline{\delta}$ as in the above proposition. The theorem follows from Proposition 3.1.

4 Bounds at Large Time

In this section, we establish sharp asymptotical behavior of solutions of (1.3) if $m>1$; and obtain sharp integral bounds if $\frac{n-2\sigma}{n+2\sigma}\le m<1$.

First, we need a comparison principle.

Lemma 4.1 (Comparison principle) *Let $m>0$. Suppose that $f_1,f_2\in C^1([0,T];C^0(M))$ are nonnegative functions satisfying*

$$\partial_t f_1^m\ge\mathcal{K}_{g_0}(f_1),\quad\partial_t f_2^m\le\mathcal{K}_{g_0}(f_2)\quad\text{on } M\times(0,T]$$

and

$$f_1(0) \geq f_2(0) \quad \text{on } M \quad \text{and} \quad f_1(0) > f_2(0) \quad \text{somewhere.}$$

Then $f_1 > f_2$ *on* $M \times (0, T]$.

Proof This proof is quite elementary. We omit it.

Proposition 4.1 *Assume as in Lemma* 3.1 *If* $m > 1$, *then* $T^* = \infty$ *and*

$$\Big\| \frac{u}{U_0} - 1 \Big\|_{C(M)} \leq \frac{C}{t} \quad \text{for } t > 1.$$

Proof Let U_c be the separable solution defined in (2.6). Let c be large so that

$$0 \equiv U_0(0) \leq u_0 < U_c(0).$$

By Lemma 4.1, we have

$$U_0 \leq u \leq U_c \quad \text{in } M \times (0, T^*).$$

Since U_c is locally uniformly bounded, by Lemma 3.1 we must have $T^* = \infty$. The two sides bounds also imply that

$$0 \leq \frac{u}{U_0} - 1 \leq \frac{U_c}{U_0} - 1 = \Big(\frac{\frac{c}{t} + \frac{m-1}{m}}{\frac{m-1}{m}} \Big)^{\frac{1}{m-1}} - 1 = O\Big(\frac{1}{t}\Big).$$

The proposition is proved.

When $m = \frac{n-2\sigma}{n+2\sigma}$, we will need to use the modulus

$$\omega_t(\rho) := \sup_{X \in M} \int_{\mathcal{B}_\rho(X)} u^{m+1}(t) \,\mathrm{d}\,\mathrm{vol}_{g_0}, \quad \rho > 0.$$

Lemma 4.2 *Suppose that* $\frac{n-2\sigma}{n+2\sigma} \leq m < 1$ *and*

$$\|u(T)\|_{L^{m+1}(M)} \leq \overline{C}$$

for some $0 < T \leq T^*$. *Then*

$$\max_{[0,T]} \|u\|_{C^0(M)} \leq C,$$

where $C > 0$ *depends only on* $M, \sigma, \Lambda, m, T, \overline{C}$ *and* $\|u_0\|_{C^0(M)}$, *and further on the modulus* $\omega_T(\cdot)$ *when* $m = \frac{n-2\sigma}{n+2\sigma}$.

Proof Let

$$\widetilde{V}(X, t) = u(X, t)^{1-m}.$$

Note that $u(X, t)$ is increasing in t for any fixed X. We have $\omega_t(\rho) \leq \omega_T(\rho)$ for $t \leq T$. Moreover, if $m > \frac{n-2\sigma}{n+2\sigma}$, for any $\overline{X} \in M$ and $\rho > 0$, by the Hölder inequality

$$\int_{\mathcal{B}_\rho(\overline{X})} |\widetilde{V}(t)|^{\frac{n}{2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0} \leq \int_{\mathcal{B}_\rho(\overline{X})} |\widetilde{V}(T)|^{\frac{n}{2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0}$$
$$\leq \Big(\int_{\mathcal{B}_\rho(\overline{X})} |u(T)|^{m+1} \,\mathrm{d}\,\mathrm{vol}_{g_0} \Big)^{\frac{n(1-m)}{2\sigma(m+1)}} (\mathrm{vol}_{g_0}(\mathcal{B}_\rho(\overline{X})))^{\frac{2\sigma(m+1)}{(n+2\sigma)m-(n-2\sigma)}}$$

$$\leq C\overline{C}^{\frac{n(1-m)}{2\sigma(m+1)}}\rho^{\frac{2n\sigma(m+1)}{(n+2\sigma)m-(n-2\sigma)}}, \tag{4.1}$$

which can be made to be small by choosing small ρ but independent of modulus $\omega_T(\cdot)$. Then the lemma is a consequence of Lemma 3.2, Proposition 3.4 and a bootstrap argument.

Corollary 4.1 *If $\frac{n-2\sigma}{n+2\sigma} \leq m < 1$, then $T^* < \infty$ and*

$$\lim_{t\to T^*}\int_M u^{m+1}(t)\,\mathrm{d}\,\mathrm{vol}_{g_0} = \infty.$$

Proof For a small $t_0 > 0$, choose a large c such that

$$U_c(X,t_0) = \Big(c - \frac{1-m}{m}t_0\Big)^{-\frac{1}{1-m}} S < u(X,t_0) \quad \text{on } M.$$

By the comparison principle, we have

$$u \geq U_c \quad \text{in } M\times[t_0,T^*).$$

Hence, $T^* \leq \frac{cm}{1-m} < \infty$.

As $u(X,t)$ is increasing in t for any fixed $X\in M$, the limit in the lemma exists. If the limit is finite, by the monotone convergence theorem we have

$$\lim_{t\to T^*} u(\cdot,t) = u(\cdot,T^*) \in L^{m+1}(M).$$

Using Lemma 4.2, we obtain

$$\max_{[0,T^*]}\|u\|_{C^0(M)} \leq C,$$

which contradicts to Lemma 3.1. Therefore, the corollary holds.

Proposition 4.2 *If $\frac{n-2\sigma}{n+2\sigma} \leq m < 1$, then*

$$\frac{1}{C}(T^*-t)^{-\frac{1}{1-m}} \leq \|u(t)\|_{L^{m+1}(M)} \leq C(T^*-t)^{-\frac{1}{1-m}},$$

where $C > 0$ is a constant.

Proof By a direct computation using the first equation in (1.3), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M u^{m+1}\,\mathrm{d}\,\mathrm{vol}_{g_0} &= \frac{m+1}{m}\int_M u\mathcal{K}_{g_0}(u)\,\mathrm{d}\,\mathrm{vol}_{g_0} \geq 0,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_M u\mathcal{K}_{g_0}(u)\,\mathrm{d}\,\mathrm{vol}_{g_0} &= 2\int_M \partial_t u\mathcal{K}_{g_0}(u)\,\mathrm{d}\,\mathrm{vol}_{g_0}\\ &= \frac{2}{m}\int_M |\mathcal{K}_{g_0}(u)|^2u^{1-m}\,\mathrm{d}\,\mathrm{vol}_{g_0}.\end{aligned}$$

We may drop $\mathrm{d}\,\mathrm{vol}_{g_0}$ in the follow integrals. Hence,

$$\frac{\mathrm{d}J_m(u)}{\mathrm{d}t} = \frac{2}{m}\Big(\int_M u^{m+1}\Big)^{-\frac{2}{m+2}}\Big[\int_M |\mathcal{K}_{g_0}(u)|^2u^{1-m} - \frac{\big(\int_M u\mathcal{K}_{g_0}(u)\big)^2}{\int_M u^{m+1}}\Big].$$

