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A Rigidity Result of Spacelike Self-Shrinkers in Pseudo-Euclidean Spaces*

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Abstract In this paper, the author proves that the spacelike self-shrinker which is closed with respect to the Euclidean topology must be flat under a growth condition on the mean curvature by using the Omori-Yau maximum principle.

Keywords Self-Shrinker, Rigidity, Omori-Yau maximum principle, Pseudo-distance
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1 Introduction

The mean curvature flow (MCF for short) in Euclidean space is a one-parameter family of immersions $X_t = X(\cdot, t) : M^m \to \mathbb{R}^{m+n}$ with corresponding images $M_t = X_t(M)$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(x,t) = H(x,t), \quad x \in M, \\ X(x,0) = X(x) \end{cases} \tag{1.1}$$

is satisfied, where $H(x,t)$ is the mean curvature vector of M_t at $X(x,t)$ in $\mathbb{R}^{m+n}$.

Self-similar shrinkers to the above MCF play an important role in understanding the behavior of the flow since they often occur as singularities. M^m is said to be a self-shrinker if it satisfies a system of quasilinear elliptic PDE of the second order

$$H = -\frac{1}{2} X^N, \tag{1.2}$$

where X^N is the normal part of X.

The corresponding MCF could also be studied for the ambient pseudo-Euclidean space $\mathbb{R}^{m+n}_n$ (see e.g. [8–12, 16]). In this setting, M^m is also called as a self-shrinker if it satisfies (1.2). Ding-Wang [6] investigated self-shrinking graphs with high codimensions in pseudo-Euclidean space and obtained rigidity results under subexponential decay condition. Chau-Chen-Yuan [2] and Huang-Wang [13] showed that any spacelike entire graphic Lagrangian self-shrinkers must be flat under the decay condition on the Hessian of the potential function respectively. Ding-Xin [7] proved that such Lagrangian self-shrinkers must be affine plane which removed the

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additional condition in [2, 13]. Later, Liu-Xin [15] derived the rigidity of spacelike self-shrinkers under two different conditions, more specifically, if the spacelike self-shrinker is complete (or a closed subset with respect to the Euclidean topology of the pseudo-Euclidean space, see [5, 14]), then it is an affine plane under a growth condition on the w-function (or mean curvature). Some rigidity and classification results were also obtained in [1, 3] for spacelike self-shrinkers under various conditions. Recently, Chen-Qiu [4] proved that any complete m-dimensional spacelike self-shrinkers in $\mathbb{R}^{m+n}_n$ must be flat by using the Omori-Yau maximum principle, which implies that under the completeness condition, the growth conditions in the previous mentioned results on the spacelike self-shrinkers can be removed. It is natural to ask that how about the corresponding rigidity results when the spacelike self-shrinker is a closed subset (with respect to the Euclidean topology) of $\mathbb{R}^{m+n}_n$.

Along this direction, in the present paper, by establishing a new Omori-Yau maximum principle (see Theorem 2.1), we demonstrate that the spacelike self-shrinker which is closed with respect to the Euclidean topology must be flat under a growth condition on the mean curvature (see Theorem 3.1).

2 An Omori-Yau Maximum Principle for Spacelike Self-shrinkers

The pseudo-Euclidean space $\mathbb{R}^{m+n}_n$ is the linear space $\mathbb{R}^{m+n}$ endowed with the metric

$$ds^2 = \sum_{i=1}^{m}(dx^i)^2 - \sum_{\alpha=m+1}^{m+n}(dx^\alpha)^2.$$

Let $X : M \to \mathbb{R}^{m+n}_n$ be a spacelike m-submanifold in $\mathbb{R}^{m+n}_n$ with the second fundamental form B defined by

$$B_{UW} := (\overline{\nabla}_U W)^N$$

for $U, W \in \Gamma(TM)$. We use the notation $(\cdot)^T$ and $(\cdot)^N$ for the orthogonal projections into the tangent bundle TM and the normal bundle NM, respectively. For $\nu \in \Gamma(NM)$, we define the shape operator $A_\nu : TM \to TM$ by

$$A_\nu(U) := -(\overline{\nabla}_U \nu)^T.$$

Taking the trace of B gives the mean curvature vector H of M in $\mathbb{R}^{m+n}_n$, i.e.

$$H := \mathrm{trace}(B) = \sum_{i=1}^{m} B_{e_i e_i},$$

where $\{e_i\}$ is a local orthonormal frame field of M.

