

Periodic Solutions for N -Body-Type Problems*

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Abstract The authors consider non-autonomous N -body-type problems with strong force type potentials at the origin and sub-quadratic growth at infinity. Using Ljusternik-Schnirelmann theory, the authors prove the existence of unbounded sequences of critical values for the Lagrangian action corresponding to non-collision periodic solutions.

Keywords Periodic solutions, N -body type problems, Variational methods

2000 MR Subject Classification 34C15, 34C25, 58F, 70F07

1 Introduction and Main Results

The 1975 paper by Gordon [1] exhibits the first prominent use of variational methods in the study of periodic solutions of the following Newtonian equations with singular potential $V(t, x) \in C^1([0, T] \times (\mathbb{R}^n \setminus S), \mathbb{R})$,

$$\begin{cases} \ddot{x} + V'(t, x) = 0, & x \in \mathbb{R}^n \setminus S, \\ x(t + T) = x(t), \end{cases} \quad (1.1)$$

where the potential $V(t, x)$ satisfies $V(t + T, x) = V(t, x)$ and Gordon's strong-force (SF for short) condition which stipulates that there exists a neighbourhood N of the compact set S and a function $U \in C^2(N \setminus S, \mathbb{R})$ such that

- (i) $U(x) \rightarrow -\infty$ as $x \rightarrow S$;
- (ii) $-V(t, x) \geq |\nabla U(x)|^2, \forall x \in N \setminus S$.

Remark 1.1 For a simple example, let $V(t, x) = \frac{-a}{|x|^\alpha}$ ($a > 0, \alpha \geq 2$), and take $U(x) = \sqrt{a} \ln |x|$. Then $\nabla U(x) = \frac{\sqrt{a}x}{|x|^2}$ and $-V(t, x) = \frac{a}{|x|^\alpha} \geq \frac{a}{|x|^2} = |\nabla U(x)|^2$ when $|x| \leq 1$.

The function $U(x)$ is introduced to control the potential $V(t, x)$ and force the Lagrange functional of the system (1.1) to satisfy the Palais-Smale condition. This is a significant step in utilizing the calculus of variations to obtain the following result.

Theorem 1.1 (Gordon) *Under the above conditions and the following condition*

$$(G_1) : V(t, x) < 0, x \neq 0,$$

Manuscript received March 3, 2018.

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*This work was supported by the National Natural Science Foundation of China (Nos. 11701463, 11671278).

there exist periodic solutions of (1.1) which tie (wind around) S and have arbitrary given topological (homotopy) type and given period.

Ambrosetti-Coti Zelati [2–3] used Morse theory to generalize Gordon’s result and obtained the following theorem.

Theorem 1.2 *Assume $V \in C^2([0, T] \times \mathbb{R}^n, \mathbb{R})$ satisfies $V(t + T, x) = V(t, x)$, Gordon’s strong force condition and the following condition:*

$$(A) : |V(t, x)|, |V_x(t, x)| \rightarrow 0 \text{ uniformly for all } t \text{ as } \|x\| \rightarrow \infty, \\ \text{and } \exists R_1 > 0 \text{ such that } V(t, x) < 0, \forall \|x\| \geq R_1,$$

then (1.1) has infinitely many T -periodic solutions.

Motivated by Gordon [1] and Ambrosetti-Coti Zelati [2–3], Jiang [4] applied Morse theory and proved the existence of infinitely many periodic solutions using a weaker condition than above condition (A), proving the following result.

Theorem 1.3 *Let Ω be an open subset in \mathbb{R}^n with compact complement $C = \mathbb{R}^n \setminus \Omega$, $n \geq 2$. Assume $V \in C^2([0, 2\pi] \times \Omega, \mathbb{R})$, $V(t + 2\pi, x) = V(t, x)$, and*

- (A₁) *there exists R_0 such that $\sup\{|V(t, x)| + |V'_x(t, x)| \mid (t, x) \in [0, 2\pi] \times (\mathbb{R}^n \setminus B_{R_0})\} < +\infty$;*
- (A₂) *V satisfies Gordon’s strong force condition (i) and (ii).*

Then (1.1) has infinitely many 2π -periodic solutions.

As an application of Ljusternik-Schirelman theory, the following result of Majer [5] can be seen as an improvement of the above condition (A₁).