By the Hölder inequality,

$$\Big(\int_M u\mathcal{K}_{g_0}(u)\Big)^2 = \Big(\int_M u^{\frac{1+m}{2}}(u^{\frac{1-m}{2}}\mathcal{K}_{g_0}(u))\Big)^2 \le \Big(\int_M |\mathcal{K}_{g_0}(u)|^2 u^{1-m}\Big)\int_M u^{m+1}.$$

It follows that

$$\frac{\mathrm{d}J_m(u)}{\mathrm{d}t} \ge 0.$$

Set

$$Z(t) = \Big(\int_M u^{m+1}\Big)^{-\frac{2}{m+1}+1}.$$

Then

$$Z'(t) = \frac{m-1}{m}J_m(u) \ge -\frac{1-m}{m}\overline{J}_m.$$

By integration, for $0 < t < T < T^*$,

$$Z(t) \le \frac{1-m}{m}\overline{J}_m(T-t) + Z(T).$$

Sending T to T^*, by Corollary 4.4 we have

$$Z(t) \le \frac{1-m}{m}\overline{J}_m \cdot (T^* - t).$$

On the other hand,

$$Z''(t) = \frac{m-1}{m}\frac{\mathrm{d}}{\mathrm{d}t}J_m(u) \le 0.$$

For any $0 < s < t < T < T^*$, we have

$$Z(t) \ge Z(T) + \frac{Z(s) - Z(T)}{s - T}(t - T).$$

Sending $s \to 0$ and $T \to T^*$, by Corollary 4.1 we obtain

$$Z(t) \ge \frac{Z(0)}{T^*}(T^* - t).$$

Therefore, the proposition is proved.

In order to study the blow up profile of u near T^*, let us introduce a re-normalization of u as follows. For $0 < m < 1$, let

$$\widetilde{u}(X,\tau) = (T^* - t)^{\frac{1}{1-m}} u(X,t), \quad \tau = -\ln\frac{T^* - t}{T^*}. \tag{4.2}$$

Then we have

$$\partial_\tau \widetilde{u}^m = \mathcal{K}_{g_0}(\widetilde{u}) - \frac{m}{1-m}\widetilde{u}^m \quad \text{in } M \times (0,\infty). \tag{4.3}$$

From Proposition 4.2, we know that

$$\frac{1}{C^{m+1}} \le \int_M \widetilde{u}(\tau)^{m+1}\,\mathrm{d}\,\mathrm{vol}_{g_0} \le C^{m+1}, \quad \tau \in [0,\infty).$$

Proposition 4.3 *If* $\frac{n-2\sigma}{n+2\sigma} < m < 1$, *then*

$$\frac{1}{C} \le \|\widetilde{u}(\tau)\|_{L^\infty(M)} \le C, \quad \tau \in [1, \infty)$$

and thus

$$\frac{1}{C}(T^* - t)^{-\frac{1}{1-m}} \le \|u(t)\|_{L^\infty(M)} \le C(T^* - t)^{-\frac{1}{1-m}}, \quad T^*(1 - \mathrm{e}^{-1}) \le t < T^*,$$

where $C > 0$ *is a constant.*

Proof By integrating (4.3), we have

$$\widetilde{u}(X,\tau)^m = \mathrm{e}^{-\frac{m}{1-m}\tau}\widetilde{u}(X,0)^m + \int_0^\tau \mathrm{e}^{\frac{m}{1-m}(s-\tau)}\mathcal{K}_{g_0}(\widetilde{u})(X,s)\,\mathrm{d}s.$$

Since $m > \frac{n-2\sigma}{n+2\sigma}$, we have (4.1). Making use of Lemma 3.2, Proposition 3.1 and a bootstrap argument, we then can obtain

$$\widetilde{u}(X,\tau) \le C_1$$

for some $C_1 > 1$.

On the other hand,

$$\int_M \widetilde{u} \ge \frac{1}{C_1^m}\int_M \widetilde{u}^{m+1} \ge \frac{1}{CC_1^m},$$

and using (K-2),

$$\mathcal{K}_{g_0}(\widetilde{u})(X,s) \ge \frac{1}{C}$$

for some C independent of s. It follows that

$$\widetilde{u}(X,\tau)^m \ge \frac{1}{C}\int_0^\tau \mathrm{e}^{\frac{m}{1-m}(s-\tau)}\,\mathrm{d}s \ge \frac{1}{C}\frac{1-m}{m}(1 - \mathrm{e}^{-\frac{m}{1-m}}),$$

if $\tau \ge 1$. Hence, the first conclusion is verified. The second one then follows from the definition of $\widetilde{u}$. Therefore, the proposition is proved.

5 The Critical Regime

In this section, we set $m = \frac{n-2\sigma}{n+2\sigma}$.

We may further normalize $\widetilde{u}$ in (4.3) as

$$w(s) = \widetilde{u}(t)/\|\widetilde{u}(t)\|_{L^{m+1}(M)}, \quad t = \beta(s) \text{ with } \beta'(s) = \|\widetilde{u}(t)\|_{L^{m+1}(M)}^{m-1},$$

which turns out to be a solution of the normalized equation (1.8) on $M \times (0, \infty)$.

On the other hand, by changing variables Lemma 3.1 implies that (1.8) and (1.9) admits a unique positive solution u satisfying $u^m \in C^1([0, T^*); C^0(M))$, where $0 < T^* \le \infty$ is taken to be the maximal existence time of the solution.

5.1 Long time existence and concentration compactness

Let u be a positive solution of (1.8) and $g = u^{\frac{4}{n+2\sigma}} g_0$. Set

$$V(t) = \int_M u(t)^{\frac{2n}{n+2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0}$$

and

$$M_q(t) = \int_M |Q^g_{K_0} - a(t)|^q \,\mathrm{d}\,\mathrm{vol}_g = \int_M |Q^g_{K_0} - a(t)|^q u^{\frac{2n}{n+2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0}, \quad q \geq 1,$$

where $Q^g_{K_0} = \mathcal{K}_g(1)$ as defined in (1.5).

Lemma 5.1 *Along the flow, we have*

(i)

$$\frac{\partial}{\partial t} u^{\frac{2n}{n+2\sigma}} = \frac{2n}{n-2\sigma}(Q^g_{K_0} - a(t)) u^{\frac{2n}{n+2\sigma}}$$

and thus $V'(t) = 0$;

(ii)

$$\frac{\partial}{\partial t}(Q^g_{K_0} - a(t)) = \frac{n+2\sigma}{n-2\sigma}\mathcal{K}_g(Q^g_{K_0} - a) - (Q^g_{K_0} - a)^2 - a(Q^g_{K_0} - a) - a';$$

(iii)

$$\frac{\mathrm{d}}{\mathrm{d}t} J_{\frac{n-2\sigma}{n+2\sigma}}(u) = \frac{2(n+2\sigma)}{n-2\sigma} V(t)^{-\frac{n+2\sigma}{n}} \cdot M_2(t) \geq 0,$$

where $J_{\frac{n-2\sigma}{n+2\sigma}}$ *is as defined in* (2.1).

