Chin. Ann. Math. Ser. B
42(4), 2021, 529–550
DOI: 10.1007/s11401-021-0276-3

Chinese Annals of Mathematics, Series B

Some Gradient Estimates and Liouville Properties of the Fast Diffusion Equation on Riemannian Manifolds*

Wen WANG[1] Rulong XIE[2] Pan ZHANG[3]

Abstract In the paper, the authors provide a new proof and derive some new elliptic type (Hamilton type) gradient estimates for fast diffusion equations on a complete noncompact Riemannian manifold with a fixed metric and along the Ricci flow by constructing a new auxiliary function. These results generalize earlier results in the literature. And some parabolic type Liouville theorems for ancient solutions are obtained.

Keywords Gradient estimate, Fast diffusion equation, Ricci flow, Liouville theorem
2000 MR Subject Classification 58J35, 35K05, 53C21

1 Introduction and Main Results

In this paper, we continue to consider the fast diffusion equation (FDE for short)

$$u_t = \Delta_{g(t)} u^{\alpha}, \quad 0 < \alpha < 1, \tag{1.1}$$

on a family of Riemannian manifolds $(M, g(t))$ for two cases: The one is that $g(t)$ is some fixed metric, and the other one is $g(t)$ deformed by the Ricci flow:

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)).$$

Li and Yau [16] established a famous space-time gradient estimate for positive solutions to the heat equation. In 1993, Hamilton [9] proved the space-only gradient estimate for closed manifolds, which was extended by Souplet and Zhang in [21] to the complete noncompact manifolds. Bailesteanu, Cao and Pulemotov [1] generalized the Hamilton's gradient estimates to the Ricci flow. For the developments, see [4, 10, 12, 17–18, 20, 22–24, 27]. In 2009, Lu, Ni, Vázquez and Villani [19] studied the FDE (1.1) on Riemannian manifolds, and derived a local space-time gradient estimates. Later in [28], Zhu studied the FDE (1.1) on complete

Manuscript received September 20, 2020. Revised March 29, 2021.
[1]School of Mathematics and Statistics, Hefei Normal University, Hefei 230601, China.
E-mail: wwen2014@mail.ustc.edu.cn, wenwang1985@163.com
[2]School of Mathematics and Statistics, Chaohu University, Chaohu 238024, Anhui, China.
E-mail: xierl@ustc.edu.cn.
[3]School of Mathematical Sciences, Anhui University, Hefei 230601, China; School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China. E-mail: panzhang20100@ahu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11721101, 11971026), the Natural Science Foundation of Anhui Province (Nos. 1908085QA04, 2008085QA08) and Natural Science Foundation of Education Committee of Anhui Province (Nos. KJ2017A454, KJ2019A0712, KJ2019A0713), Excellent Young Talents Foundation of Anhui Province (Nos. GXYQ2017048, GXYQ2017070, GXYQ2020049) and the research project of Hefei Normal University (No. 2020PT26).

noncompact Riemannian manifolds, and derived the following space-only gradient estimate (Hamilton type gradient estimate) and Liouville type theorem.

Theorem A (see [28]) *Let (M^n, g) be a Riemannian manifold with $n \geq 2$ and $\mathrm{Ric}(M^n) \geq -k$ for some $k \geq 0$. Suppose that u is an arbitrary positive solution to the FDE* (1.1) *in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$ and $v \leq M$. Then for $1 - \frac{2}{n} < \alpha < 1$,*

$$\frac{|\nabla v|}{v^{\frac{1}{2}}} \leq C M^{\frac{1}{2}} \Big(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \Big). \tag{1.2}$$

Recently, Xu [26], Huang and Ma [11] improved the result of Zhu [28]. Huang and Ma [10] proved a gradient estimate and Liouville theorem for FDE (1.1) with $1 - \frac{1}{-3+2\sqrt{n+3}} < \alpha < 1 - \frac{3}{n+3}, n \neq 6$. Xu [26] derived the gradient estimate for the FDE (1.1) with $1 - \frac{4}{n+3} < \alpha < 1$. Cao and Zhu [3] proved Li-Yau-Hamilton type differential Harnack estimates for positive solutions of the FDE (1.1). Li, Bai and Zhang [13] proved Hamilton type gradient estimates for the fast diffusion equations under the Ricci flow.

Our results of this paper are encouraged by the work in [1, 3, 8, 10–11, 13, 21–22, 25, 28–29]. We consider the FDE (1.1), and derive some elliptic type (Hamilton-Souplet-Zhang type) gradient estimates.

To prove the propertity of the positive solution of the FDE (1.1), we will use the following transformation: Let $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$. Then

$$v_t = (1-\alpha) v \Delta v - |\nabla v|^2. \tag{1.3}$$

Our paper is organized as follows: We show our main results in Section 1. We will give some lemmas and the proof of the main results on Riemannian manifolds with a fixed metric in Section 2. The proof of the main results on Riemannian manifolds along the Ricci flow will be given in Section 3.

Theorem 1.1 *Let $(\mathrm{M^n}, g)$ be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}(\mathrm{M^n}) \geq -K$ for some $K \geq 0$ in $B_{x_0,R}$, which is a geodesic ball centered at some fixed point x_0 in $\mathrm{M^n}$ with radius R. Assume that v is any positive solution to* (1.3) *in $Q_{R,T} = B_{x_0,R} \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$ with $0 < \delta \leq v \leq A$ for some constants δ and A.*

(1) *If $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$, then*

$$\frac{|\nabla v|^2}{v^{2-\beta}} \leq C\delta^{\beta} \Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{C}{\delta T} \Big) \quad \text{in } Q_{\frac{R}{2}, \frac{T}{2}}. \tag{1.4}$$

(2) *If $1 - \frac{4}{n+4} < \alpha < 1$, then*

$$\frac{|\nabla v|^2}{v^2} \leq C \Big(K + \frac{1}{R^2} + \frac{1}{\delta T} \Big) \quad \text{in } Q_{\frac{R}{2}, \frac{T}{2}}. \tag{1.5}$$

Here $\beta = -\frac{\alpha}{2(1-\alpha)}$ and $C = C(n, \alpha)$ is a positive constant.

By applying Theorem 1.1, we deduce the following Liouville type theorem.

Theorem 1.2 *Let* $(\mathrm{M}^{\mathrm{n}}, g)$ *be an* n*-dimensional complete, noncompact manifold with nonnegative Ricci curvature. Let* u *be a positive solution to* (1.1) *and* $d(x)$ *be the geodesic distance of* g*.*

(1) *If* $1-\frac{3+\sqrt{16+2n}}{7+2n}<\alpha<1$ *and* $\frac{1}{u(x,t)}=o\big([d(x)+|t|]^{\frac{1}{1-\alpha}}\big)$ *near infinity, then* u *is a constant.*

(2) *If* $1-\frac{4}{n+4}<\alpha<1$ *and* $\frac{1}{u(x,t)}=o\big([d(x)+|t|]^{\frac{1}{1-\alpha}}\big)$ *near infinity, then* u *is a constant.*

Remark 1.1 (1) When $n\geq 2$, we have

$$\begin{aligned}&\frac{3+\sqrt{16+2n}}{7+2n}-\frac{1}{-3+2\sqrt{n+3}}\\ =\ &\frac{6n-12+(4n+3)\sqrt{16+2n}-(4n+14)\sqrt{n+3}}{(7+2n)(4n+3)}>0,\end{aligned}$$

that is

$$1-\frac{3+\sqrt{16+2n}}{7+2n}<1-\frac{1}{-3+2\sqrt{n+3}}.$$

(2) When $n\geq 4$, we have

$$\frac{3+\sqrt{16+2n}}{7+2n}-\frac{2}{n}=\frac{n\sqrt{16+2n}-n-14}{n(7+2n)}>0,$$

that is

$$1-\frac{3+\sqrt{16+2n}}{7+2n}<1-\frac{2}{n}.$$

(3) When $n\geq 7$, we have

$$\frac{3+\sqrt{16+2n}}{7+2n}-\frac{4}{n+3}=\frac{(n+3)\sqrt{16+2n}-5n-19}{(n+3)(7+2n)}>0,$$

that is

$$1-\frac{3+\sqrt{16+2n}}{7+2n}<1-\frac{4}{n+3}.$$

Hence, Theorems 1.1–1.2 generalize some known results in [11, 26, 28].

Remark 1.2 Our proof is a little different from the proofs of Zhu [28], Huang, Ma [11] and Xu [26]. We derive the evolution equation of quantity $\log\frac{A}{v}$ with $v\leq A$.

Remark 1.3 When $n\geq 4$, we have

$$1-\frac{4}{n+4}\leq 1-\frac{2}{n}.$$

When $3\leq n\leq 29$, we have

$$\frac{4}{n+4}-\frac{1}{-3+2\sqrt{n+3}}=\frac{13n-2(n+4)\sqrt{n+3}}{(n+4)(4n+3)}>0,$$

that is

$$1-\frac{4}{n+4}<1-\frac{1}{-3+2\sqrt{n+3}}.$$

So, (1.5) and Theorem 1.2 generalize the results of Zhu [28], Huang and Ma [11].

Remark 1.4 The upper bound of the gradient estimate (1.5) does not contain the upper bound of v.

Theorem 1.3 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}(\mathrm{M^n}) \geq -K$ *for some* $K \geq 0$ *in* $B_{x_0,R}$*, which is a geodesic ball centered at some fixed point* x_0 *in* $\mathrm{M^n}$ *with radius* R*. Assume that* v *is any positive solution to* (1.3) *in* $Q_{R,T} = B_{x_0,R} \times [t_0 - T, t_0] \subset \mathrm{M^n} \times (-\infty, \infty)$ *with* $v \leq A$*. Let* $1 - \frac{2}{n+4} < \alpha < 1$*. Then there exist a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla v|^2}{v} \leq CA\Big(K + \frac{1}{R^2}\Big) + \frac{C}{T} \tag{1.6}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$.