We denote the absolute value of $|H|^2$ by $\|H\|^2$, which is nonnegative. Let $V := -\frac{1}{2}X^T$ and $\Delta_V := \Delta + \langle V, \nabla\cdot\rangle$.

In the following, we show that the Omori-Yau maximum principle concerning the operator Δ_V is applicable in the situation of the spacelike self-shrinker which is closed with respect to the Euclidean topology under certain condition.

Theorem 2.1 *Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a spacelike self-shrinker, which is closed with respect to the Euclidean topology. Assume that the origin $o \in M$. If there exists a constant*

$C>0$, such that $\|H\|\le C(z+1)$, where $z=\langle X,X\rangle$ is the pseudo-distance function. Then for any $f\in C^2(M)$ with $\lim\limits_{x\to\infty}\frac{f(x)}{\log(z(x)+1)}=0$, there exists $\{x_j\}\subset M$, such that

$$\lim_{j\to\infty} f(x_j)=\sup f,\quad \lim_{j\to\infty}|\nabla f|(x_j)=0,\quad \lim_{j\to\infty}\Delta_V f(x_j)\le 0. \tag{2.1}$$

Proof Let $\{\epsilon_j\}$ be a sequence of positive real numbers such that $\epsilon_j\to 0$ as $j\to\infty$. Define

$$f_j(x)=f(x)-\epsilon_j\log(z(x)+1),\quad \forall\, j.$$

By [14, Proposition 3.1] (see also in [17]), the pseudo-distance function z is proper, together with the condition on f, we know $f_j\to-\infty$ as $x\to\infty$, and the set $\{x\in M\mid z(x)\le C_1\}$ is compact for any constant $C_1>0$, so f_j has a lower bound, say A, on it. Then there is a constant $C_2\ge C_1$ such that $f_j(x)<A$ for $x\in\{x\in M\mid z(x)\ge C_2\}$, thus f_j attains its maximum at some point $x_j\in\{x\in M\mid z(x)\le C_2\}$. If $\{z(x_j)\}$ is bounded, then there is a subsequence of $\{x_j\}$ converging to some point $x\in M$, at which f attains its maximum, in this case, the conclusions follow easily. Now we assume that $z(x_j)\to+\infty$ as $j\to+\infty$. Consequently, we have

$$\nabla f_j(x_j)=0,\quad \Delta_V f_j(x_j)\le 0. \tag{2.2}$$

Direct computation gives

$$\Delta_V z=2m-z,\quad |\nabla z|^2=4(z+4\|H\|^2). \tag{2.3}$$

By using (2.2)–(2.3) and $\|H\|\le C(z+1)$, we obtain

$$\lim_{j\to\infty}|\nabla f|(x_j)=\lim_{j\to\infty}\epsilon_j\frac{|\nabla z|(x_j)}{z(x_j)+1}=\lim_{j\to\infty}\epsilon_j\frac{2\sqrt{z(x_j)+4\|H\|^2(x_j)}}{z(x_j)+1}=0$$

and

$$\begin{aligned}\lim_{j\to\infty}\Delta_V f(x_j)&=\lim_{j\to\infty}\Big(\Delta_V f_j(x_j)+\epsilon_j\frac{\Delta_V z(x_j)}{z(x_j)+1}-\epsilon_j\frac{|\nabla z|^2(x_j)}{(z(x_j)+1)^2}\Big)\\&\le\lim_{j\to\infty}\Big(\epsilon_j\frac{\Delta_V z(x_j)}{z(x_j)+1}-\epsilon_j\frac{|\nabla z|^2(x_j)}{(z(x_j)+1)^2}\Big)\\&=\lim_{j\to\infty}\Big(\epsilon_j\frac{2m-z(x_j)}{z(x_j)+1}-4\epsilon_j\frac{z(x_j)+4\|H\|^2(x_j)}{(z(x_j)+1)^2}\Big)=0\end{aligned}$$

It remains to prove $\lim\limits_{j\to+\infty}f(x_j)=\sup f$. If there exists a subsequence $\{x_{j_k}\}\ne\{x_j\}$, such that $\lim\limits_{k\to+\infty}f(x_{j_k})=\sup f$, then by still denoting $\{x_{j_k}\}$ as x_j, our proof is completed. Otherwise, we claim that $\lim\limits_{j\to+\infty}f(x_j)=\sup f$ (If $\sup f=\infty$, then we claim that $\lim\limits_{j\to+\infty}\sup f(x_j)=\infty$). Indeed, if this was not true, there would exist $\widehat{x}\in M$ and $\delta>0$, such that

$$f(\widehat{x})>f(x_j)+\delta \tag{2.4}$$

for each $j\ge j_0$ sufficiently large.