Proof By (1.8), we have

$$\frac{\partial_t u}{u} = \frac{n+2\sigma}{n-2\sigma}(Q^g_{K_0} - a(t)). \tag{5.1}$$

The first item follows. Using the definition of $Q^g_{K_0}$ and (5.1), we have

$$\begin{aligned}
\frac{\partial}{\partial t} Q^g_{K_0} &= \frac{\partial}{\partial t}(u^{-\frac{n-2\sigma}{n+2\sigma}} \mathcal{K}_{g_0}(u)) \\
&= -\frac{n-2\sigma}{n+2\sigma}\frac{\partial_t u}{u} Q^g_{K_0} + u^{-\frac{n-2\sigma}{n+2\sigma}} \mathcal{K}_{g_0}(\partial_t u) \\
&= -(Q^g_{K_0} - a) Q^g_{K_0} + \frac{n+2\sigma}{n-2\sigma} u^{-\frac{n-2\sigma}{n+2\sigma}} \mathcal{K}_{g_0}(u(Q^g_{K_0} - a)) \\
&= -(Q^g_{K_0} - a) Q^g_{K_0} + \frac{n+2\sigma}{n-2\sigma} \mathcal{K}_g(Q^g_{K_0} - a) \\
&= -(Q^g_{K_0} - a)^2 - a(Q^g_{K_0} - a) + \frac{n+2\sigma}{n-2\sigma}\mathcal{K}_g(Q^g_{K_0} - a).
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} J_{\frac{n-2\sigma}{n+2\sigma}}(u) &= \frac{2(n+2\sigma)}{n-2\sigma} V(t)^{-\frac{n+2\sigma}{n}} \cdot \Big[\int_{\mathbb{S}^n} |\mathcal{K}_{g_0}(u)|^2 u^{\frac{4\sigma}{n+2\sigma}} - a(t) u \mathcal{K}_{g_0}(u) \,\mathrm{d}\,\mathrm{vol}_{g_0}\Big] \\
&= \frac{2(n+2\sigma)}{n-2\sigma} V(t)^{-\frac{n+2\sigma}{n}} \cdot \int |\mathcal{K}_{g_0}(u) - a(t) u^{\frac{n-2\sigma}{n+2\sigma}}|^2 u^{\frac{4\sigma}{n+2\sigma}} \,\mathrm{d}\,\mathrm{vol}_{g_0}
\end{aligned}$$

$$= \frac{2(n+2\sigma)}{n-2\sigma} V(t)^{-\frac{n+2\sigma}{n}} \cdot M_2(t).$$

The lemma is proved.

Without loss of generality, we assume from now on

$$V(t) = 1. \tag{5.2}$$

Lemma 5.2 *We have*

$$a'(t) = \frac{2(n+2\sigma)}{n-2\sigma} M_2(t) \geq 0$$

and

$$0 < J_{\frac{n-2\sigma}{n+2\sigma}}(u_0) \leq a(t) \leq \overline{J}_{\frac{n-2\sigma}{n+2\sigma}}.$$

Hence, $\lim\limits_{t\to\infty} a(t) =: a_\infty$ *exists.*

Proof Since $V(t) = 1$,

$$a(t) = J_{\frac{n-2\sigma}{n+2\sigma}}(u) \leq \overline{J}_{\frac{n-2\sigma}{n+2\sigma}}.$$

By item (iii) of Lemma 5.1,

$$a'(t) = \frac{2(n+2\sigma)}{n-2\sigma} M_2(t) \geq 0.$$

Thus $a(t) \geq a(0) = J_{\frac{n-2\sigma}{n+2\sigma}}(u_0)$. The lemma is proved.

Lemma 5.3 *We have* $T^* = \infty$.

Proof By (1.8), we have

$$\partial_t (\mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s} u(t)^m) = \mathrm{e}^{-\int_0^t a(s)\,\mathrm{d}s} \mathcal{K}_{g_0}(u)(t) \geq 0.$$

If $T^* < \infty$,

$$\mathrm{e}^{\int_0^{T^*} a(s)\,\frac{\mathrm{d}s}{m}} u(T^*) := \lim_{t\to T^*} \mathrm{e}^{m^{-1}\int_0^t a(s)\,\mathrm{d}s} u(t)$$

exists and belongs to $L^{m+1}(M)$. Integrating (1.8), we obtain

$$u(t)^m = u_0^m + \int_0^t \mathrm{e}^{-\int_s^t a(\tau)\,\mathrm{d}\tau} \mathcal{K}_{g_0}(u)(s)\,\mathrm{d}s. \tag{5.3}$$

By Lemma 5.2,

$$u(t)^m \leq u_0^m + \int_0^t \mathrm{e}^{J_{\frac{n-2\sigma}{n+2\sigma}}(u_0)\cdot(s-t)} \mathcal{K}_{g_0}(u)(s)\,\mathrm{d}s.$$

It follows from Lemma 3.2, Proposition 3.1 and a bootstrap argument that

$$\|u^m\|_{C(M\times[0,T^*])} < \infty.$$

This contradicts to the definition of T^*. Hence, T^* can not be a finite positive number. The proof is thus finished.

Next, we compute the derivative of M_q. By Lemma 5.1,

$$\frac{\mathrm{d}}{\mathrm{d}t} M_q(t) = \int_M \Big[q|Q_{K_0}^g - a|^{q-2} (Q_{K_0}^g - a) \partial_t (Q_{K_0}^g - a)$$

$$
\begin{aligned}
&+\frac{2n}{n-2\sigma}|Q^g_{K_0}-a|^q(Q^g_{K_0}-a)\Big]\,\mathrm{d}\,\mathrm{vol}_g\\
=&\int_M\Big[\frac{(n+2\sigma)q}{n-2\sigma}|Q^g_{K_0}-a|^{q-2}(Q^g_{K_0}-a)\mathcal{K}_g(Q^g_{K_0}-a)\\
&+\Big(\frac{2n}{n-2\sigma}-q\Big)|Q^g_{K_0}-a|^q(Q^g_{K_0}-a)-aq|Q^g_{K_0}-a|^q\\
&-qa'|Q^g_{K_0}-a|^{q-2}(Q^g_{K_0}-a)\Big]\,\mathrm{d}\,\mathrm{vol}_g.
\end{aligned}\tag{5.4}
$$

Denote the first term as

$$
N_q:=\int_M\frac{(n+2\sigma)q}{n-2\sigma}|Q^g_{K_0}-a|^{q-2}(Q^g_{K_0}-a)\mathcal{K}_g(Q^g_{K_0}-a)\,\mathrm{d}\,\mathrm{vol}_g,
$$

which is a 'bad' term to us because of (1.13). Using Hardy-Littlewood-Sobolev inequality,

$$
\begin{aligned}
|N_q|&=\frac{(n+2\sigma)q}{n-2\sigma}\Big|\int_M|Q^g_{K_0}-a|^{q-2}(Q^g_{K_0}-a)\mathcal{K}_g(Q^g_{K_0}-a)\,\mathrm{d}\,\mathrm{vol}_g\Big|\\
&=\frac{(n+2\sigma)q}{n-2\sigma}\Big|\int_M|Q^g_{K_0}-a|^{q-2}u(Q^g_{K_0}-a)\mathcal{K}_{g_0}(u(Q^g_{K_0}-a))\,\mathrm{d}\,\mathrm{vol}_{g_0}\Big|\\
&\le\frac{(n+2\sigma)q}{n-2\sigma}C\||Q^g_{K_0}-a|^{q-2}u(Q^g_{K_0}-a)\|_{L^{\frac{2n}{n+2\sigma}}}\|u(Q^g_{K_0}-a)\|_{L^{\frac{2n}{n+2\sigma}}}\\
&=\frac{(n+2\sigma)q}{n-2\sigma}CM^{\frac{n+2\sigma}{2n}}_{\frac{2n(q-1)}{n+2\sigma}}M^{\frac{n+2\sigma}{2n}}_{\frac{2n}{n+2\sigma}}.
\end{aligned}\tag{5.5}
$$