Moreover, if $(\mathrm{M^n}, g)$ *has nonnegative Ricci curvature and* u *is any positive solution to* (1.1) *on* $\mathrm{M^n} \times (0, \infty)$*, then there exists a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla v|^2}{v} \leq \frac{C}{T}. \tag{1.7}$$

By applying Theorem 1.3, we deduce the following Liouville type theorem.

Theorem 1.4 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete, noncompact manifold with nonnegative Ricci curvature. Let* u *be a positive solution to* (1.1) *with* $1 - \frac{2}{n+4} < \alpha < 1$ *such that* $\frac{1}{u(x,t)} = o\big([d(x) + |t|]^{\frac{1}{1-\alpha}}\big)$ *near infinity, where* $d(x)$ *is the geodesic distance of* g*. Then* u *is a constant.*

When $u(x,t)$ is independent on t, by (1.6), we can derive the following Liouville type theorem.

Theorem 1.5 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete, noncompact manifold with nonnegative Ricci curvature. Let* u *be a positive solution to the equation*

$$\Delta u^m = 0, \quad 1 - \frac{2}{n+4} < \alpha < 1. \tag{1.8}$$

Assume that $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$ *with* $0 < v \leq A$ *for some constant* A*. Then* u *is a constant.*

Daskalopoulos et al. [6–7] observed that the metric $g = u^{\frac{4}{n-2}} \mathrm{d}y^2$ satisfies the Yamabe flow (see [2])

$$\frac{\partial g}{\partial t} = -Rg$$

on $\mathbb{R}^n$, $n \geq 3$, for $0 < t < T$, where R is the scalar curvature of the metric g, if and only if u satisfies

$$u_t = \frac{(n-1)(n+2)}{n-2} \Delta u^{\frac{n-2}{n+2}}.$$

When $n \geq 17$, we have $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \frac{n-2}{n+2} < 1$. Therefore, we obtain the following theorem.

Theorem 1.6 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}(\mathrm{M^n}) \geq -K$ *for some* $K \geq 0$ *in* $B_{x_0,R}$*, which is a geodesic ball centered at some fixed point* x_0 *in* $\mathrm{M^n}$ *with radius* R*. Assume that* u *is any positive solution to the equation*

$$u_t = \Delta u^{\frac{n-2}{n+2}}, \quad n \geq 17 \tag{1.9}$$

in $Q_{R,T} = B_{x_0,R} \times [t_0 - T, t_0] \subset \mathrm{M^n} \times (-\infty, \infty)$. *Assume also that* $v = \frac{n-2}{4} u^{-\frac{4}{n+2}}$ *with* $0 < \delta \le v \le A$ *for some constants* δ *and* A. *Then there exists a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla v|^2}{v^{2-\beta}} \le C\delta^\beta \Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{C}{\delta T}\Big) \tag{1.10}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$, *where* $\beta = -\frac{\alpha}{2(1-\alpha)}$.

By using Theorem 1.6, we deduce the following Liouville type theorem.

Theorem 1.7 *Let* $(\mathrm{M^n}, g)$ *be an n-dimensional complete, noncompact manifold with non-negative Ricci curvature. Let u be a positive solution to* (1.9) *such that* $\frac{1}{u(x,t)} = o\big([d(x)+|t|]^{\frac{n+2}{4}}\big)$ *near infinity, where* $d(x)$ *is the geodesic distance of* g. *If* $n \ge 17$, *then u is a constant.*

Next, we state our estimates for the FDE coupled with the Ricci flow (see [5, 9]), which are similar to the fixed metric case.

Let $(\mathrm{M^n}, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)). \tag{1.11}$$

Theorem 1.8 *Let* $(\mathrm{M^n}, g(x,t))_{t\in[0,T]}$ *be a complete solution to* (1.11). *Suppose that* $|\mathrm{Ric}(x,t)| \le K$ *for some* $K \ge 0$ *and all* $(x,t) \in B_{R,T} = B(x_0, R) \times (0, T]$ *for some fixed* $x_0 \in \mathrm{M^n}$. *Assume that v is any positive solution to the equation*

$$v_t = (1-\alpha) v \Delta_{g(t)} v - |\nabla^{g(t)} v|^2.$$

Assume also that $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$ *with* $0 < \delta \le v \le A$.

(1) *If* $1 - \frac{4}{n+4} < \alpha < 1$, *then*

$$\frac{|\nabla^{g(t)} v|^2}{v^2} \le C\Big([(1-\alpha)A + 1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big) \quad \text{in } B_{\frac{R}{2},T}. \tag{1.12}$$

(2) *If* $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$, *then*

$$\frac{|\nabla^{g(t)} v|^2}{v^{2-\beta}} \le C\delta^\beta\Big([(1-\alpha)A + 1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big) \quad \text{in } B_{\frac{R}{2},T}. \tag{1.13}$$

Here $\beta = -\frac{\alpha}{2(1-\alpha)}$ *and* $C = C(n, \alpha)$ *is a positive constant.*

Remark 1.5 Since

$$1 - \frac{4}{n+4} \le 1 - \frac{4}{n+8}.$$

So, Theorem 1.8 generalizes the one of Li, Bai and Zhang [13].

Theorem 1.9 *Let* $(\mathrm{M^n}, g(x,t))_{t\in[0,T]}$ *be a complete solution to* (1.11). *Suppose that* $|\mathrm{Ric}(x,t)| \le K$ *for some* $K \ge 0$ *and all* $(x,t) \in B_{R,T} = B(x_0, R) \times (0, T]$ *for some fixed* $x_0 \in \mathrm{M^n}$. *Assume that v is any positive solution to the equation*

$$v_t = (1-\alpha) v \Delta_{g(t)} v - |\nabla^{g(t)} v|^2.$$

Assume also that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$ *with* $v \le A$. *Let* $1 - \frac{2}{n+4} < \alpha < 1$. *Then there exist a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla^{g(t)}v|^2}{v} \le CA\Big([(1-\alpha)A + 1]K + \frac{1}{R^2}\Big) + \frac{C}{t} \tag{1.14}$$

in $B_{\frac{R}{2},T}$.

When $n \ge 17$, we have $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \frac{n-2}{n+2} < 1$. Therefore, we obtain the following theorem.

Theorem 1.10 *Let* $(\mathrm{M}^{\mathrm{n}}, g(x,t))_{t\in[0,T]}$ *be a complete solution to* (1.11). *Suppose that* $|\mathrm{Ric}(x,t)| \le K$ *for some* $K \ge 0$ *and all* $(x,t) \in B_{R,T} = B(x_0, R) \times (0, T]$ *for some fixed* $x_0 \in \mathrm{M}^{\mathrm{n}}$. *Assume that* u *is any positive solution to the equation*

$$u_t = \Delta^{g(t)} u^{\frac{n-2}{n+2}}, \quad n \ge 17$$

in $B_{R,T}$. *Assume also that* $v = \frac{n-2}{4}u^{-\frac{4}{n+2}}$ *with* $0 < \delta \le v \le A$ *for some constants* δ *and* A. *Then there exists a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla^{g(t)}v|^2}{v^{2-\beta}} \le C\delta^\beta \Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{C}{\delta t}\Big) \tag{1.15}$$

in $B_{\frac{R}{2},T}$, *where* $\beta = -\frac{\alpha}{2(1-\alpha)}$.

2 FED Under the Fixed Metric

2.1 Basic lemmas

Before proving the main theorems, we need some lemmas. Consider the equation

$$v_t = (1-\alpha)v\Delta v - |\nabla v|^2 \tag{2.1}$$

on a complete Riemannian manifold $(\mathrm{M}^{\mathrm{n}}, g)$. Let $v(x,t)$ be a solution of (2.1) and $0 < v < A$ for some constant A in the cylinder

$$Q_{R,T} :\equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty),$$

here $t_0 \in \mathbb{R}$ and $T > 0$. We first introduce a new smooth function

$$g = \log \frac{A}{v}$$

in $Q_{R,T}$. Then $v = A \cdot \mathrm{e}^{-g}$,

$$\begin{aligned} v_t &= -A\mathrm{e}^{-g}g_t = -vg_t, \\ \nabla v &= -A\mathrm{e}^{-g}\nabla g = -v\nabla g, \\ \Delta v &= -A\mathrm{e}^{-g}\Delta g + A\mathrm{e}^{-g}|\nabla g|^2 = -v\Delta g + v|\nabla g|^2. \end{aligned} \tag{2.2}$$

From (2.1), we have

$$\begin{aligned} g_t &= -\frac{1}{A\mathrm{e}^{-g}}v_t = -\frac{1}{v}[(1-\alpha)v\Delta v - |\nabla v|^2] \\ &= -(1-\alpha)[-v\Delta g + v|\nabla g|^2] + v|\nabla g|^2 \\ &= (1-\alpha)v\Delta g + \alpha v|\nabla g|^2. \end{aligned} \tag{2.3}$$

By utilizing the above equation (2.3), we can derive the following lemma.

Lemma 2.1 *Let $\omega = |\nabla g|^2$. Then for any $(x,t) \in Q_{R,T}$,*

$$(1-\alpha)v\Delta\omega - \omega_t = bv\omega^2 - 2K(1-\alpha)v\omega - 2\alpha v\langle \nabla\omega, \nabla g\rangle, \tag{2.4}$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$.