Theorem 1.4 *Assume $W \in C^1([0, T] \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R})$ satisfies*

- (i) $W(t + T, x) = W(t, x)$;
- (ii) $\exists c \in \mathbb{R}, \theta < 2, r > 0$, such that

$$W(t, x) \leq c|x|^\theta, \quad W'(t, x)x - 2W(t, x) \leq c|x|^\theta, \quad \forall |x| > r, \forall t > 0;$$

- (iii) $a < (\frac{\pi}{T})^2$.

Then the equation $\ddot{u} + au + W'(t, u) = 0$ has infinitely many T -periodic solutions.

For a 3-body type problem, Bahri-Rabinowitz [6] used Morse theory to prove the following result.

Theorem 1.5 (Bahri-Rabinowitz) *Let $V(q) = \frac{1}{2} \sum_{1 \leq i \neq j \leq 3} V_{ij}(q_i - q_j)$. Assume V_{ij} satisfies*

- (B₁) $V_{ij} \in C^2(\mathbb{R}^l \setminus \{0\}, \mathbb{R})$;
- (B₂) $V_{ij} < 0$;
- (B₃) $V_{ij}(q), V'_{ij}(q) \rightarrow 0$ as $|q| \rightarrow \infty$;
- (B₄) $V_{ij}(q) \rightarrow -\infty$ as $q \rightarrow 0$;
- (B₅) for $\forall M > 0, \exists R > 0$, s.t. $V'_{ij}(q) \cdot q > M|V_{ij}(q)|, |q| > R$;
- (B₆) $\exists U_{ij} \in C^1(\mathbb{R}^l \setminus \{0\}, \mathbb{R})$, s.t. $U_{ij}(q) \rightarrow \infty$ as $q \rightarrow 0$, and $-V_{ij} \geq |U'_{ij}|^2$.

Then for any given $T > 0$, the equation

$$\ddot{q}_i + \frac{\partial V(q)}{\partial q_i} = 0 \tag{1.2}$$

has infinitely many T -periodic noncollision solutions.

We say that a function $X(t) = (x_1(t), \dots, x_N(t)) \in C^2(\mathbb{R}, (\mathbb{R}^k)^N)$ is a non-collision T -periodic solution of (1.2) if $X(t)$ satisfies $x_i(t) \neq x_j(t)$ for all $i \neq j$ and $t \in \mathbb{R}$, satisfies equation (1.2) and is T -periodic.

Majer-Terracini [7] generalized the result of Bahri-Rabinowitz to the following n -body type problems:

$$\ddot{x}_i(t) + \nabla_{x_i} V(t, x_1(t), \dots, x_N(t)) = 0, \quad x_i(t) \in \mathbb{R}^k, \quad i = 1, \dots, N. \tag{1.3}$$

Their principal theorem is the following.

Theorem 1.6 *Assume $k \geq 3$, and $V_{ij} \in C^1((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}, \mathbb{R})$ are T -periodic in t , and V satisfies*

$$(M_1) \quad V_{ij}(t, x) = V_{ji}(t, -x), \quad \forall x \in \mathbb{R}^k \setminus \{0\};$$

$$(M_2) \quad V_{ij}(t, x) \leq 0, \quad \forall x \in \mathbb{R}^k \setminus \{0\};$$

(M_3) $V_{ij}(t, \xi) \rightarrow -\infty$ uniformly in t as $|\xi| \rightarrow 0$, for all $1 \leq i \neq j \leq N$, and V_{ij} satisfies Gordon's strong force condition with $S = \{0\}$;

$$(M_4) \quad \exists \rho > 0, \exists \theta \in [0, \frac{\pi}{2}), \text{ s.t. any } (\nabla V_{ij}(t, x), x) \leq \theta, \quad \forall x, |x| > \rho.$$

Then (1.3) has at least one T -periodic non-collision solution.

In the case of symmetric potentials, Fadell-Husseini [8] proved the following result.