Furthermore, for $q>2$ and $\nu\ge\frac{2n(q-1)}{n+2\sigma}$, by the Hölder inequality with using $V(t)=1$,

$$
\begin{aligned}
|N_q|&\le CM^{\frac{n+2\sigma}{2n}}_{\frac{2n(q-1)}{n+2\sigma}}M^{\frac{n+2\sigma}{2n}}_{\frac{2n}{n+2\sigma}}\\
&\le CM_\nu^{\frac{q-1}{\nu}}M_2^{\frac12}\\
&\le\varepsilon M_\nu+\frac{C}{\varepsilon^{\frac{q-1}{\nu-q+1}}}M_2^{\frac{\nu}{2(\nu-q+1)}},
\end{aligned}\tag{5.6}
$$

where $\varepsilon>0$ can be very small, and the Young inequality is used in the last inequality. As for the second term, since $Q^g_{K_0}>0$, we have $(Q^g_{K_0}-a)\ge-\overline{J}_{\frac{n-2\sigma}{n+2\sigma}}$ and thus

$$
|Q^g_{K_0}-a|^q(Q^g_{K_0}-a)\ge|Q^g_{K_0}-a|^{q+1}-\overline{J}_{\frac{n-2\sigma}{n+2\sigma}}|Q^g_{K_0}-a|^q.\tag{5.7}
$$

So it is a 'good' term to us. As for the last term, we have the estimate, using Lemma 5.2,

$$
\Big|\int_M qa'|Q^g_{K_0}-a|^{q-2}(Q^g_{K_0}-a)\,\mathrm{d}\,\mathrm{vol}_g\Big|\le CM_2M_{q-1}.\tag{5.8}
$$

Proposition 5.1 *We have*

$$
M_q(t)\to0\quad\text{as } t\to\infty,\quad\text{if } 1\le q<\frac{2n}{n-2\sigma}+\frac{n+2\sigma}{n-2\sigma}.
$$

Proof Step 1 We consider $q=2$.

By (5.5) and the Hölder inequality, we have

$$
|N_2|\le CM_2.
$$

Noting that

$$\int_M a'(Q^g_{K_0} - a)\,\mathrm{d}\,\mathrm{vol}_g = 0,$$

by (5.4) and (5.7) we obtain

$$\frac{4\sigma}{n-2\sigma}M_3 - CM_2 \le \frac{\mathrm{d}}{\mathrm{d}t}M_2(t) \le \frac{4\sigma}{n-2\sigma}M_3 + CM_2,$$

which implies that

$$\int_1^\infty M_3\,\mathrm{d}t \le \frac{n-2\sigma}{4\sigma}\Big(M_2(1) + C\int_1^\infty M_2\,\mathrm{d}t\Big)$$

and

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}t}M_2(t)\Big\|_{L^1([1,\infty))} \le C\int_1^\infty (M_2 + M_3)\,\mathrm{d}t < \infty.$$

Therefore, $\lim\limits_{t\to\infty} M_2 = 0$.

Step 2 We consider $2 < q \le \frac{2n}{n-2\sigma}$, which implies $\frac{2n(q-1)}{n+2\sigma} \le q$.

By taking $\nu = q$ and $\varepsilon = 1$ in (5.6), we obtain

$$|N_q| \le M_q + CM_2^{\frac{q}{2}} \le M_q + CM_2.$$

It follows from (5.4) and (5.7)–(5.8) that

$$\begin{aligned} &-C(M_q + M_2(1+M_{q-1})) + \Big(\frac{2n}{n-2\sigma} - q\Big)M_{q+1} \\ &\le \frac{\mathrm{d}}{\mathrm{d}t}M_q(t) \le \Big(\frac{2n}{n-2\sigma} - q\Big)M_{q+1} + C(M_q + M_2(1+M_{q-1})). \end{aligned} \tag{5.9}$$

If, in addition, $q \le 3$, then $M_q \le M_3 + M_2 \in L^1([1,\infty))$ and $M_{q-1} \le M_2^{\frac{q-1}{2}} \le C$. Using the left part of the above inequality first, we have

$$M_{q+1} \in L^1([1,\infty)), \quad q \le 3 \text{ and } q < \frac{2n}{n-2\sigma}.$$

Hence, both the lower and upper bound of $\frac{\mathrm{d}}{\mathrm{d}t}M_q$ in (5.9) belong to $L^1([1,\infty))$, so does it. Repeating this process, we will conclude that

$$\begin{aligned} \lim_{t\to\infty} M_q(t) = 0 \quad &\text{for all } 2 \le q \le \frac{2n}{n-2\sigma}, \\ M_{q+1} \in L^1([1,\infty)) \quad &\text{for all } 2 \le q < \frac{2n}{n-2\sigma}. \end{aligned}$$

Step 3 We consider $\frac{2n}{n-2\sigma} < q < \frac{2n}{n-2\sigma} + \frac{n+2\sigma}{n-2\sigma}$, which implies $\frac{2n(q-1)}{n+2\sigma} \le q+1$.

By taking $\nu = \frac{2n(q-1)}{n+2\sigma}$ and $\varepsilon = \frac{1}{2}\big(q - \frac{2n}{n-2}\big)$ in (5.6), in view of that

$$\frac{\nu}{2(\nu-q+1)} = \frac{n(q-1)}{2n(q-1) - (q-1)(n+2\sigma)} = \frac{n}{n-2\sigma} > 1,$$

then we have

$$|N_q| \le \varepsilon M_\nu + CM_2^{\frac{n}{n-2\sigma}} \le \varepsilon(M_{q+1} + M_2) + CM_2^{\frac{n}{n-2\sigma}}$$

$$\leq \varepsilon M_{q+1} + CM_2$$
$$= \frac{1}{2}\Big(q - \frac{2n}{n-2}\Big)M_{q+1} + CM_2.$$

It follows from (5.4) and (5.7)–(5.8) that

$$\begin{aligned} & - C(M_q + M_2(1 + M_{q-1})) \\ & \leq \frac{\mathrm{d}}{\mathrm{d}t}M_q(t) + \frac{1}{2}\Big(q - \frac{2n}{n-2\sigma}\Big)M_{q+1} \leq C(M_q + M_2(1 + M_{q-1})). \end{aligned} \tag{5.10}$$

Arguing as in Step 2, we will again conclude that $M_q(t) \to 0$ as $t \to \infty$. The proposition is proved.

Let $DJ_{\frac{n-2\sigma}{n+2\sigma}}(f, \cdot) : L^{\frac{2n}{n-2\sigma}}(M) \to \mathbb{R}$ be the Frechét differential of the functional $J_{\frac{n-2\sigma}{n+2\sigma}}$ at $f \in L^{\frac{2n}{n+2\sigma}}(M)$.

Corollary 5.1 *Along the flow,*

$$DJ_{\frac{n-2\sigma}{n+2\sigma}}(u, \cdot) \to 0 \quad \textit{as } t \to \infty.$$

Hence, the flow is a Palais-Smale flow line.

Proof For any $\varphi \in L^{\frac{2n}{n+2\sigma}}(M)$, we have

$$\begin{aligned} DJ_{\frac{n-2\sigma}{n+2\sigma}}(u, \varphi) &= 2\int_M (\mathcal{K}_{g_0}(u) - a(t)u^{\frac{n-2\sigma}{n+2\sigma}})\varphi)\,\mathrm{d}\,\mathrm{vol}_{g_0} \\ &= 2\int_M (Q^g_{K_0} - a)u^{\frac{n-2\sigma}{n+2\sigma}}\varphi\,\mathrm{d}\,\mathrm{vol}_{g_0} \leq 2M_{\frac{2n}{n-2\sigma}}^{\frac{n-2\sigma}{2n}}\|\varphi\|_{L^{\frac{2n}{n+2\sigma}}(M)}. \end{aligned}$$

The corollary follows from Proposition 5.1.