Proof By using the Bochner-Weitzenböck formula

$$\Delta|\nabla g|^2 = 2|\nabla^2 g|^2 + 2\mathrm{Ric}(\nabla g, \nabla g) + 2\langle \nabla\Delta g, \nabla g\rangle, \tag{2.5}$$

we have

$$\begin{aligned}(1-\alpha)v\Delta\omega - \omega_t = {}& 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g)\\ &+ 2(1-\alpha)v\langle \nabla\Delta g, \nabla g\rangle - \omega_t.\end{aligned}$$

By (2.3), we obtain

$$\begin{aligned}&(1-\alpha)v\Delta\omega - \omega_t\\ ={}& 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) + 2v\Big\langle \nabla\Big(\frac{g_t}{v} - \alpha|\nabla g|^2\Big), \nabla g\Big\rangle - \omega_t\\ ={}& 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) + 2\langle \nabla g_t, \nabla g\rangle\\ &- \frac{2g_t}{v}\langle \nabla v, \nabla g\rangle - 2\alpha v\langle \nabla\omega, \nabla g\rangle - \omega_t\\ ={}& 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g)\\ &- \frac{2g_t}{v}\langle \nabla v, \nabla g\rangle - 2\alpha v\langle \nabla\omega, \nabla g\rangle.\end{aligned} \tag{2.6}$$

By applying (2.2)–(2.3) and the Cauchy inequality, we have

$$\begin{aligned}&2(1-\alpha)v|\nabla^2 g|^2 - \frac{2g_t}{v}\langle \nabla v, \nabla g\rangle\\ ={}& 2(1-\alpha)v|\nabla^2 g|^2 + 2g_t|\nabla g|^2\\ \geq{}& 2(1-\alpha)v\frac{(\Delta g)^2}{n} + 2|\nabla g|^2[(1-\alpha)v\Delta g + \alpha v|\nabla g|^2]\\ \geq{}& -\frac{n(1-\alpha)}{2}v|\nabla g|^4 + 2\alpha v|\nabla g|^4.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) and noting that $\mathrm{Ric} \geq -K$, we have

$$\begin{aligned}&(1-\alpha)v\Delta\omega - \omega_t\\ \geq{}& 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) - \frac{n(1-\alpha)}{2}v|\nabla g|^4 + 2\alpha v|\nabla g|^4 - 2\alpha v\langle \nabla\omega, \nabla g\rangle\\ ={}& \Big[2\alpha - \frac{n(1-\alpha)}{2}\Big]v\omega^2 - 2K(1-\alpha)v\omega - 2\alpha v\langle \nabla\omega, \nabla g\rangle\\ ={}& bv\omega^2 - 2K(1-\alpha)v\omega - 2\alpha v\langle \nabla\omega, \nabla g\rangle,\end{aligned}$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$. The proof is completed.

Lemma 2.2 *Let $\varpi = v^\beta|\nabla g|^2$ with $\beta = -\frac{\alpha}{2(1-\alpha)}$. Then*

$$(1-\alpha)v\Delta\varpi - \varpi_t \geq -2K(1-\alpha)v\varpi + \gamma v^{1-\beta}\varpi^2 - \alpha v\langle \nabla\varpi, \nabla g\rangle, \tag{2.8}$$

where $\gamma = \frac{\alpha^2}{4(1-\alpha)} + 2\alpha - \frac{n(1-\alpha)}{2} > 0$ and $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$.

Proof Applying (2.5), we have

$$\begin{aligned}(1-\alpha)v\Delta\varpi-\varpi_t&=(1-\alpha)v^\beta v\Delta|\nabla g|^2+(1-\alpha)v|\nabla g|^2\Delta v^\beta\\&\quad+2(1-\alpha)v\nabla|\nabla g|^2\nabla v^\beta-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad+2(1-\alpha)v^{\beta+1}\langle\nabla\Delta g,\nabla g\rangle+(1-\alpha)v|\nabla g|^2\Delta v^\beta\\&\quad+2(1-\alpha)v\langle\nabla|\nabla g|^2,\nabla v^\beta\rangle-\varpi_t.\end{aligned}$$

By utilizing (2.1) and (2.3), we have

$$\begin{aligned}&(1-\alpha)v\Delta\varpi-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad+2v^{\beta+1}\Big\langle\nabla\Big(\frac{g_t}{v}-\alpha|\nabla g|^2\Big),\nabla g\Big\rangle-2(1-\alpha)\beta v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle\\&\quad+(1-\alpha)\beta(\beta-1)v^{\beta-1}|\nabla v|^2|\nabla g|^2+\beta(1-\alpha)v^\beta|\nabla g|^2\Delta v-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad+2v^\beta\langle\nabla g_t,\nabla g\rangle-2v^{\beta-1}g_t\langle\nabla v,\nabla g\rangle-2\alpha v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle\\&\quad-2(1-\alpha)\beta v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle+(1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4\\&\quad+\beta v^{\beta-1}|\nabla g|^2(v_t+|\nabla v|^2)-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad-2v^{\beta-1}g_t\langle\nabla v,\nabla g\rangle-2\alpha v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle\\&\quad-2(1-\alpha)\beta v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle+(1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4\\&\quad+\beta v^{\beta+1}|\nabla g|^4.\end{aligned}\tag{2.9}$$

By the Cauchy inequality, we have

$$\begin{aligned}&2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2-2v^{\beta-1}g_t\langle\nabla v,\nabla g\rangle\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2v^\beta g_t|\nabla g|^2\\&\geq\frac{2}{n}(1-\alpha)v^{\beta+1}(\Delta g)^2+2(1-\alpha)v^{\beta+1}|\nabla g|^2\Delta g+2\alpha v^{\beta+1}|\nabla g|^4\\&\geq-\frac{n(1-\alpha)}{2}v^{\beta+1}|\nabla g|^4+2\alpha v^{\beta+1}|\nabla g|^4.\end{aligned}\tag{2.10}$$

Combining (2.9) and (2.10), we have

$$\begin{aligned}&(1-\alpha)v\Delta\varpi-\varpi_t\\&\geq-2K(1-\alpha)v^{\beta+1}|\nabla g|^2\\&\quad+\Big[(1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha)\Big]v^{\beta+1}|\nabla g|^4\\&\quad-[2\alpha+2\beta(1-\alpha)]v\langle\nabla\varpi,\nabla g\rangle,\end{aligned}\tag{2.11}$$

where we use the fact that

$$\langle\nabla\varpi,\nabla g\rangle=v^\beta\langle\nabla|\nabla g|^2,\nabla g\rangle-\beta v^\beta|\nabla g|^4.$$

In order to obtain the gradient estimates, we need to require the coefficient $f(\beta)$ of $|\nabla g|^4$ to be positive. In fact,

$$\begin{aligned}f(\beta) &= (1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha)\\&= -(1-\alpha)\Big[\beta+\frac{\alpha}{2(1-\alpha)}\Big]^2+\frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2},\end{aligned}$$

we choose $\beta=-\frac{\alpha}{2(1-\alpha)}$, then $f(\beta)>0$ when $1-\frac{3+\sqrt{16+2n}}{7+2n}<\alpha<1$. Therefore, (2.11) can be written as

$$(1-\alpha)v\Delta\varpi-\varpi_t\geq -2K(1-\alpha)v\varpi+\gamma v^{1-\beta}\varpi^2-\alpha v\langle\nabla\varpi,\nabla g\rangle,$$

where $\gamma=\frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2}>0$. The proof is completed.

Taking $\beta=1$ in (2.11), the following lemma is derived.

Lemma 2.3 *Let* $\widetilde{\omega}=v|\nabla g|^2$. *Then*

$$(1-\alpha)v\Delta\widetilde{\omega}-\partial_t\widetilde{\omega}\geq\epsilon\widetilde{\omega}^2-2K(1-\alpha)v\widetilde{\omega}-2v\langle\nabla\widetilde{\omega},\nabla g\rangle, \tag{2.12}$$

where $\epsilon=2\alpha-1-\frac{n(1-\alpha)}{2}>0$ *with* $1-\frac{2}{n+4}<\alpha<1$.

We next introduce a smooth cut-off function (see [10, 17, 24]), which will be used in the proof of our main theorems.

Lemma 2.4 (see [16, 21, 28]) *We use the geodesic polar coordinate here. Assume that a function* $\varphi=\varphi(x,t)$ *is a smooth cut-off function supported in* $Q_{R,T}$, *satisfying the following properties*:

(1) $\varphi=\varphi(d(x,x_0),t)\equiv\varphi(r,t)$; $\varphi(r,t)=1$ *in* $Q_{\frac{R}{2},\frac{T}{2}}$, $0\leq\varphi\leq1$.
(2) φ *is decreasing as a radial function in the spatial variables.*
(3) $\frac{|\partial_r\varphi|}{\varphi^a}\leq\frac{C_a}{R}$, $\frac{|\partial_r^2\varphi|}{\varphi^a}\leq\frac{C_a}{R^2}$ *when* $0<a<1$.
(4) $\frac{|\partial_t\varphi|}{\varphi^{\frac12}}\leq\frac{C}{T}$.

2.2 The proof of theorems

In this section, we will prove our main theorems by Lemma 2.4.