Since

$$f(x_j)-\epsilon_j\log(z(x_j)+1)=f_j(x_j)\ge f_j(\widehat{x})=f(\widehat{x})-\epsilon_j\log(z(\widehat{x})+1), \tag{2.5}$$

we have

$$f(x_j) \geq f(\widehat{x}) + \epsilon_j(\log(z(x_j)+1) - \log(z(\widehat{x})+1)).$$

If $z(x_j) \to +\infty$ as $j \to +\infty$, then for j large enough, we have $\log(z(x_j)+1) > \log(z(\widehat{x})+1)$, that is $f(x_j) > f(\widehat{x})$, which contradicts with (2.4).

If $\{z(x_j)\}$ is bounded, then for some subsequence of j, x_j converges to a point $\overline{x}$, so that $f(\widehat{x}) \geq f(\overline{x}) + \delta$. On the other hand, we can deduce from (2.5) that

$$f(\overline{x}) \geq f(\widehat{x}).$$

This is also a contradiction. This proves (2.1).

3 Rigidity Results

We will consider the corresponding rigidity of the spacelike self-shrinker which is closed with respect to the Euclidean topology by using the Omori-Yau maximum principle as follows.

Theorem 3.1 *Let $X: M^m \to \mathbb{R}^{m+n}_n$ be a spacelike self-shrinker, which is closed with respect to the Euclidean topology. Assume that the origin $o \in M$. If there exists a constant $C > 0$, such that $\|H\| \leq C(z+1)$, then M^m is a linear subspace.*

Proof By [16, Proposition 2.1],

$$\Delta|B|^2 = 2|\nabla B|^2 + 2\langle\nabla_i\nabla_j H, B_{ij}\rangle + 2\langle B_{ij}, H\rangle\langle B_{ik}, B_{jk}\rangle + 2|R^\perp|^2 - 2\sum_{\alpha\beta} S^2_{\alpha\beta}, \tag{3.1}$$

where $R^\perp$ denotes the curvature of the normal bundle, $S_{\alpha\beta} = h_{\alpha ij}h_{\beta ij}$ and $B_{ij} = (\overline{\nabla}_{e_i}e_j)^N = -h_{\alpha ij}e_\alpha$, here $\{e_\alpha\}$ is a local orthonormal normal frame field near the considered point.

From the self-shrinker equation (1.2), we get

$$\nabla_{e_j}H = -\frac{1}{2}(\overline{\nabla}_{e_j}(X - \langle X, e_k\rangle e_k))^N = \frac{1}{2}\langle X, e_k\rangle B_{jk}$$

and

$$\nabla_{e_i}\nabla_{e_j}H = \frac{1}{2}B_{ij} - \langle H, B_{ik}\rangle B_{jk} + \frac{1}{2}\langle X, e_k\rangle\nabla_{e_i}B_{jk}. \tag{3.2}$$

Combining (3.1) and (3.2), by using the Codazzi equation, it follows

$$\Delta_V|B|^2 = \Delta|B|^2 + \langle V, \nabla|B|^2\rangle = 2|\nabla B|^2 + |B|^2 + 2|R^\perp|^2 - 2\sum_{\alpha,\beta} S^2_{\alpha\beta}.$$

Let $\|B\|^2$ be the square of the norm of the second fundamental form of M in $\mathbb{R}^{m+n}_n$, which is nonnegative. We use the same notation for other timelike quantities. Then the above equality implies that

$$\begin{aligned}\Delta_V\|B\|^2 = -\Delta_V|B|^2 &= -2|\nabla B|^2 - |B|^2 - 2|R^\perp|^2 + 2\sum_{\alpha,\beta} S^2_{\alpha\beta}\\ &= 2\|\nabla B\|^2 + \|B\|^2 + 2\|R^\perp\|^2 + 2\sum_{\alpha,\beta} S^2_{\alpha\beta}.\end{aligned} \tag{3.3}$$