Proof of Theorem 1.2 It follows from Lemma 5.2 and Proposition 5.1.

5.2 Global bound via the moving spheres method

Theorem 5.1 *Suppose that (M, g_0) is the standard sphere, $\mathcal{K}_{g_0} = (P^{g_0}_\sigma)^{-1}$ in (1.10). If u is positive solution of (1.8) on $\mathbb{S}^n \times (0, \infty)$ and $u(0)^m \in C^1(\mathbb{S}^n)$ is not identical to zero, then $u \in C^1(\mathbb{S}^n \times (0, \infty))$ and the differential Harnack inequality holds*

$$|\nabla \ln u| \leq C \quad \textit{on } \mathbb{S}^n \times [1, \infty),$$

and thus

$$\frac{1}{C} \leq u \leq C \quad \textit{on } \mathbb{S}^n \times [1, \infty),$$

where $C > 0$ depends on $u(t)$ with $t \in [\frac{1}{2}, 1]$.

The global existence follows from (5.3) and the C^1 regularity follows from a bootstrap argument for (5.3).

Next, we shall use the moving spheres method in [27] to prove the differential Harnack inequality. Pick any point $\xi_0 \in \mathbb{S}^n$ as the south pole and let F be the inverse of the stereographic projection with Jacobi determinant $|J_F|$. Let

$$v(x) = |J_F|^{\frac{n-2\sigma}{2n}}u(F(x))^{\frac{n-2\sigma}{n+2\sigma}}.$$

Then we have

$$\frac{1}{A(t)}\partial_t[A(t)v(x,t)] = c_{n,\sigma}\int_{\mathbb{R}^n}\frac{v(y,t)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y \quad \text{in } \mathbb{R}^n\times[0,\infty), \tag{5.11}$$

where

$$A(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}.$$

For $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, denote

$$v_{x_0,\lambda}(x,t) = \Big(\frac{\lambda}{|x-x_0|}\Big)^{n-2\sigma} v(x^{x_0,\lambda},t), \quad \text{where } x^{x_0,\lambda} = x_0 + \frac{\lambda^2(x-x_0)}{|x-x_0|^2}$$

as the generalized Kelvin transform of v with respect to the sphere $\partial B_\lambda(x_0)$. Using the following two identities (see, e.g., [27, page 162]),

$$\Big(\frac{\lambda}{|x-x_0|}\Big)^{n-2\sigma}\int_{|z-x_0|\ge\lambda}\frac{v(z,t)^{\frac{n+2\sigma}{n-2\sigma}}}{|x^{x_0,\lambda}-z|^{n-2\sigma}}\,\mathrm{d}z = \int_{|z-x_0|\le\lambda}\frac{v_{x_0,\lambda}(z,t)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-z|^{n-2\sigma}}\,\mathrm{d}z \tag{5.12}$$

and

$$\Big(\frac{\lambda}{|x-x_0|}\Big)^{n-2\sigma}\int_{|z-x_0|\le\lambda}\frac{v(z,t)^{\frac{n+2\sigma}{n-2\sigma}}}{|x^{x_0,\lambda}-z|^{n-2\sigma}}\,\mathrm{d}z = \int_{|z-x_0|\ge\lambda}\frac{v_{x_0,\lambda}(z,t)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-z|^{n-2\sigma}}\,\mathrm{d}z, \tag{5.13}$$

it is easy to see that $v_{x_0,\lambda}$ is also a solution of (5.11) in $(\mathbb{R}^n\setminus\{x_0\})\times[0,\infty)$. Notice that

$$\begin{aligned}&\frac{1}{A(t)}\partial_t[A(t)(v(x,t)-v_{x_0,\lambda}(x,t))]\\ &= \int_{|z-x_0|\ge\lambda} K(x_0,\lambda;x,z)[v(z,t)^{\frac{n+2\sigma}{n-2\sigma}} - v_{x_0,\lambda}(z,t)^{\frac{n+2\sigma}{n-2\sigma}}]\,\mathrm{d}z, \quad x\in\mathbb{R}^n\setminus\overline{B}_\lambda(x_0),\end{aligned} \tag{5.14}$$

where

$$K(x_0,\lambda;x,z) = \frac{1}{|x-z|^{n-2\sigma}} - \Big(\frac{\lambda}{|x-x_0|}\Big)^{n-2\sigma}\frac{1}{|x^{x_0,\lambda}-z|^{n-2\sigma}}.$$

It is elementary to check that

$$\begin{aligned} K(x_0,\lambda;x,z) > 0, &\quad \forall\ |x-x_0|,|z-x_0|>\lambda>0,\\ K(x_0,\lambda;x,z) = 0, &\quad \forall\ |x-x_0|=\lambda,\\ \nabla_x K(x_0,\lambda;x,z)\cdot(x-x_0) > 0, &\quad \forall\ |x-x_0|=\lambda,\ |z-x_0|>\lambda.\end{aligned}$$

Lemma 5.4 *There exist positive constants λ_0 and ε_0 such that for each $x_0\in B_1$, there holds*

$$v_{x_0,\lambda}(x,t) < v(x,t), \quad \forall\ 0<\lambda\le\lambda_0, \quad |x-x_0|>\lambda, \quad t\in\Big[\frac12,1\Big], \tag{5.15}$$

$$v(x,t) - v_{x_0,\lambda}(x,t) \ge \frac{\varepsilon_0}{|x|^{n-2\sigma}}, \quad \forall\ |x|\ge\lambda_0+1, \quad t\in\Big[\frac12,1\Big] \tag{5.16}$$

and

$$v(x,t) - v_{x_0,\lambda}(x,t) \ge \varepsilon_0(|x-x_0|-\lambda), \quad \forall\ \lambda\le|x|\le\lambda_0+1, \quad t\in\Big[\frac34,1\Big]. \tag{5.17}$$

Proof This follows from a direct computation. See the proof of [27, Lemma 3.1].

We shall show that (5.5) holds for all $t \in [1, \infty)$.

Fix an arbitrary $T > \frac{3}{2}$ and a point $x_0 \in B_1$. Without loss of generality, we assume $x_0 = 0$ and write $v_\lambda = v_{0,\lambda}$ for brevity. Similar to Lemma 5.4, we have the following lemma which asserts that one can start the moving spheres procedure up to T.

Lemma 5.5 *There exists $\lambda_T \in (0, \lambda_0]$ depending on T such that*

$$v_\lambda(x,t) < v(x,t), \quad \forall\ 0 < \lambda \le \lambda_T, \quad |x| \ge \lambda, \quad \frac{3}{4} \le t \le T.$$

Define

$$\overline{\lambda} = \sup\{\mu \le \lambda_0 : v_\lambda(x,t) \le v(x,t),\ \forall\ 0 < \lambda \le \mu,\ |x| \ge \lambda,\ 1 \le t \le T\}.$$

Obviously, $\overline{\lambda} \ge \lambda_T$.