Theorem 1.7 *Assume that V_{ij} satisfies the following conditions:*

$$(V_1) \quad V(t, x) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(t, x_i - x_j);$$

$$(V_2) \quad V_{ij} \in C^1(\mathbb{R} \times (\mathbb{R}^k - \{0\}); \mathbb{R}) \text{ for all } 1 \leq i \neq j \leq N;$$

$$(V_3) \quad V_{ij}(t, \xi) \rightarrow -\infty \text{ uniformly in } t \text{ as } |\xi| \rightarrow 0, \text{ for all } 1 \leq i \neq j \leq N;$$

$$(V_4) \quad V_{ij}(t, \xi) \leq 0, 1 \leq i \neq j \leq N, \xi \neq 0;$$

$$(V_5) \quad \text{the strong force condition (see [13]) holds for } V_{ij} \text{ with } S = \{0\};$$

$$(V_6) \quad V_{ij}(t + \frac{T}{2}, -\xi) = V_{ij}(t, \xi).$$

Then there exist unbounded sequences of critical values for the Lagrangian action corresponding to non-collision periodic solutions for (1.3).

In this paper, we consider a relaxation of condition (V_4) which requires the potentials to be non-positive, but still maintain that the potentials have some growth so that the result in Theorem 1.7 still holds. We will make use of Majer's abstract critical point theorem to study the N -body-type problem. The key difficulty is in proving the local Palais-Smale condition, but we are able to obtain the following result.

Theorem 1.8 *Assume V_{ij} satisfies (V_1) – (V_3) , (V_5) with $S = \{0\}$, (V_6) and*

$$(V'_4) : \exists g > 0, \theta < 2, r > 0 \text{ such that } V_{ij}(t, \xi) \leq gm_i m_j |\xi|^\theta, |\xi| > r.$$

Then there exist unbounded sequences of critical values for the Lagrangian action corresponding to non-collision periodic solutions for (1.3).

Notice that the condition (V'_4) in our Theorem 1.8 is a kind of growth condition which weakens the ordinary condition on potentials which requires them to be non-positive. We also obtain the following corollary.

Corollary 1.1 *Let $\alpha \geq 2$, $r_1 > 0$, $r_2 > r_1$, $a, g > 0$, $\theta < 2$ and $V(t, x) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(x_i - x_j)$. Assume further that $V_{ij}(\xi) \in C^1(\mathbb{R}^k - \{0\}, \mathbb{R})$ satisfies $V_{ij}(\xi) = -am_i m_j |\xi|^{-\alpha}$, $|\xi| < r_1$ and $V_{ij}(\xi) = gm_i m_j |\xi|^\theta$, $|\xi| \geq r_2 > r_1$. Then the assumptions and the result of Theorem 1.8 hold, but V_{ij} does not satisfy the assumption (V_4) in Theorem 1.7.*

We would like to remark that for Newtonian type potentials there is a rich literature which includes (see [9–23]).

2 Some Lemmas

We introduce the spaces

$$E = \left\{ (x_1, \dots, x_N) \mid x_i \in H^1(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^k), x_i\left(t + \frac{T}{2}\right) = -x_i(t) \right\},$$

$$\Delta = \{ (x_1, \dots, x_N) \mid x_i \in H^1(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^k), x_i(t) \neq x_j(t), \forall t, i \neq j \},$$

where $H^1(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^k)$ is the metric completion of smooth T -periodic functions for the norm $\|x\|_{H^1} = \left(\int_0^T (|x(t)|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}}$, and the functional $f : \Delta \rightarrow \mathbb{R}$ is defined by

$$f(x_1, \dots, x_N) = \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i(t)|^2 dt - \int_0^T V(t, x_1(t), \dots, x_N(t)) dt.$$

Clearly, E is a closed subspace of $H^1(\mathbb{R}/T\mathbb{Z}; (\mathbb{R}^k)^N)$, and so a Hilbert space, while Δ is an open subset of E .

Using a standard argument (for instance, see [3]), it is easy to prove the following lemma.

Lemma 2.1 *Suppose (V_1) – (V_2) and (V_6) hold, then a critical point of f in Δ is a non-collision solution of (1.3).*

The closed subset $\Gamma = E - \Delta$ of E will be called the collision set, and a standard argument can be applied to show that the strong force assumption (V_3) implies that $f(X) \rightarrow +\infty$ when X approaches the collision set Γ . More precisely, we have the following lemma.