The formula (3.3) and

$$\sum_{\alpha,\beta} S_{\alpha\beta}^2 \geq \frac{1}{n}\Big(\sum_{\alpha} S_{\alpha\alpha}\Big)^2 = \frac{1}{n}\|B\|^4$$

give us

$$\Delta_V \|B\|^2 \geq 2\sum_{\alpha,\beta} S_{\alpha\beta}^2 \geq \frac{2}{n}\|B\|^4. \tag{3.4}$$

It follows that

$$\begin{aligned}\Delta_V\Big(-\frac{1}{\sqrt{1+\|B\|^2}}\Big) &= \frac{\Delta_V\|B\|^2}{2(1+\|B\|^2)^{\frac{3}{2}}} - \frac{3|\nabla\|B\|^2|^2}{4(1+\|B\|^2)^{\frac{5}{2}}} \\ &\geq \frac{\|B\|^4}{n(1+\|B\|^2)^{\frac{3}{2}}} - \frac{3|\nabla\|B\|^2|^2}{4(1+\|B\|^2)^{\frac{5}{2}}}. \end{aligned} \tag{3.5}$$

Dividing both sides of (3.5) by $\sqrt{1+\|B\|^2}$, we have

$$\frac{\|B\|^4}{n(1+\|B\|^2)^2} \leq \frac{1}{\sqrt{1+\|B\|^2}}\Delta_V\Big(-\frac{1}{\sqrt{1+\|B\|^2}}\Big) + \frac{3|\nabla\|B\|^2|^2}{4(1+\|B\|^2)^3}. \tag{3.6}$$

Applying Theorem 2.1 to $-\frac{1}{\sqrt{1+\|B\|^2}}$, we can conclude that for j sufficiently large, there exist points $\{x_j\} \subset M$, such that

$$\begin{aligned}\frac{1}{\sqrt{1+\|B\|^2}}(x_j) &< \inf\Big(\frac{1}{\sqrt{1+\|B\|^2}}\Big) + \frac{1}{j}, \\ \frac{|\nabla\|B\|^2|^2}{4(1+\|B\|^2)^3}(x_j) &< \frac{1}{j}, \\ \Delta_V\Big(-\frac{1}{\sqrt{1+\|B\|^2}}\Big)(x_j) &< \frac{1}{j}.\end{aligned}$$

Combining with (3.6), it follows that

$$\frac{\|B\|^4}{n(1+\|B\|^2)^2}(x_j) < \frac{1}{j}\Big(\inf\Big(\frac{1}{\sqrt{1+\|B\|^2}}\Big) + \frac{1}{j}\Big) + \frac{3}{j}.$$

When $j \to \infty$, $\frac{1}{\sqrt{1+\|B\|^2}}(x_j)$ goes to its infimum and $\|B\|^2(x_j)$ goes to its supremum. Therefore,

$$\frac{\Big(\sup_M \|B\|^2\Big)^2}{\Big(1+\sup_M \|B\|^2\Big)^2} \leq 0.$$

If $\sup_M \|B\|^2 = \infty$, then we have

$$\frac{\Big(\sup_M \|B\|^2\Big)^2}{\Big(1+\sup_M \|B\|^2\Big)^2} = \frac{1}{\Big(1+\frac{1}{\sup_M \|B\|^2}\Big)^2} = 1.$$

This yields the contradiction. Thus $\sup_M \|B\|^2 < \infty$, it follows that $B \equiv 0$. Hence M^m is a linear subspace.

Remark 3.1 In [15], the authors show that if $\|H\|^2 \leq e^{\alpha z}$ $\left(\alpha < \frac{1}{8}\right)$, then M is a linear subspace. Note that the two curves $y = (x-1)^2$ and $y = e^{\alpha x}$ $\left(\alpha < \frac{1}{8}\right)$ shall meet at two distinct points, the one is $(0, 1)$ and the other one is far away from the origin in the first quadrant. Over the interval between these two points, the function graph of $y = (x-1)^2$ stays above that of $y = e^{\alpha x}$ $\left(\alpha < \frac{1}{8}\right)$. Hence in this interval, the above condition on the mean curvature is weaker than the one in [15].

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