Lemma 5.6 *There exists $\varepsilon_2 > 0$ such that*

$$v(x,t) - v_{\overline{\lambda}}(x,t) \ge \frac{\varepsilon_2}{|x|^{n-2\sigma}}, \quad \forall\ |x| \ge \overline{\lambda} + 1,\ t \in [1,T]$$

and

$$v(x,t) - v_{\overline{\lambda}}(x,t) \ge \varepsilon_2(|x| - \overline{\lambda}), \quad \forall\ \overline{\lambda} \le |x| \le \overline{\lambda} + 1,\ t \in [1,T].$$

Proof Let

$$\xi(z,t,\lambda) = \begin{cases} \dfrac{v(z,t)^{\frac{n+2\sigma}{n-2\sigma}} - v_\lambda(z,t)^{\frac{n+2\sigma}{n-2\sigma}}}{v(z,t) - v_\lambda(z,t)}, & v(z,t) \ne v_\lambda(z,t), \\ 0, & v(z,t) = v_\lambda(z,t) \end{cases}$$

and $w^\lambda(z,t) = v(z,t) - v_\lambda(z,t)$. By (5.14), we have

$$w^{\overline{\lambda}}(x,t) = A(t)^{-1} w^{\overline{\lambda}}\Big(x, \frac{3}{4}\Big) + \int_{\frac{3}{4}}^{t} \int_{B_\lambda(x_0)^c} \frac{A(s)}{A(t)} K(x_0, \overline{\lambda}; x, z) w(z,t)^{\overline{\lambda}}\, \mathrm{d}z \mathrm{d}s$$

for $(x,t) \in B_\lambda(x_0)^c \times (\frac{3}{4}, T]$. Since $w^\lambda \ge 0$, we obtain

$$w^{\overline{\lambda}}(x,t) \ge A(t)^{-1} w^{\overline{\lambda}}\Big(x, \frac{3}{4}\Big) + \int_{\frac{3}{4}}^{1} \int_{B_\lambda(x_0)^c} \frac{A(s)}{A(t)} K(x_0, \overline{\lambda}; x, z) w(z,t)^{\overline{\lambda}}\, \mathrm{d}z \mathrm{d}s.$$

The lemma then follows from Lemma 5.4.

Lemma 5.7 *We have $\overline{\lambda} = \lambda_0$.*

Proof If not, by the above lemma,

$$\partial_r [v(\cdot,t) - v_{\overline{\lambda}}(x,t)]\big|_{|x| = \overline{\lambda}} \ge \varepsilon_2 \quad \text{for } t \in \Big[\frac{3}{4}, T\Big].$$

By the continuity of ∇v, there exists a small $\varepsilon_3 > 0$ so that

$$\partial_r [(v(\cdot,t) - v_\lambda(x,t))]|_{\partial B_r} \ge \frac{\varepsilon_2}{2} \quad \text{for } \overline{\lambda} \le \lambda,\ r \le \overline{\lambda} + \varepsilon,\ \frac{3}{4} \le t \le T.$$

Since $v(\cdot,t)-v_\lambda(x,t)=0$ on ∂B_λ, we have

$$v(\cdot,t)-v_\lambda(x,t)>0 \quad \text{for } \overline{\lambda}\le\lambda<|x|\le\overline{\lambda}+\varepsilon_3,\ \frac{3}{4}\le t\le T.$$

Using the first lower bound in Lemma 5.6 and choosing $\lambda-\overline{\lambda}$ to be very small,

$$v(\cdot,t)-v_\lambda(x,t)>0 \quad \text{for } |x|>\overline{\lambda}+\varepsilon_3,\ \frac{3}{4}\le t\le T.$$

We obtain a contradiction and the lemma follows.

Proof of Theorem 5.1 By the above lemmas, we have, for all $x_0\in B_1$,

$$v(x,t)\ge v_{x_0,\lambda}(x,t),\quad \forall\, 0<\lambda<\lambda_0,\ |x-x_0|\ge\lambda,\ t\in\Big[\frac{3}{4},T\Big].$$

By [27, Lemmas A.1–A.2], we have

$$|\nabla\ln v(t)|\le C \quad \text{in } B_{\frac{1}{2}},\quad \frac{3}{4}\le t\le T.$$

Since $\xi_0\in\mathbb{S}^n$ and T are arbitrarily chosen, the differential Harnack inequality follows. Since the flow keeps the volume, the uniform positive lower and upper bounds follow. The proof is finished.

6 Convergence

It is important to mention that the normalized flow, which preserves the volume $\int_M u^{m+1}\,\mathrm{d\,ovl}_{g_0}$, is equivalent to (4.3) upon a variable change. In this section, we aim to demonstrate the convergence of solutions of (4.3), including the scenario where $m>1$ with replacing the τ variable by $\tau=\ln(t+1)$ in (4.2), provided that the solutions are consistently bounded between positive constants. Namely, suppose that

$$\partial_t u^m=\mathcal{K}_{g_0}(u)-\frac{m}{|1-m|}u^m \quad \text{on } M\times(0,\infty) \tag{6.1}$$

and

$$\frac{1}{C_0}\le u\le C_0 \quad \text{on } M\times(0,\infty), \tag{6.2}$$

where K_0 satisfies (K-1)–(K-3), $C_0\ge1$ is a constant and $m\in(0,1)\cup(1,\infty)$. We shall prove that u converges to a steady solution

$$\mathcal{K}_{g_0}(\varphi)=\frac{m}{|1-m|}\varphi^m \quad \text{on } M,\ \varphi>0. \tag{6.3}$$

First we need two lemmas.

Lemma 6.1 *Let φ be a solution of* (6.3) *and $\zeta=u-\varphi$. Then there exists a constant depends only M,g,n,σ,Λ and C_0 such that*

$$\|\zeta(\cdot,t+\tau)\|_{L^2(M)}\le C\mathrm{e}^{Ct}\|\zeta(\cdot,t)\|_{L^2(M)},\quad \forall t\ge1,\ \tau\ge0.$$

Proof By the equation of u and φ, a direct computation yields

$$mu^{m-1}\partial_t\zeta = \mathcal{K}_{g_0}\zeta + \frac{m}{|1-m|}\eta\zeta, \tag{6.4}$$

where

$$\eta = \int_0^1 m((1-\lambda)\varphi + \lambda u)^{m-1}\,\mathrm{d}\lambda.$$

By (6.1)–(6.2), η and $\partial_t u$ are uniformly bounded. Multiplying both sides of (6.4) by ζ and integrating over M, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \zeta^2 u^{m-1}\,\mathrm{d}\,\mathrm{vol}_{g_0} \le C\int_M \zeta^2 u^{m-1}\,\mathrm{d}\,\mathrm{vol}_{g_0}.$$

By applying Gronwall's inequality and taking into account (6.2), we finish the proof.

Lemma 6.2 *Assume as in Lemma* 6.1*. Then we have*

$$\|\partial_t\zeta(\cdot,t+1)\|_{C^0(M)} + \|\zeta(\cdot,t+1)\|_{C^0(M)} \le C\|\zeta(\cdot,t)\|_{L^2(M)}, \quad t\ge 1,$$

where $C>0$ depends only on M, g, n, σ, Λ and C_0.

Proof Since ζ satisfies the linear equation (6.4), the lemma follows from a bootstrap argument with using Lemma 6.1 as a starting point.

Theorem 6.1 *Assume as above. Suppose that $u \in C^1([0,\infty); C^0(M))$ is a positive solution of* (6.1) *satisfying* (6.2)*. Then*

$$\lim_{t\to\infty} u(t) = S \quad \text{uniformly on } M,$$

where S is a positive solution of (6.3)*.*

Proof Since the flow possesses a gradient structure, the idea of Simon [35] can be adapted. We follow the proof of [25, Theorem 1.2] with some slight deviation in establishing local estimates.