Proof of Theorem 1.1 Part 1: Assume that the maximum of $\varphi\varpi$ is arrived at a point (x_1,t_1). By [16], we can suppose, without loss of generality, that x_1 is not on the cut-locus of M^{n}. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\varpi)\leq0$, $(\varphi\varpi)_t\geq0$ and $\nabla(\varphi\varpi)=0$. By

$$0=\nabla(\varphi\varpi)=\varpi\nabla\varphi+\varphi\nabla\varpi,$$

then

$$\nabla\varpi=-\frac{\nabla\varphi}{\varphi}\varpi.$$

Hence, by (2.8) and a straightforward calculation, it yields that

$$\begin{aligned}0&\geq(1-\alpha)v\Delta(\varphi\varpi)-(\varphi\varpi)_t\\&=\varphi[(1-\alpha)v\Delta\varpi-\varpi_t]+(1-\alpha)v\varpi\Delta\varphi+2(1-\alpha)v\nabla\varphi\nabla\varpi-\varpi\varphi_t\end{aligned}$$

$$\begin{aligned}
&\geq \gamma v^{1-\beta}\varphi\varpi^2 - 2K(1-\alpha)v\varphi\varpi - \alpha v\varphi\langle\nabla\varpi,\nabla g\rangle\\
&\quad + (1-\alpha)v\varpi\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\varpi - \varpi\varphi_t\\
&= \gamma v^{1-\beta}\varphi\varpi^2 - 2K(1-\alpha)v\varphi\varpi + \alpha v\varpi\langle\nabla\varphi,\nabla g\rangle\\
&\quad + (1-\alpha)v\varpi\Delta\varphi - 2(1-\alpha)v\varpi\frac{|\nabla\varphi|^2}{\varphi} - \varpi\varphi_t. \qquad (2.13)
\end{aligned}$$

This implies

$$\begin{aligned}
2\varphi\varpi^2 \leq{}& \frac{4}{\gamma}K(1-\alpha)v^\beta\varphi\varpi - \frac{2\alpha}{\gamma}\langle\nabla\varphi,\nabla g\rangle v^\beta\varpi\\
&- \frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi + \frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi + \frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t. \qquad (2.14)
\end{aligned}$$

We next estimate upper bounds for each term of the right hand side of (2.14). Applying the Young inequality, we have

$$\frac{4}{\gamma}K(1-\alpha)v^\beta\varphi\varpi \leq \frac{1}{5}\varphi\varpi^2 + C\varphi K^2\delta^{2\beta} \leq \frac{1}{5}\varphi\varpi^2 + CK^2\delta^{2\beta}, \qquad (2.15)$$

$$-\frac{2\alpha}{\gamma}\langle\nabla\varphi,\nabla g\rangle v^\beta\varpi \leq \frac{2\alpha}{\gamma}|\nabla\varphi|\cdot\varpi^{\frac{3}{2}}v^\beta \leq \frac{1}{5}\varphi\varpi^2 + C\frac{|\nabla\varphi|^4}{\varphi^3}\delta^{4\beta} \leq \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{4\beta}}{R^4}, \qquad (2.16)$$

$$\begin{aligned}
-\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi &= -\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big)\\
&\leq Cv^\beta\varpi\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big)\\
&\leq C\delta^\beta\varpi\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big|\\
&\leq \frac{1}{5}\varphi\varpi^2 + C\delta^{2\beta}\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big), \qquad (2.17)
\end{aligned}$$

$$\frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi \leq C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}v^\beta\varpi \leq \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta}}{R^4} \qquad (2.18)$$

and

$$\frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t \leq \frac{C}{\gamma}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}v^{\beta-1}\varpi \leq \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta-2}}{T^2}. \qquad (2.19)$$

We substitute (2.15)–(2.19) into (2.14), and have

$$\varphi\varpi^2 \leq C\delta^{2\beta}\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2T^2}\Big) \qquad (2.20)$$

at (x_1,t_1). Therefore, for all $(x,t)\in Q_{R,T}$, we obtain

$$\begin{aligned}
(\varphi\varpi)^2(x,t) &\leq (\varphi\varpi)^2(x_1,t_1) \leq \varphi\varpi^2(x_1,t_1)\\
&\leq C\delta^{2\beta}\Big(K^2 + \frac{1+\delta^{2\beta}}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2T^2}\Big). \qquad (2.21)
\end{aligned}$$

Notice that $\varphi(x,t)=1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $\varpi = v^\beta|\nabla g|^2 = v^\beta\frac{|\nabla v|^2}{v^2}$, we get that

$$\frac{|\nabla v|^2}{v^{2-\beta}} \leq C\delta^\beta\Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{1}{\delta T}\Big).$$

This proves part 1 of the theorem.

Part 2 Assume that the maximum of $\varphi\omega$ is arrived at a point (x_1, t_1). By [16], we can suppose, without loss of generality, that x_1 is not on the cut-locus of M^{n}. Therefore, at (x_1, t_1), it yields $\Delta(\varphi\omega) \le 0$, $(\varphi\omega)_t \ge 0$ and $\nabla(\varphi\omega) = 0$. By $0 = \nabla(\varphi\omega) = \omega\nabla\varphi + \varphi\nabla\omega$, then $\nabla\omega = -\frac{\nabla\varphi}{\varphi}\omega$. Hence, by (2.4) and a straightforward calculation, it yields that

$$\begin{aligned}
0 &\ge (1-\alpha)v\Delta(\varphi\omega) - (\varphi\omega)_t \\
&= \varphi[(1-\alpha)v\Delta\omega - \omega_t] + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\omega - \omega\varphi_t \\
&\ge bv\varphi\omega^2 - 2K(1-\alpha)v\varphi\omega - 2\alpha v\varphi\langle\nabla\omega, \nabla g\rangle \\
&\quad + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\omega - \omega\varphi_t \\
&= bv\varphi\omega^2 - 2K(1-\alpha)v\varphi\omega + 2\alpha v\omega\langle\nabla\varphi, \nabla g\rangle \\
&\quad + (1-\alpha)v\omega\Delta\varphi - 2(1-\alpha)v\omega\frac{|\nabla\varphi|^2}{\varphi} - \omega\varphi_t.
\end{aligned} \tag{2.22}$$

This implies

$$\begin{aligned}
2\varphi\omega^2 &\le \frac{4}{b}K(1-\alpha)\varphi\omega - \frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega \\
&\quad - \frac{2(1-\alpha)}{b}\omega\Delta\varphi + \frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega + \frac{2}{bv}\omega\varphi_t.
\end{aligned} \tag{2.23}$$

We next estimate upper bounds for each term of the right hand side of (2.23). Applying the Young inequality, we have

$$\frac{4}{b}K(1-\alpha)\varphi\omega \le \frac{1}{5}\varphi\omega^2 + C\varphi K^2 \le \frac{1}{5}\varphi\omega^2 + CK^2, \tag{2.24}$$

$$\begin{aligned}
-\frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega &\le \frac{4\alpha}{b}|\nabla\varphi|\cdot\omega^{\frac{3}{2}} \\
&\le \frac{1}{5}\varphi\omega^2 + C\frac{|\nabla\varphi|^4}{\varphi^3} \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4},
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
-\frac{2(1-\alpha)}{b}\omega\Delta\varphi &= -\frac{2(1-\alpha)}{b}\omega\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big) \\
&\le C\omega\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\
&\le C\omega\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\
&\le \frac{1}{5}\varphi\omega^2 + C\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big),
\end{aligned} \tag{2.26}$$

$$\frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega \le C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4} \tag{2.27}$$

and

$$\frac{2}{bv}\omega\varphi_t \le \frac{C}{\delta}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{\delta^2T^2} \tag{2.28}$$

We substitute (2.24)–(2.28) into (2.23), and have

$$\varphi\omega^2 \le C\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{\delta^2T^2} \tag{2.29}$$

at (x_1, t_1). Therefore, for all $(x,t) \in Q_{R,T}$, we obtain

$$\begin{aligned}(\varphi\omega)^2(x,t) &\le (\varphi\omega)^2(x_1,t_1) \le \varphi\omega^2(x_1,t_1)\\ &\le C\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{\delta^2 T^2}.\end{aligned} \tag{2.30}$$

Notice that $\varphi(x,t) = 1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $\omega = |\nabla g|^2 = \frac{|\nabla v|^2}{v^2}$, we get that

$$\frac{|\nabla v|^2}{v^2} \le C\Big(K + \frac{1}{R^2} + \frac{1}{\delta T}\Big).$$

The proof is completed.

Proof of Theorem 1.2 Part 1 From (1.4), we know that, when v is a positive ancient solution to (2.1) such that $v(x,t) = o([d(x,x_0) + |t|])$, then v is a constant. Notice that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$, so when u is a positive ancient solution to (1.1) such that $\frac{1}{u(x,t)} = o([d(x,x_0)+|t|]^{\frac{1}{1-\alpha}})$, then u is a constant. This ends the part 1.

Part 2 From (1.5), we know that, when v is a positive ancient solution to (2.1) such that $v(x,t) = o([d(x,x_0) + |t|])$, then v is a constant. Notice that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$, so when u is a positive ancient solution to (1.1) such that $\frac{1}{u(x,t)} = o([d(x,x_0) + |t|]^{\frac{1}{1-\alpha}})$, then u is a constant. This ends the proof of Theorem 1.2.