Lemma 2.2 (see [1, 24]) *Assume V satisfies (V_1) – (V_3) and (V_5) . If $\{X^n\}$ is a sequence in Δ such that $X^n \rightarrow X \in \Gamma$ in both the C^0 topology and weak topology of E , then $f(X^n) \rightarrow +\infty$.*

Lemma 2.3 *Assume V satisfies (V_1) – (V_3) , (V'_4) and (V_5) – (V_6) , then there is a constant λ_0 depending on g, m_i, r, θ , such that f satisfies the $(PS)_c$ condition for $c \geq \lambda_0$; that is, any sequence $\{x^k\} \subset \Delta$ satisfying $f(x^k) \rightarrow c$ and $f'(x^k) \rightarrow 0$ is pre-compact in H^1 .*

Proof We notice that the arguments for one-body problem with center-forces in Jiang [4] and Majer [5] cannot be directly generalized to the N -body ($N \geq 3$) case because the kinetic energy and potential energy are translational invariance for positions in N -body problems, and the Lagrangian action for the N -body ($N \geq 3$) case is also translational invariance for positions, but for one-body problem with center-forces, the potential is not translational invariance for positions, the arguments Jiang [4] and Majer [5] used this.

Here we must consider the differences and use different arguments. By Holder’s inequality, we have

$$\sum_{i < j} m_i m_j |x_i - x_j|^\theta \leq \left(\sum_{i < j} m_i m_j \right)^{\frac{2-\theta}{2}} \left(\sum_{i < j} m_i m_j |x_i - x_j|^2 \right)^{\frac{\theta}{2}}.$$

We then obtain that

$$\begin{aligned} \sum_{i < j} m_i m_j |x_i - x_j|^2 &= \frac{1}{2} \sum_{1 \leq i, j \leq N} m_i m_j |x_i - x_j|^2 \\ &= \sum_{i=1}^N m_i \sum_{i=1}^N m_i |x_i|^2 - \left(\sum_{i=1}^N m_i x_i \right)^2 \\ &\leq \sum_{i=1}^N m_i \sum_{i=1}^N m_i |x_i|^2 \end{aligned}$$

and

$$f(x^k) = f(x_1^k, \dots, x_N^k) = \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - \int_0^T V(t, x_1^k(t), \dots, x_N^k(t)) dt.$$

Let us use the notation

$$\xi_{ij}^k(t) = x_i^k(t) - x_j^k(t).$$

We have three possibilities:

(i) For all $1 \leq i, j \leq N$ and for all $t \in [0, T]$, $|\xi_{ij}^k(t)| > r$ when k is large, then by (V'_4) and the above inequality we have

$$\begin{aligned} &\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - \int_0^T V(t, x_1^k(t), \dots, x_N^k(t)) dt \\ &\geq \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - g \left(\sum_{i < j} m_i m_j \right)^{\frac{2-\theta}{2}} \left[\sum_{i=1}^N m_i \right]^{\frac{\theta}{2}} \int_0^T \left[\sum_{i=1}^N m_i |x_i^k|^2 \right]^{\frac{\theta}{2}} dt. \end{aligned}$$

Since $x^k(t + \frac{T}{2}) = -x^k(t)$ implies $\int_0^T x^k(t) dt = 0$, by Wirtinger’s inequality and $f(x^k) \rightarrow c \leq d$ we get

$$d \geq \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - g \left[\sum_{i < j} m_i m_j \right]^{\frac{2-\theta}{2}} \left[\sum_{i=1}^N m_i \right]^{\frac{\theta}{2}} \left[\frac{T}{2\pi} \right]^\theta \int_0^T \left[\sum_{i=1}^N m_i |\dot{x}_i^k|^2 \right]^{\frac{\theta}{2}} dt.$$

By the assumption (V'_4) , we know that $\theta < 2$, hence we have $e > 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt \leq e.$$

(ii) There are $1 \leq i_0, j_0 \leq N$ such that for all $t \in [0, T]$ there holds $|\xi_{i_0 j_0}^k(t)| \leq r$ when k is large. Then by Lemma 2.2 and (V_2) , we have $a > -\infty$ and $0 < b < +\infty$ such that for all $t \in [0, T]$,

$$a \leq V_{i_0 j_0}(t, \xi_{i_0 j_0}^k(t)) \leq c m_{i_0} m_{j_0} |\xi_{i_0 j_0}^k(t)|^\theta \leq b.$$