Let $\beta = \frac{m}{|1-m|}$. By (6.1), we have, for any $0 \le t_0 < t$,

$$u(X,t)^m = \mathrm{e}^{-\beta(t-t_0)}u(X,t_0)^m + \int_{t_0}^t \mathrm{e}^{\beta(s-t)}\mathcal{K}_{g_0}(u)(X,s)\,\mathrm{d}s. \tag{6.5}$$

By the potential estimates, we have

$$\begin{aligned} &u(X,t)^m - u(Y,t)^m \\ &\le \mathrm{e}^{-\beta(t-t_0)}(u(X,t_0)^m - u(Y,t_0)^m) + C|X-Y|^\alpha \int_{t_0}^t \mathrm{e}^{\beta(s-t)}\,\mathrm{d}s \\ &\le \mathrm{e}^{-\beta(t-t_0)}(u(X,t_0)^m - u(Y,t_0)^m) + C|X-Y|^\alpha, \quad \forall\, X,Y\in M, \end{aligned} \tag{6.6}$$

where C and α are positive constants depending only on M, g_0, K_0, m and C_0 in (6.2). By taking $t_0 = 0$, it follows that $u(t)$ is uniformly continuous for $t\in[0,\infty)$ and there exists a sequence $t_j\to\infty$ and a positive function $u_\infty \in C(M)$ such that

$$u(t_j) \to u_\infty \quad \text{in } C^0(M) \quad \text{as } j\to\infty.$$

Claim u_∞ is a solution of (6.3).

Indeed, let

$$G_m(u) = \int_M \Big(\frac{1}{2}u\mathcal{K}_{g_0}u - \frac{\beta}{m+1}u^{m+1}\Big)\,\mathrm{d}\,\mathrm{vol}_{g_0}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}G_m(u) = \int_M (\mathcal{K}_{g_0}u - \beta u^m)\partial_t u\,\mathrm{d}\,\mathrm{vol}_{g_0} = m\int_M |\partial_t u|^2 u^{m-1}\,\mathrm{d}\,\mathrm{vol}_{g_0} \geq 0. \tag{6.7}$$

By the assumption (6.2), $G(u)$ is bounded and hence,

$$\lim_{t\to\infty} G(u(t)) = G(u_\infty).$$

For $\tau > 0$, we have

$$\begin{aligned}
&\int_M |u(t_j+\tau)^{\frac{m+1}{2}} - u(t_j)^{\frac{m+1}{2}}|^2\,\mathrm{d}\,\mathrm{vol}_{g_0}\\
&= \int_M \Big|\int_{t_j}^{t_j+\tau} \partial_s u(t_j+s)^{\frac{m+1}{2}}\,\mathrm{d}s\Big|^2\,\mathrm{d}\,\mathrm{vol}_{g_0}\\
&\leq \tau\frac{(m+1)^2}{4}\int_M\int_{t_j}^{t_j+\tau} |\partial_s u(t_j+s)|u(t_j+s)^{m-1}\,\mathrm{d}s\mathrm{d}\,\mathrm{vol}_{g_0}\\
&= \tau\frac{(m+1)^2}{4m}(G(u(t_j+\tau)) - G(u(t_j))).
\end{aligned}$$

Using the pointwise estimate

$$|u(X,t_j+\tau)^{\frac{m+1}{2}} - u(X,t_j)^{\frac{m+1}{2}}| \leq |u(X,t_j+\tau) - u(X,t_j)|^{\frac{m+1}{2}},$$

we have $u(X,t_j+\tau) \to u_\infty$ in L^{m+1} uniformly in τ. By interpolation inequality, we have, for any $m+1 < q < \infty$,

$$u(X,t_j+\tau) \to u_\infty \quad \text{in } L^q(M)$$

uniformly in τ. By (6.5),

$$u(X,t_j+1)^m = \mathrm{e}^{-\beta}u(X,t_j)^m + \int_{t_j}^{t_j+1} \mathrm{e}^{\beta(s-t_j-1)}\mathcal{K}_{g_0}(u)(X,s)\,\mathrm{d}s.$$

Sending $j\to\infty$, we obtain

$$u_\infty^m = \mathrm{e}^{-\beta}u_\infty^m + \mathcal{K}_{g_0}(u_\infty)\lim_{j\to\infty}\int_{t_j}^{t_j+1} \mathrm{e}^{\beta(s-t_j-1)}\,\mathrm{d}s,$$

i.e.,

$$\beta u_\infty^m = \mathcal{K}_{g_0}(u_\infty).$$

The claim is verified.

Since the functional $G(\cdot)$ is real analytic on

$$\omega_C = \Big\{f \in L^{m+1}(M) : \frac{1}{C} \leq f \leq C\Big\},$$

the so-called Lojasiewicz-Simon gradient inequality holds; see Chill [10]. Then one can use the standard argument to show that

$$u(t) \to u_\infty \quad \text{in } C(M) \quad \text{as } t \to \infty.$$

In fact, armed with Lemmas 6.1–6.2, one can mimic the corresponding proof in [25] to establish the aforementioned full convergence. We omit the details. The theorem is proved.

As in [25], the linearized operator at S will play a crucial role in the convergence rate. We may consider the eigenvalue problem

$$\mathcal{K}_{g_0}(\phi) = \lambda S^{m-1}\phi \quad \text{on } M. \tag{6.8}$$

To seek a symmetry structure, we introduce

$$\mathrm{d}\mu = S^{1-m}\mathrm{d}\,\mathrm{vol}_{g_0}, \quad \mathcal{K}^\mu(f)(X) = \int_M K_0(X,Y)f(Y)\,\mathrm{d}\mu$$

and $L^2(M, \mathrm{d}\mu)$ space equipped with the inner product

$$\langle f, h\rangle_{L^2(M,\mathrm{d}\mu)} = \int_M fh\,\mathrm{d}\mu.$$

Note that

$$\langle \mathcal{K}^\mu(f), h\rangle_{L^2(M,\mathrm{d}\mu)} = \langle f, \mathcal{K}^\mu(h)\rangle_{L^2(M,\mathrm{d}\mu)}$$

and $\mathcal{K}^\mu : L^2(M, \mathrm{d}\mu) \to L^2(M, \mathrm{d}\mu)$ is compact. The eigenvalue problem

$$\mathcal{K}^\mu(\varphi) = \lambda\varphi \quad \text{in } L^2(M, \mathrm{d}\mu) \tag{6.9}$$

has countable many eigenvalues, which must be real. If φ is an eigenfunction, then $\phi = S^{1-m}\varphi$ will be an eigenfunction of (6.8). In line with [25], one can establish a sharp convergence rate under a further L^2 positive assumption. Namely,

$$\int_M f\mathcal{K}_{g_0}f\,\mathrm{d}\,\mathrm{vol}_{g_0} > 0, \quad \forall f \in L^2(M),\ f \neq 0. \tag{K-5}$$

We leave the details to the interested reader.

Proof of Theorem 1.1 If $m > 1$, it follows from Proposition 4.1. If $\frac{n-2\sigma}{m+2\sigma} < m < 1$, it follows from Proposition 4.3 and Theorem 6.1. If $m = \frac{n-2\sigma}{m+2\sigma}$, by Proposition 4.2 we have the lower and upper bound. Let $\widetilde{u}$ be defined as in (4.3) and

$$G(\widetilde{u}) = \int_M \Big[\frac{1}{2}\widetilde{u}\mathcal{K}_{g_0}\widetilde{u} - \frac{m}{(1-m)(1+m)}\widetilde{u}^{m+1}\Big].$$

By direct computation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \int_M \partial_t\widetilde{u}^m\partial_t\widetilde{u} \geq 0.$$

Since G is bounded, $\lim\limits_{t\to\infty} G = G_\infty$ exists. On the other hand,

$$\frac{m}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_M \widetilde{u}^{m+1} = 2G + \frac{m}{1+m}\int_M \widetilde{u}^{m+1}.$$

Thus

$$\int_M \widetilde{u}(t)^{m+1} = \mathrm{e}^t \Big(\int_M \widetilde{u}_0^{m+1} + \frac{2(m+1)}{m} \int_0^t \mathrm{e}^{-s} G(\widetilde{u}(s))\, \mathrm{d}s \Big).$$

Since $\int_M \widetilde{u}(t)^{m+1}$ is bounded, this forces

$$\frac{2(m+1)}{m} \int_0^\infty \mathrm{e}^{-s} G(\widetilde{u}(s))\, \mathrm{d}s = -\int_M \widetilde{u}_0^{m+1}.$$

It follows that

$$\int_M \widetilde{u}(t)^{m+1} = -\frac{2(m+1)}{m} \int_t^\infty \mathrm{e}^{t-s} G(\widetilde{u}(s))\, \mathrm{d}s \to -\frac{2(m+1)}{m} G_\infty$$

as $t \to \infty$. We complete the proof of Theorem 1.1.