Proof of Theorem 1.3 Assume that the maximum of $\varphi\widetilde{\omega}$ is arrived at a point (x_1,t_1). By [16], we can suppose, without loss of generality, that x_1 is not on the cut-locus of M^{n}. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\widetilde{\omega}) \le 0$, $(\varphi\widetilde{\omega})_t \ge 0$ and $\nabla(\varphi\widetilde{\omega}) = 0$. By $0 = \nabla(\varphi\widetilde{\omega}) = \widetilde{\omega}\nabla\varphi + \varphi\nabla\widetilde{\omega}$, then $\nabla\widetilde{\omega} = -\frac{\nabla\varphi}{\varphi}\widetilde{\omega}$. Hence, by (2.12) and a straightforward calculation, it yields that

$$\begin{aligned}0 &\ge (1-\alpha)v\Delta(\varphi\widetilde{\omega}) - (\varphi\widetilde{\omega})_t\\ &= \varphi[(1-\alpha)v\Delta\widetilde{\omega} - \widetilde{\omega}_t] + (1-\alpha)v\widetilde{\omega}\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\widetilde{\omega} - \widetilde{\omega}\varphi_t\\ &\ge \epsilon\varphi\widetilde{\omega}^2 - 2K(1-\alpha)v\varphi\widetilde{\omega} - 2v\varphi\langle\nabla\widetilde{\omega}, \nabla g\rangle\\ &\quad + (1-\alpha)v\widetilde{\omega}\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\widetilde{\omega} - \widetilde{\omega}\varphi_t\\ &= \epsilon\varphi\widetilde{\omega}^2 - 2K(1-\alpha)v\varphi\widetilde{\omega} + 2v\widetilde{\omega}\langle\nabla\varphi, \nabla g\rangle\\ &\quad + (1-\alpha)v\widetilde{\omega}\Delta\varphi - 2(1-\alpha)v\widetilde{\omega}\frac{|\nabla\varphi|^2}{\varphi} - \widetilde{\omega}\varphi_t.\end{aligned} \tag{2.31}$$

This implies

$$\begin{aligned}2\varphi\widetilde{\omega}^2 \le{}& \frac{4}{\epsilon}K(1-\alpha)\varphi v\widetilde{\omega} - \frac{4}{\epsilon}\langle\nabla\varphi, \nabla g\rangle v\widetilde{\omega}\\ &- \frac{2(1-\alpha)}{\epsilon}v\widetilde{\omega}\Delta\varphi + \frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\widetilde{\omega} + \frac{2}{\epsilon}\widetilde{\omega}\varphi_t.\end{aligned} \tag{2.32}$$

We next estimate upper bounds for each term of the right hand side of (2.32). Applying the Young inequality, we have

$$\frac{4}{\epsilon}K(1-\alpha)\varphi v\widetilde{\omega} \le \frac{1}{5}\varphi\widetilde{\omega}^2 + C\varphi A^2K^2 \le \frac{1}{5}\varphi\widetilde{\omega}^2 + CA^2K^2, \tag{2.33}$$

$$-\frac{4}{\epsilon}\langle\nabla\varphi, \nabla g\rangle v\widetilde{\omega} \le \frac{4}{\epsilon}|\nabla\varphi| \cdot \widetilde{\omega}^{\frac{3}{2}}\sqrt{A}$$

$$\leq \frac{1}{5}\varphi\widetilde{\omega}^2 + C\frac{|\nabla\varphi|^4}{\varphi^3}A^2 \leq \frac{1}{5}\varphi\widetilde{\omega}^2 + \frac{CA^2}{R^4}, \tag{2.34}$$

$$\begin{aligned}
-\frac{2(1-\alpha)}{\epsilon}v\widetilde{\omega}\Delta\varphi &= -\frac{2(1-\alpha)}{\epsilon}v\widetilde{\omega}\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big)\\
&\leq CA\widetilde{\omega}\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big)\\
&\leq CA\widetilde{\omega}\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big|\\
&\leq \frac{1}{5}\varphi\widetilde{\omega}^2 + CA^2\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big),
\end{aligned} \tag{2.35}$$

$$\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\widetilde{\omega} \leq C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}A\widetilde{\omega} \leq \frac{1}{5}\varphi\widetilde{\omega}^2 + \frac{CA^2}{R^4} \tag{2.36}$$

and

$$\frac{2}{\epsilon}\widetilde{\omega}\varphi_t \leq \frac{2}{\epsilon}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}\widetilde{\omega} \leq \frac{1}{5}\varphi\widetilde{\omega}^2 + \frac{C}{T^2}. \tag{2.37}$$

We substitute (2.33)–(2.37) into (2.32), and have

$$\varphi\widetilde{\omega}^2 \leq CA^2\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{T^2} \tag{2.38}$$

at (x_1, t_1). Therefore, for all $(x,t) \in Q_{R,T}$, we obtain

$$\begin{aligned}
(\varphi\widetilde{\omega})^2(x,t) \leq (\varphi\widetilde{\omega})^2(x_1,t_1) &\leq \varphi\widetilde{\omega}^2(x_1,t_1)\\
&\leq CA^2\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{T^2}.
\end{aligned} \tag{2.39}$$

Notice that $\varphi(x,t) = 1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $\widetilde{\omega} = v|\nabla g|^2 = \frac{|\nabla v|^2}{v}$, we get that

$$\frac{|\nabla v|^2}{v} \leq CA\Big(K + \frac{1}{R^2} + \frac{\sqrt{K}}{R}\Big) + \frac{C}{T}.$$

The proof is completed.

Proof of Theorem 1.4 From Theorem 1.3, we know that, when v is a positive ancient solution to (2.1) such that $v(x,t) = o([d(x,x_0) + |t|])$, then v is a constant. Notice that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$, so when u is a positive ancient solution to (1.1) such that $\frac{1}{u(x,t)} = o([d(x,x_0)+|t|]^{\frac{1}{1-\alpha}})$, then u is a constant. This ends the proof of Theorem 1.4.

3 FDE along the Ricci Flow

The Ricci flow (1.11) was first introduced by Hamilton [9], and was an important tool of analyzing the structure of manifolds. In 2010, Bailesteanu, Cao and Pulemotov [1] generalized the Hamiltons gradient estimates for the heat equation on Riemannian manifolds with a fixed metric to the Ricci flow, and proved the theorem below.

Theorem B (see [1]) *Let $(\mathrm{M}^{\mathrm{n}}, g(x,t))_{t\in(0,T]}$ be a complete solution along the Ricci flow. Let $|\mathrm{Ric}(x,t)| \leq K$ for some $K > 0$ and all $(x,t) \in B_{R,T} :\equiv B(x_0,R)\times[0,T]$. Suppose that u is a smooth positive solution to the heat equation*

$$u_t = \Delta_{g_t}u.$$

If $u \le A$ for some $A > 0$ and all $(x,t) \in B_{R,T}$, then there exists a constant $C = C(n)$ such that

$$\frac{|\nabla^{g(t)}u|}{u} \le \Big(\frac{1}{R} + \frac{1}{\sqrt{t}} + \sqrt{K}\Big)\Big(1 + \log\frac{A}{u}\Big). \tag{3.1}$$

In this section, we will derive some Hamilton type gradient estimates for fast diffusion equations (1.1) on a Riemannian manifold evolved by the Ricci flow.

3.1 Basic lemmas

Before the proof of the main theorems, we need some lemmas. Consider the equation

$$v_t = (1-\alpha)v\Delta_{g(t)}v - |\nabla^{g(t)}v|^2 \tag{3.2}$$

on a complete Riemannian manifold $(\mathrm{M^n}, g)$ along the Ricci flow. Let $v(x,t)$ be a solution of (3.2) and $0 < v < A$ for some constant A in the cylinder

$$B_{R,T} :\equiv B(x_0, R) \times (0,T] \subset \mathrm{M^n} \times (-\infty, \infty),$$

here $T > 0$.

Now, in order to simplify writing, we all set $\Delta = \Delta^{g(t)}$ and $\nabla = \nabla^{g(t)}$.

We first introduce a new smooth function

$$g = \log\frac{A}{v}$$

in $B_{R,T}$. From (3.2), we have

$$g_t = (1-\alpha)v\Delta g + \alpha v|\nabla g|^2. \tag{3.3}$$

By utilizing the above equation (3.3), we can derive the following lemma.

Lemma 3.1 *Let $\omega = |\nabla g|^2$. Then for any $(x,t) \in B_{R,T}$,*

$$(1-\alpha)v\Delta\omega - \omega_t \ge bv\omega^2 - 2[(1-\alpha)A + 1]K\omega - 2\alpha v\langle\nabla\omega, \nabla g\rangle, \tag{3.4}$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$.

Proof The Ricci flow equation (3.1) implies

$$\partial_t|\nabla g|^2 = 2\langle\nabla g, \nabla g_t\rangle + 2\mathrm{Ric}(\nabla g, \nabla g). \tag{3.5}$$

By further using the Bochner-Weitzenböck formula (2.5), we have

$$\begin{aligned}(1-\alpha)v\Delta\omega - \omega_t &= 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g)\\ &\quad + 2(1-\alpha)v\langle\nabla\Delta g, \nabla g\rangle - 2\langle\nabla g, \nabla g_t\rangle.\end{aligned}$$

By (3.3), we obtain

$$\begin{aligned}(1-\alpha)v\Delta\omega - \omega_t &= 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g)\\ &\quad + 2v\Big\langle\nabla\Big(\frac{g_t}{v} - \alpha|\nabla g|^2\Big), \nabla g\Big\rangle - 2\langle\nabla g, \nabla g_t\rangle\end{aligned}$$

$$
\begin{aligned}
&= 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g) + 2\langle \nabla g_t, \nabla g\rangle \\
&\quad - \frac{2g_t}{v}\langle \nabla v, \nabla g\rangle - 2\alpha v\langle \nabla\omega, \nabla g\rangle - 2\langle \nabla g, \nabla g_t\rangle \\
&= 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g) \\
&\quad - \frac{2g_t}{v}\langle \nabla v, \nabla g\rangle - 2\alpha v\langle \nabla\omega, \nabla g\rangle. \qquad (3.6)
\end{aligned}
$$

Substituting (2.7) into (3.6) and noting that $|\mathrm{Ric}| \le K$, we have

$$
\begin{aligned}
&(1-\alpha)v\Delta\omega - \omega_t \\
&\ge +[2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g) - \frac{n(1-\alpha)}{2}v|\nabla g|^4 + 2\alpha v|\nabla g|^4 - 2\alpha v\langle \nabla\omega, \nabla g\rangle \\
&= \Big[2\alpha - \frac{n(1-\alpha)}{2}\Big]v\omega^2 - 2[(1-\alpha)A + 1]K\omega - 2\alpha v\langle \nabla\omega, \nabla g\rangle \\
&= bv\omega^2 - 2[(1-\alpha)A + 1]K\omega - 2\alpha v\langle \nabla\omega, \nabla g\rangle,
\end{aligned}
$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$. The proof is completed.