For the remaining index pairs (i, j) and the corresponding potentials, we can use the above arguments of (i) and notice that we can add some negative terms to estimate the lower bound for the sum of all the potentials satisfying $|\xi_{ij}^k(t)| > r$:

$$-g\left(\sum_{i<j} m_i m_j\right)^{\frac{2-\theta}{2}} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k|^2\right]^{\frac{\theta}{2}}.$$

Now we can consider all cases for the index pairs. We have

$$\begin{aligned} & -V(t, x_1^k(t), \dots, x_N^k(t)) \\ & \geq \frac{N^2 - N}{2}(-b) - g\left(\sum_{i<j} m_i m_j\right)^{\frac{2-\theta}{2}} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k|^2\right]^{\frac{\theta}{2}}. \end{aligned}$$

Then taking the integral and using a similar argument as in (i), we can also find $e_1 > 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt \leq e_1.$$

(iii) There are $1 \leq i_0, j_0 \leq N$, $t_1 \in [0, T]$ and $t_2 \in [0, T]$ such that $|\xi_{i_0 j_0}^k(t_1)| > r$, $|\xi_{i_0 j_0}^k(t_2)| \leq r$ when k is large. Then

$$\begin{aligned} -V(t_1, x_1^k(t_1), \dots, x_N^k(t_1)) & \geq -g\left(\sum_{i<j} m_i m_j\right)^{\frac{2-\theta}{2}} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k(t_1)|^2\right]^{\frac{\theta}{2}}, \\ -V(t_2, x_1^k(t_2), \dots, x_N^k(t_2)) & \geq -b. \end{aligned}$$

Hence for all $t \in [0, T]$, we have

$$-V(t, x_1^k(t), \dots, x_N^k(t)) \geq -b - g\left(\sum_{i<j} m_i m_j\right)^{\frac{2-\theta}{2}} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k(t)|^2\right]^{\frac{\theta}{2}}.$$

Again, after taking the integral, we can find $e_2 > 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt \leq e_2.$$

In all cases we get the bounded property for

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt.$$

This implies $\{x^k\}$ has a weakly convergent subsequence. The proof of the strongly convergent property is more or less standard.

The following is an abstract critical point theorem which we will use in the proof of our main result. A proof of this theorem can be found in Majer [5].

Lemma 2.4 Let Δ be an open subset in a Banach space and let $Cat(\Delta)$ denote the category of Δ . Suppose f is a functional on Δ . Assume that

- (i) $Cat(\Delta) = +\infty$;
- (ii) for any sequence $\{q_n\} \subset \Delta$ and $\bar{q}_n \rightarrow q \in \partial\Delta$, we will have $f(q_n) \rightarrow +\infty$;
- (iii) for any $K \in \mathbb{R}$, $Cat_\Delta(\{q \in \Delta \mid f(q) \leq K\}) < +\infty$, and
- (iv) there exists a $\lambda_0 \in \mathbb{R}$ such that the Palais-Smale condition holds on the set $\{q \in \Delta \mid f(q) \geq \lambda_0\}$.

Then f possesses an unbounded sequence of critical values.

Fadell-Husseini [8] proved the following result.

Lemma 2.5 If Δ refers to the open subset defined in our proof of Theorem 1.8, then $Cat(\Delta) = +\infty$.

We notice that we can use similar methods as in Lemma 2.3 to prove the following lemma.

Lemma 2.6 For any $K \in \mathbb{R}$ such that $f(q) \leq K$, there is $A \geq 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i(t)|^2 dt \leq A.$$

Zhang-Zhou [24] gave the following lemma.

Lemma 2.7 For any constant $K \geq 0$, the set $D_K = \{X \in \Delta \mid \|\dot{X}\|_{L^2} \leq K\}$ is of finite category in Δ ; that is, $Cat_\Delta(D_K) < +\infty$.

By the monotone property of category and using Lemmas 2.6–2.7, we have the following lemma.

Lemma 2.8 For any $K \in \mathbb{R}$, $Cat_\Delta(\{q \in \Delta \mid f(q) \leq K\}) < +\infty$.

The proof of Theorem 1.8 now follows by Lemmas 2.1–2.5 and Lemma 2.8.

Acknowledgement We sincerely thank Professor Florin Diacu and Alexander Wires for their assistance for English expressions in the preparation of this paper.

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