Proof of Theorem 1.3 By Theorem 5.1, we have global positive upper and lower bounds. Using Lemma 5.2 and a change of variable, we can transform the normalized flow into (6.1). Theorem 1.3 then follows from Theorem 6.1.

Acknowledgement The author would like to express his gratitude to Tianling Jin at HKUST for the friendship and fruitful collaboration which has inspired some of the ideas in the current paper.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Aubin, T., Equations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* **55**, 1976, 269–296.

[2] Bahri, A. and Coron, J.-M., The scalar-curvature problem on the standard three-dimensional sphere, *J. Funct. Anal.,* **95** (1), 1991, 106–172.

[3] Bonforte, M. and Endal, J., Nonlocal nonlinear diffusion equations, Smoothing effects, Green functions, and functional inequalities, *J. Funct. Anal.,* **284**, 2023, 104.

[4] Branson, T. P., Differential operators canonically associated to a conformal structure, *Math. Scand.,* **57**(2), 1985, 295–345.

[5] Brendle, S., Convergence of the Yamabe flow for arbitrary initial energy, *J. Differential Geom.,* **69**, 2005, 217–278; Convergence of the Yamabe flow in dimension 6 and higher, *Invent. Math.,* **170**(3), 2007, 541–576.

[6] Chan, H., Sire, Y. and Sun, L., Convergence of the fractional Yamabe flow for a class of initial data, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), **21**, 2020, 1703–1740.

[7] Chang, S.-Y. and González, M., Fractional Laplacian in conformal geometry, *Adv. Math.,* **226**, 2011, 1410–1432.

[8] Chang, S.-Y. and Yang, R., On a class of non-local operators in conformal geometry, *Chinese Ann. Math. Ser. B,* **38** (1), 2017, 215–234.

[9] Chen, W., Li, C. and Ou, B., Classification of solutions for an integral equation, *Comm. Pure Appl. Math.,* **59**, 2006, 330–343.

[10] Chill, R., On the Lojasiewicz-Simon gradient inequality, *J. Funct. Anal.,* **201** (2), 2003, 572–601.

[11] Chow, B., The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature, *Comm. Pure. Appl. Math.,* **45**, 1992, 1003–1014.

[12] Daskalopoulos, P., Sire, Y. and Vázquez, J.-L., Weak and smooth solutions for a fractional Yamabe flow: The case of general compact and locally conformally flat manifolds, *Comm. Partial Differential Equations,* **42** (9), 2017, 1481–1496.

[13] Dou, J. and Zhu, M., Reversed Hardy-Littewood-Sobolev inequality, *Int. Math. Res. Not. IMRN,* **2015**, 2015, 9696–9726.

[14] Fefferman, C. and Graham, C. R., The Ambient Metric, Annals of Mathematics Studies, **178**, Princeton University Press, Princeton, NJ, 2012.

[15] Giaquinta, M. and Giusti, E., On the regularity of the minima of variational integrals, *Acta Math.,* **148**, 1982, 31–46.

[16] Gursky, M. and Malchiodi, A., A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature, *J. Eur. Math. Soc.* ($JEMS$), **17** (9), 2015, 2137–2173.

[17] Graham, C. R., Jenne, R., Mason, L. J. and Sparling, G. A. J., Conformally invariant powers of the Laplacian, I, Existence, *J. London Math. Soc.*(2), **46**, 1992, 557–565.

[18] Graham, C. R. and Zworski, M., Scattering matrix in conformal geometry, *Invent. Math.,* **152**, 2003, 89–118.

[19] Han, Y. and Zhu, M., Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications, *J. Differential Equations,* **260**, 2016, 1–25.

[20] Hang, F. and Yang, P., Q-curvature on a class of manifolds of dimension at least 5, *Comm. Pure Appl. Math.,* **69** (8), 2016, 1452–1491.

[21] Hebey, E. and Robert, F., Compactness and global estimates for the geometric Paneitz equation in high dimensions, *Electron. Res. Announc. AMS,* **10**, 2004, 135–141.

[22] Jin, T., Li, Y. Y. and Xiong, J., The Nirenberg problem and its generalizations: A unified approach, *Math. Ann.,* **369** (1–2), 2017, 109–151.

[23] Jin, T. and Xiong, J., A fractional Yamabe flow and some applications, *J. Reine Angew. Math.,* **696** 2014, 187–223.

[24] Jin, T. and Xiong, J., Asymptotic symmetry and local behavior of solutions of higher order conformally invariant equations with isolated singularities, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **38**, 2021, 1167–1216.

[25] Jin, T., Xiong, J. and Yang, X., Stability of the separable solutions for a nonlinear boundary diffusion problem, *J. Math. Pures Appl.*(9), 183, 2024, 1–43.

[26] Li, A. and Li, Y. Y., On some conformally invariant fully nonlinear equations, II, Liouville, Harnack and Yamabe, *Acta Math.,* **195**, 2005, 117–154.

[27] Li, Y. Y., Remark on some conformally invariant integral equations: The method of moving spheres, *J. Eur. Math. Soc.* ($JEMS$), **6**, 2004, 153–180.

[28] Li Y. Y. and Xiong, J., Compactness of conformal metrics with constant Q-curvature, I, *Adv. Math.,* **345**, 2019, 116–160.

[29] Lieb, E. H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.* (2), **118**, 1983, 349–374.

[30] Malchiodi, A., On conformal metrics with constant Q-curvature, *Anal. Theory Appl.,* **35**, 2019, 117–143.

[31] Paneitz, S., A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, *SIGMA Symmetry Integrability Geom. Methods Appl.,* **4**, 2008.

[32] Qing, J. and Raske, D., On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds, *Int. Math. Res. Not.,* 2006, 20 PP.

[33] Schoen, R., Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Diff. Geom.,* **20**, 1984, 479–495.

[34] Schwetlick, H. and Struwe, M., Convergence of the Yamabe flow for "large" energies, *J. Reine Angew. Math.,* **562**, 2003, 59–100.

[35] Simon, L., Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems, *Ann. of Math.* (2), **118**, 1983, 525–571.

[36] Struwe, M., A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.,* **187**(4), 1984, 511–517.

[37] Trudinger, N., Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Annali Scuola Norm. Sup. Pisa,* **22**, 1968, 265–274.

[38] Yamabe, H., On a deformation of Riemannian structures on compact manifolds, *Osaka Math J.,* **12**, 1960, 21–37.

[39] Ye, R., Global existence and convergence of Yamabe flow, *J. Differential Geom.,* **39**, 1994, 35–50.

[40] Zhu, M., Prescribing integral curvature equation, *Differential Integral Equations,* **29**, 2016, 889–904.