Lemma 3.2 *Let $\varpi = v^\beta|\nabla g|^2$ with $\beta = -\frac{\alpha}{2(1-\alpha)}$. Then*

$$
(1-\alpha)v\Delta\varpi - \varpi_t \ge -2[(1-\alpha)A + 1]K\varpi + \gamma v^{1-\beta}\varpi^2 - \alpha v\langle \nabla\varpi, \nabla g\rangle, \qquad (3.7)
$$

where $\gamma = \frac{\alpha^2}{4(1-\alpha)} + 2\alpha - \frac{n(1-\alpha)}{2} > 0$ and $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$.

Proof Applying (2.5), we have

$$
\begin{aligned}
(1-\alpha)v\Delta\varpi - \varpi_t &= (1-\alpha)v^\beta v\Delta|\nabla g|^2 + (1-\alpha)v|\nabla g|^2\Delta v^\beta \\
&\quad + 2(1-\alpha)v\nabla|\nabla g|^2\nabla v^\beta - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + 2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g, \nabla g) \\
&\quad + 2(1-\alpha)v^{\beta+1}\langle \nabla\Delta g, \nabla g\rangle + (1-\alpha)v|\nabla g|^2\Delta v^\beta \\
&\quad + 2(1-\alpha)v\langle \nabla|\nabla g|^2, \nabla v^\beta\rangle - \varpi_t.
\end{aligned}
$$

By utilizing (2.1), (2.3) and (3.5), we have

$$
\begin{aligned}
&(1-\alpha)v\Delta\varpi - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + 2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g, \nabla g) \\
&\quad + 2v^{\beta+1}\Big\langle \nabla\Big(\frac{g_t}{v} - \alpha|\nabla g|^2\Big), \nabla g\Big\rangle - 2(1-\alpha)\beta v^{\beta+1}\langle \nabla|\nabla g|^2, \nabla g\rangle \\
&\quad + (1-\alpha)\beta(\beta-1)v^{\beta-1}|\nabla v|^2|\nabla g|^2 + \beta(1-\alpha)v^\beta|\nabla g|^2\Delta v - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + 2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g, \nabla g) \\
&\quad + 2v^\beta\langle \nabla g_t, \nabla g\rangle - 2v^{\beta-1}g_t\langle \nabla v, \nabla g\rangle - 2\alpha v^{\beta+1}\langle \nabla|\nabla g|^2, \nabla g\rangle \\
&\quad - 2(1-\alpha)\beta v^{\beta+1}\langle \nabla|\nabla g|^2, \nabla g\rangle + (1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4 \\
&\quad + \beta v^{\beta-1}|\nabla g|^2(v_t + |\nabla v|^2) - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + \big[2(1-\alpha)v - 2\big]v^\beta\mathrm{Ric}(\nabla g, \nabla g) \\
&\quad - 2v^{\beta-1}g_t\langle \nabla v, \nabla g\rangle - 2\alpha v^{\beta+1}\langle \nabla|\nabla g|^2, \nabla g\rangle \\
&\quad - 2(1-\alpha)\beta v^{\beta+1}\langle \nabla|\nabla g|^2, \nabla g\rangle + (1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4
\end{aligned}
$$

$$+ \beta v^{\beta+1}|\nabla g|^4. \tag{3.8}$$

Therefore, by (2.10) we have

$$\begin{aligned}(1-\alpha)v\Delta\varpi - \varpi_t \geq & -2[(1-\alpha)A+1]v^\beta|\nabla g|^2K \\ & + \Big[(1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha)\Big]v^{\beta+1}|\nabla g|^4 \\ & - [2\alpha+2\beta(1-\alpha)]v\langle\nabla\varpi,\nabla g\rangle,\end{aligned} \tag{3.9}$$

where we use the fact that

$$\langle\nabla\varpi,\nabla g\rangle = v^\beta\langle\nabla|\nabla g|^2,\nabla g\rangle - \beta v^\beta|\nabla g|^4.$$

In order to obtain the gradient estimates, we need to require the coefficient $f(\beta)$ of $|\nabla g|^4$ to be positive. In fact,

$$\begin{aligned}f(\beta) &= (1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha) \\ &= -(1-\alpha)\Big[\beta+\frac{\alpha}{2(1-\alpha)}\Big]^2+\frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2}.\end{aligned}$$

We choose $\beta = -\frac{\alpha}{2(1-\alpha)}$, then $f(\beta) > 0$ when $1-\frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$. Therefore, (3.9) can be written as

$$(1-\alpha)v\Delta\varpi - \varpi_t \geq -2[(1-\alpha)A+1]K\varpi + \gamma v^{1-\beta}\varpi^2 - \alpha v\langle\nabla\varpi,\nabla g\rangle,$$

where $\gamma = \frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2} > 0$. The proof is completed.

Taking $\beta = 1$ in (3.9),the following lemma is derived.

Lemma 3.3 *Let $\omega_1 = v|\nabla g|^2$. Then*

$$(1-\alpha)v\Delta\omega_1 - \partial_t\omega_1 \geq \epsilon\omega_1^2 - 2[(1-\alpha)A+1]K\omega_1 - 2v\langle\nabla\omega_1,\nabla g\rangle, \tag{3.10}$$

where $\epsilon = 2\alpha-1-\frac{n(1-\alpha)}{2} > 0$ with $1-\frac{2}{n+4} < \alpha < 1$.

We next introduce a smooth cut-off function (see [1, 16]), which will be used in the proof of our main theorems.

Lemma 3.4 (see [1, 16]) *We use the geodesic polar coordinate here. Given $\tau \in (0,T]$, there exists a smooth function $\overline{\Psi} : [0,\infty)\times[0,T] \to \mathrm{R}$ satisfying the following requirements*:

1. *The support of $\overline{\Psi}(r,t)$ is a subset of $[0,R]\times[0,T]$, and $0 \leq \overline{\Psi}(r,t) \leq 1$ in $[0,R]\times[0,T]$.*
2. *The equalities $\overline{\Psi}(r,t) = 1$ and $\frac{\partial\overline{\Psi}}{\partial r}(r,t) = 0$ hold in $[0,\frac{R}{2}]\times[\tau,T]$ and $[0,\frac{R}{2}]\times[0,T]$, respectively.*
3. *The estimate $\big|\frac{\partial\overline{\Psi}}{\partial t}\big| \leq \frac{\overline{C}\,\overline{\Psi}^{\frac{1}{2}}}{\tau}$ is satisfied on $[0,\infty)\times[0,T]$ for some $\overline{C} > 0$, and $\overline{\Psi}(r,0) = 0$ for all $r \in [0,\infty)$.*
4. *The inequalities $\frac{-C_a\overline{\Psi}^a}{R} \leq \frac{\partial\overline{\Psi}}{\partial r} \leq 0$ and $\big|\frac{\partial^2\overline{\Psi}}{\partial r^2}\big| \leq \frac{C_a\overline{\Psi}^a}{R^2}$ hold on $[0,\infty)\times[0,T]$ for every $a \in (0,1)$ with some constant C_a dependent on a.*

3.2 The proof of theorems

In this section, we will prove our main theorems by Lemma 3.4. Let $\mathrm{dist}(x, x_0, t)$ be the distance between $x \in \mathrm{M^n}$ and x_0 with respect to the metric $g(x, t)$.

Proof of Theorem 1.8 Part 1: In order to derive the result, we also need a cut-off function φ by Li-Yau [16] on $B_{R,T}$. Define a smooth function $\varphi : \mathrm{M^n} \times [0, T] \to \mathrm{R}$ by $\varphi(x, t) = \overline{\Psi}(\mathrm{dist}(x, x_0, t), t)$ supported in $B_{R,T}$, where $\overline{\Psi}$ satisfies Lemma 3.4.

Let $\omega = |\nabla g|^2$. Assume that the function $\varphi\omega$ arrives its maximum at a point (x_1, t_1) and x_1 is not in the cut-locus of $\mathrm{M^n}$ by [15]. Therefore, at (x_1, t_1), it yields $\Delta(\varphi\omega) \le 0$, $(\varphi\omega)_t \ge 0$ and $\nabla(\varphi\omega) = 0$.

By $0 = \nabla(\varphi\omega) = \omega\nabla\varphi + \varphi\nabla\omega$, then we have $\nabla\omega = -\frac{\nabla\varphi}{\varphi}\omega$. Hence, by (3.4) and a straightforward calculation, it yields that

$$\begin{aligned} 0 &\ge (1-\alpha)v\Delta(\varphi\omega) - (\varphi\omega)_t \\ &= \varphi[(1-\alpha)v\Delta\omega - \omega_t] + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\langle\nabla\varphi, \nabla\omega\rangle - \omega\varphi_t \\ &\ge bv\varphi\omega^2 - 2[(1-\alpha)A + 1]K\varphi\omega - 2\alpha v\varphi\langle\nabla\omega, \nabla g\rangle \\ &\quad + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\langle\nabla\varphi, \nabla\omega\rangle - \omega\varphi_t \\ &= bv\varphi\omega^2 - 2[(1-\alpha)A + 1]K\varphi\omega + 2\alpha v\omega\langle\nabla\varphi, \nabla g\rangle \\ &\quad + (1-\alpha)v\omega\Delta\varphi - 2(1-\alpha)v\omega\frac{|\nabla\varphi|^2}{\varphi} - \omega\varphi_t. \end{aligned} \tag{3.11}$$

This implies

$$\begin{aligned} 2\varphi\omega^2 \le{}& \frac{4}{bv}[(1-\alpha)A + 1]K\varphi\omega - \frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega \\ &- \frac{2(1-\alpha)}{b}\omega\Delta\varphi + \frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega + \frac{2}{bv}\omega\varphi_t. \end{aligned} \tag{3.12}$$

We next estimate upper bounds for each term of the right hand side of (3.12). Applying the Young inequality, we have

$$\begin{aligned} \frac{4}{bv}[(1-\alpha)A + 1]K\varphi\omega &\le \frac{1}{5}\varphi\omega^2 + C[(1-\alpha)A + 1]^2\varphi\frac{K^2}{\delta^2} \\ &\le \frac{1}{5}\varphi\omega^2 + C[(1-\alpha)A + 1]^2\frac{K^2}{\delta^2}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} -\frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega &\le \frac{4\alpha}{b}|\nabla\varphi| \cdot \omega^{\frac{3}{2}} \\ &\le \frac{1}{5}\varphi\omega^2 + C\frac{|\nabla\varphi|^4}{\varphi^3} \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} -\frac{2(1-\alpha)}{b}\omega\Delta\varphi &= -\frac{2(1-\alpha)}{b}\omega\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi \cdot \partial_r(\log\sqrt{g})\Big) \\ &\le C\omega\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\ &\le C\omega\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\ &\le \frac{1}{5}\varphi\omega^2 + C\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big) \end{aligned} \tag{3.15}$$

and

$$\frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega \le C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4}. \tag{3.16}$$

For the last term, by [1], we have

$$\frac{2}{bv}\omega\varphi_t \le \frac{C}{\delta}\Big|\frac{\partial\varphi}{\partial t}\Big|\omega + \frac{C}{\delta}\Big|\frac{\partial\varphi}{\partial r}\Big|\Big|\frac{\partial}{\partial t}\mathrm{dist}\Big|\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{\delta^2 t^2}. \tag{3.17}$$

We substitute (3.13)–(3.17) into (3.12), and have

$$\varphi\omega^2 \le C\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2 t^2}\Big) \tag{3.18}$$

at (x_1,t_1). Therefore, for all $(x,t)\in B_{\frac{R}{2},T}$, we obtain

$$\begin{aligned}(\varphi\omega)^2(x,t) &\le (\varphi\omega)^2(x_1,t_1) \le \varphi\omega^2(x_1,t_1)\\ &\le C\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2 t^2}\Big).\end{aligned} \tag{3.19}$$

Notice that $\varphi(x,t)=1$ in $B_{\frac{R}{2},T}$ and $\omega = |\nabla g|^2 = \frac{|\nabla v|^2}{v^2}$, we get that

$$\frac{|\nabla v|^2}{v^2} \le C\Big([(1-\alpha)A+1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big).$$

Part 2: Define a smooth function

$$\varphi : \mathrm{M}^{\mathrm{n}}\times[0,T]\to\mathrm{R}$$

by $\varphi(x,t)=\overline{\Psi}(\mathrm{dist}(x,x_0,t),t)$ supported in $B_{R,T}$, where $\overline{\Psi}$ satisfies Lemma 3.4.

Let $\varpi = v^\beta|\nabla g|^2$. Assume that the function $\varphi\varpi$ arrives its maximum at a point (x_1,t_1) and x_1 is not in the cut-locus of M^{n} by [16]. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\varpi)\le 0$, $(\varphi\varpi)_t \ge 0$ and $\nabla(\varphi\varpi)=0$.

By $0=\nabla(\varphi\varpi)=\varpi\nabla\varphi+\varphi\nabla\varpi$, then we have $\nabla\varpi = -\frac{\nabla\varphi}{\varphi}\varpi$. Hence, by (3.7) and a straightforward calculation, it yields that

$$\begin{aligned}0 &\ge (1-\alpha)v\Delta(\varphi\varpi) - (\varphi\varpi)_t\\ &= \varphi[(1-\alpha)v\Delta\varpi - \varpi_t] + (1-\alpha)v\varpi\Delta\varphi + 2(1-\alpha)v\langle\nabla\varphi,\nabla\varpi\rangle - \varpi\varphi_t\\ &\ge \gamma v^{1-\beta}\varphi\varpi^2 - 2[(1-\alpha)A+1]K\varphi\varpi - \alpha v\varphi\langle\nabla\varpi,\nabla g\rangle\\ &\quad + (1-\alpha)v\varpi\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\varpi - \varpi\varphi_t\\ &= \gamma v^{1-\beta}\varphi\varpi^2 - 2[(1-\alpha)A+1]K\varphi\varpi + \alpha v\varpi\langle\nabla\varphi,\nabla g\rangle\\ &\quad + (1-\alpha)v\varpi\Delta\varphi - 2(1-\alpha)v\varpi\frac{|\nabla\varphi|^2}{\varphi} - \varpi\varphi_t.\end{aligned} \tag{3.20}$$

This implies

$$\begin{aligned}2\varphi\varpi^2 \le{}& \frac{4}{\gamma}[(1-\alpha)A+1]Kv^{\beta-1}\varphi\varpi - \frac{2\alpha}{\gamma}\langle\nabla\varphi,\nabla g\rangle v^\beta\varpi\\ &- \frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi + \frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi + \frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t.\end{aligned} \tag{3.21}$$

We next estimate upper bounds for each term of the right hand side of (3.21). Applying the Young inequality, we have

$$\begin{aligned}
\frac{4[(1-\alpha)A+1]}{\gamma}Kv^{\beta-1}\varphi\varpi &\le \frac{1}{5}\varphi\varpi^2 + C\varphi K^2\delta^{2\beta-2}[(1-\alpha)A+1]^2 \\
&\le \frac{1}{5}\varphi\varpi^2 + CK^2\delta^{2\beta-2}[(1-\alpha)A+1]^2, \qquad (3.22)\\
-\frac{2\alpha}{\gamma}\langle\nabla\varphi,\nabla g\rangle v^\beta\varpi &\le \frac{2\alpha}{\gamma}|\nabla\varphi|\cdot\varpi^{\frac{3}{2}}v^{\frac{\beta}{2}} \\
&\le \frac{1}{5}\varphi\varpi^2 + C\frac{|\nabla\varphi|^4}{\varphi^3}\delta^{2\beta} \le \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta}}{R^4}, \qquad (3.23)\\
-\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi &= -\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big) \\
&\le Cv^\beta\varpi\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\
&\le C\delta^\beta\varpi\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\
&\le \frac{1}{5}\varphi\varpi^2 + C\delta^{2\beta}\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big) \qquad (3.24)\\
\frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi &\le C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}v^\beta\varpi \le \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta}}{R^4}, \qquad (3.25)
\end{aligned}$$

and

$$\frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t \le \frac{C}{\gamma}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}v^{\beta-1}\varpi \le \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta-2}}{t^2}. \qquad (3.26)$$

We substitute (3.22)–(3.26) into (3.21), and have

$$\varphi\varpi^2 \le C\delta^{2\beta}\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2 t^2}\Big) \qquad (3.27)$$

at (x_1,t_1). Therefore, for all $(x,t)\in B_{\frac{R}{2},T}$, we obtain

$$\begin{aligned}
(\varphi\varpi)^2(x,t) &\le (\varphi\varpi)^2(x_1,t_1) \le \varphi\varpi^2(x_1,t_1) \\
&\le C\delta^{2\beta}\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2t^2}\Big). \qquad (3.28)
\end{aligned}$$

Notice that $\varphi(x,t)=1$ in $B_{\frac{R}{2},T}$ and $\varpi = v^\beta|\nabla g|^2 = \frac{|\nabla v|^2}{v^{2-\beta}}$, we get that

$$\frac{|\nabla v|^2}{v^{2-\beta}} \le C\delta^\beta\Big([(1-\alpha)A+1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big).$$

So, we prove Theorem 1.8.

Proof of Theorem 1.9 Define a smooth function $\varphi : \mathrm{M}^{\mathrm{n}}\times[0,T]\to\mathrm{R}$ by $\varphi(x,t) = \overline{\Psi}(\mathrm{dist}(x,x_0,t),t)$ supported in $B_{R,T}$, where $\overline{\Psi}$ satisfies Lemma 3.4.

Let $\omega_1 = v|\nabla g|^2$. Assume that the function $\varphi\omega_1$ arrives its maximum at a point (x_1,t_1) and x_1 is not in the cut-locus of M^{n} by [16]. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\omega_1)\le 0$, $(\varphi\omega_1)_t \ge 0$

and $\nabla(\varphi\omega_1)=0$. By $0=\nabla(\varphi\omega_1)=\omega_1\nabla\varphi+\varphi\nabla\omega_1$, then we have $\nabla\omega_1=-\frac{\nabla\varphi}{\varphi}\omega_1$. Hence, by (3.10) and a straightforward calculation, it yields that

$$\begin{aligned}
0 &\geq (1-\alpha)v\Delta(\varphi\omega_1)-(\varphi\omega_1)_t \\
&= \varphi[(1-\alpha)v\Delta\omega_1-(\omega_1)_t]+(1-\alpha)v\omega_1\Delta\varphi+2(1-\alpha)v\langle\nabla\varphi,\nabla\omega_1\rangle-\omega_1\varphi_t \\
&\geq \epsilon\varphi\omega_1^2-2[(1-\alpha)A+1]K\varphi\omega_1-2v\varphi\langle\nabla\omega_1,\nabla g\rangle \\
&\quad +(1-\alpha)v\omega_1\Delta\varphi+2(1-\alpha)v\nabla\varphi\nabla\omega_1-\omega_1\varphi_t \\
&= \epsilon\varphi\omega_1^2-2[(1-\alpha)A+1]K\varphi\omega_1+2v\omega_1\langle\nabla\varphi,\nabla g\rangle \\
&\quad +(1-\alpha)v\omega_1\Delta\varphi-2(1-\alpha)v\omega_1\frac{|\nabla\varphi|^2}{\varphi}-\omega_1\varphi_t
\end{aligned} \tag{3.29}$$

This implies

$$\begin{aligned}
2\varphi\omega_1^2 \leq{}& \frac{4}{\epsilon}[(1-\alpha)A+1]K\varphi\omega_1-\frac{4}{\epsilon}\langle\nabla\varphi,\nabla g\rangle v\omega_1 \\
&-\frac{2(1-\alpha)}{\epsilon}v\omega_1\Delta\varphi+\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\omega_1+\frac{2}{\epsilon}\omega_1\varphi_t.
\end{aligned} \tag{3.30}$$

We next estimate upper bounds for each term of the right hand side of (3.30). Applying the Young inequality, we have

$$\begin{aligned}
\frac{4}{\epsilon}[(1-\alpha)A+1]K\varphi\omega_1 &\leq \frac{1}{5}\varphi\omega_1^2+C\varphi[(1-\alpha)A+1]^2K^2 \\
&\leq \frac{1}{5}\varphi\omega_1^2+C[(1-\alpha)A+1]^2K^2, &(3.31)\\
-\frac{4}{\epsilon}\langle\nabla\varphi,\nabla g\rangle v\omega_1 &\leq \frac{4}{\epsilon}|\nabla\varphi|\cdot\omega_1^{\frac{3}{2}}\sqrt{A} \\
&\leq \frac{1}{5}\varphi\omega_1^2+C\frac{|\nabla\varphi|^4}{\varphi^3}A^2\leq\frac{1}{5}\varphi\omega_1^2+\frac{CA^2}{R^4}, &(3.32)\\
-\frac{2(1-\alpha)}{\epsilon}v\omega_1\Delta\varphi &= -\frac{2(1-\alpha)}{\epsilon}v\omega_1\Big(\partial_r^2\varphi+(n-1)\frac{\partial_r\varphi}{r}+\partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big) \\
&\leq CA\omega_1\Big(|\partial_r^2\varphi|+(n-1)\frac{|\partial_r\varphi|}{r}+\sqrt{K}|\partial_r\varphi|\Big) \\
&\leq CA\omega_1\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}}+(n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}}+\sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\
&\leq \frac{1}{5}\varphi\omega_1^2+CA^2\Big(\frac{1}{R^4}+\frac{K}{R^2}\Big), &(3.33)\\
\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\omega_1 &\leq C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}A\omega_1\leq\frac{1}{5}\varphi\omega_1^2+\frac{CA^2}{R^4} &(3.34)
\end{aligned}$$

and

$$\frac{2}{\epsilon}\omega_1\varphi_t\leq\frac{2}{\epsilon}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}\omega_1\leq\frac{1}{5}\varphi\omega_1^2+\frac{C}{t^2}. \tag{3.35}$$

We substitute (3.31)–(3.35) into (3.30), and have

$$\varphi\omega_1^2\leq CA^2\Big([(1-\alpha)A+1]^2K^2+\frac{1}{R^4}+\frac{K}{R^2}\Big)+\frac{C}{t^2} \tag{3.36}$$

at (x_1, t_1). Therefore, for all $(x,t) \in B_{\frac{R}{2},T}$, we obtain

$$\begin{aligned}(\varphi\omega_1)^2(x,t) &\le (\varphi\omega_1)^2(x_1,t_1) \le \varphi\omega_1^2(x_1,t_1)\\ &\le CA^2\Big(\frac{1}{R^4}+\frac{K}{R^2}\Big)+C[(1-\alpha)A+1]^2K^2+\frac{C}{t^2}.\end{aligned} \tag{3.37}$$

Notice that $\varphi(x,t)=1$ in $B_{\frac{R}{2},T}$ and $\omega_1 = v|\nabla g|^2 = \frac{|\nabla v|^2}{v}$, we get that

$$\frac{|\nabla v|^2}{v} \le CA\Big([(1-\alpha)A+1]K+\frac{1}{R^2}+\frac{\sqrt{K}}{R}\Big)+\frac{C}{t}.$$

Acknowledgements We are grateful to Professors Jiayu Li and Xiaobao Zhu for their support and encouragement. The first author would like to thank Professor Qi S Zhang for introducing this topic in the summer course at USTC. We appreciate the referees for the valuable suggestions and the very careful reading of the original manuscript.

References

[1] Bailesteanu, M., Cao, X. D. and Pulemotov A., Gradient estimates for the heat equation under the Ricci flow, *J. Funct. Anal.*, **258**, 2010, 3517–3542.

[2] Brendle, S., Convergence of the Yamabe flow in dimension 6 and higher, *Invent. Math.*, **170**, 2007, 541–576.

[3] Cao, H. and Zhu, M., Aronson-Bénilan estimates for the fast diffusion equation under the Ricci flow, *Nonlinear Analysis*, **170**, 2018, 258–281.

[4] Chen, D. G. and Xiong, C. W., Gradient estimates for doubly nonlinear diffusion equations, *Nonlinear Analysis*, **112**, 2015, 156–164.

[5] Chow, B., Lu, P. and Ni, L., Hamiltons Ricci Flow, American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[6] Daskalopoulos, P., del Pino, M., King, J. and Sesum, N. Type I ancient compact solutions of the Yamabe flow, *Nonlinear Anal. Theory Methods Appl.*, **137**, 2016, 338–356.

[7] del Pino, M. and Sáez, M., On the extinction profile for solutions of $u_t = \Delta u^{\frac{n-2}{N+2}}$, *Indiana Univ. Math. J.*, **50**(1), 2001, 611–628.

[8] Dung, H. T. and Dung, N. T., Sharp gradient estimates for a heat equation in Riemannian manifolds, *Proc. Amer. Math. Soc.*, **147**, 2019, 5329–5338.

[9] Hamilton, R. S., A matrix Harnack estimates for the heat equation, *Comm. Anal. Geom.*, **1**, 1993, 113–126.

[10] Hou, S. B. and Zou, L., Harnack estimate for a semilinear parabolic equation, *Sci. China Math.*, **60**, 2017, 833–840.

[11] Huang, G. Y. and Ma, B. Q., Hamilton's gradient estimates of porous medium and fast diffusion equations, *Geom. Dedicate*, **188**, 2017, 1–16.

[12] Jiang, X. R., Gradient estimates for a nonlinear heat equation on Riemannnian manifolds, *Proc. Amer. Math. Soc.*, **144**, 2016, 3635–3642.

[13] Li, H., Bai, H. and Zhang, G., Hamilton's gradient estimates for fast diffusion equations under the Ricci flow, *J. Math. Anal. Appl.*, **444**(2), 2016, 1372–1379.

[14] Li, J. F. and Xu, X., Defferential Harnack inequalities on Riemannian manifolds I: Linear heat equation, *Adv. Math.*, **226**, 2001, 4456–4491.

[15] Li, J. Y., Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equationson Riemannian manifolds, *J. Funct. Anal.*, **100**, 1991, 233–256.

[16] Li, P. and Yau, S. T., On the parabolic kernel of the Schrodinger operator, *Acta Math.*, **156**, 1986, 153–201.

[17] Li, S. Z. and Li, X. D., Hamilton differential Harnack inequality and W-entropy for Witten Laplacian on Riemannian manifolds, *J. Func. Anal.*, **274**(11), 2018, 3263–3290.

[18] Li, Y. and Zhu, X. R., Harnack estimates for a heat-type equation under the Ricci flow, *J. Differential Equations*, **260**, 2016, 3270–3301.

[19] Lu, P., Ni, L., Vázquez, J. L. and Villani, C., Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds, *J. Math. Pures Appl.*, **91**, 2009, 1–19.

[20] Ma, L., Zhao, L. and Song, X. F., Gradient estimate for the degenerate parabolic equation $u_t = \Delta F(u) + H(u)$ on manifolds, *J. Differential Equations*, **224**, 2008, 1157–1177.

[21] Souplet, P. and Zhang, Q. S., Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds, *Bull. London Math. Soc.,* **38**, 2006, 1045–1053.

[22] Wang, L. F., Liouville theorems and gradient estimates for a nonlinear elliptic equation, *J. Differential Equations*, 2015, Available from:

[23] Wang, W., Complement of gradient estimates and Liouville theorems for nonlinear Parabolic equations on noncompact Riemannian manifolds, *Math. Methods Appl. Sci.*, **40**(6), 2017, 2078–2083.

[24] Wang, Y. Z. and Chen, W. Y., Gradient estimates and entropy monotonicity formula for doubly nonlinear diffusion equations on Riemannian manifolds, *Math. Methods Appl. Sci.*, **37**, 2014, 2772–2781.

[25] Wu, J. Y., Elliptic gradient estimates for a nonlinear heat equation and applications, *Nonlinear Analysis*: *TMA*, **151**, 2017, 1–17.

[26] Xu, X., Gradient estimates for $u_t = \Delta F(u)$ on manifolds and some Liouville-type theorems, *J. Differential Equations*, **252**, 1403–1420.

[27] Yang, Y. Y., Gradient estimates for the equation $\Delta u + cu^{-\alpha} = 0$ on Riemannian manifolds, *Acta Mathematica Sinica, English Series*, **26**(6), 2010, 1177–1182.

[28] Zhu, X. B., Hamilton's gradient estimates and Liouville theorems for fast diffusion equations on noncompact Riemannian manifolds, *Proceedings of the American Mathematical Society*, **139**(5), 2011, 1637–1644.

[29] Zhu, X. B., Hamilton's gradient estimates and Liouville theorems for porous medium equations on noncompact Riemannnian manifolds, *J. Math. Anal. Appl.,* **402**, 2013, 201–206.