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Reducibility for Schrödinger Operator with Finite Smooth and Time-Quasi-periodic Potential*

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Abstract In this paper, the author establishes a reduction theorem for linear Schrödinger equation with finite smooth and time-quasi-periodic potential subject to Dirichlet boundary condition by means of KAM (Kolmogorov-Arnold-Moser) technique. Moreover, it is proved that the corresponding Schrödinger operator possesses the property of pure point spectra and zero Lyapunov exponent.

Keywords Reducibility, Quasi-periodic Schrödinger operator, KAM theory, Finite smooth potential, Lyapunov exponent, Pure-Point spectrum
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1 Introduction

In recent years there have been many literatures to investigate the reducibility for the linear Schrödinger equation of quasi-periodic potential, of the form

$$\mathrm{i}\,\dot{u} = (H_0 + \varepsilon W(\omega t, x, -\mathrm{i}\nabla))u, \quad x \in \mathbb{R}^d \text{ or } x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d \tag{1.1}$$

or of the more general form, where $H_0 = -\triangle + V(x)$ or an abstract self-adjoint (unbounded) operator, and the perturbation W is quasi-periodic in time t and it may or may not depend on x or/and ∇. From the reducibility it is proved immediately that the corresponding Schrödinger operator is of the pure point spectrum property and zero Lyapunov exponent.

When $x \in \mathbb{R}^d$, there are many interesting and important results. See [1–2, 4, 8, 14–15, 19] and the references therein.

When $x \in \mathbb{T}^d$ with any integer $d \geq 1$, there are relatively less results. In [11], it is proved that

$$\dot{u} = \mathrm{i}((-\triangle + \varepsilon W(\phi_0 + \omega t, x; \omega))u), \quad x \in \mathbb{T}^d \tag{1.2}$$

is reduced to an autonomous equation for most values of the frequency vector ω, where W is analytic in (t, x) and quasi-periodic in time t with frequency vector ω. The reduction is made by means of Töplitz-Lipschitz property of operator and very hard KAM technique. As a special case of (1.2) with $d = 1$, the reduction can be automatically derived from the earlier KAM

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theorem for nonlinear partial differential equations, while assuming that W is analytic in (t, x). See [13] and [16], for example.

As we know, the spectrum property depends heavily on the smoothness of the perturbation for the discrete Schrödinger operator. For example, the Anderson localization and positivity of the Lyapunov exponent for one frequency discrete quasi-period Schrödinger operator with analytic potential occur in non-perturbative sense (The largeness of the potential does not depend on the Diophantine condition. See [6] for the detail.). However, one can only get perturbative results when the analytic property of the potential is weaken to Gevrey regularity (see [12]). Comparing with the discrete Schrödinger operator, a natural question is whether the spectral property of the continuous Schrödinger operator depends on the smoothness of the potential.

Actually in his pioneer work, by reducibility Combescure [8] studied the quantum stability problem for one-dimensional harmonic oscillator with a time-periodic perturbation. The techniques in [8] were extended in [9–10], in order to deal with an abstract Schrödinger operator $-\mathrm{i}\partial_t + H_0 + \beta W(\omega t)$, where H_0, a self-adjoint operator acting in some Hilbert space, has a simple discrete spectrum $\lambda_n < \lambda_{n+1}$ obeying a gap condition of the type $\inf\{n^{-\alpha}(\lambda_{n+1} - \lambda_n) : n = 1, 2, \cdots\} > 0$ for a given $\alpha > 0$, $\beta \in \mathbb{R}$, and $W = W(t)$ is periodic in t and r times strongly continuously differentiable as a bounded operator.

In this paper, we will extend the time-periodic W to time-quasi-periodic one. Let us consider a linear Schrödinger equation with quasi-periodic coefficient

$$\mathcal{L}u \triangleq \mathrm{i}u_t - u_{xx} + Mu + \varepsilon W(\omega t, x)u = 0 \tag{1.3}$$

subject to the Dirichlet boundary condition

$$u(t, 0) = u(t, \pi) = 0. \tag{1.4}$$

Given $p \geq 0$, let $\mathcal{H}_0^p[0, \pi]$ be the usual Sobolev space with the boundary condition (1.4), where the space is understood as $L_0^2[0, \pi]$ when $p = 0$.

Assumption A *Assume that there is a hull function*

$$\mathcal{W}(\theta, x) : \ \mathbb{T}^n \times [0, \pi] \to \mathbb{R}, \quad \mathbb{T}^n = \mathbb{R}^n/\pi\mathbb{Z}^n$$

with

$$\mathcal{W}(\theta, \cdot) \in C^N(\mathbb{T}^n, \mathcal{H}_0^p[0, \pi])$$

such that

$$W(\omega\, t, x) = \mathcal{W}(\theta, x)|_{\theta = \omega t}\,,$$

where $N > 180\, n$. This implies that W is quasi-periodic in time t with frequency $\omega \in \mathbb{R}^n$.

Assumption B *Assume that W is an even function of x.*

Assumption C *Assume $\omega = \tau\omega_0$, where ω_0 is Diophantine*:

$$|\langle k, \omega_0 \rangle| \geq \frac{\gamma}{|k|^{n+1}}, \quad k \in \mathbb{Z}^n \setminus \{0\} \tag{1.5}$$

with $0 < \gamma \ll 1$ a constant, and $\tau \in [1, 2]$ is a parameter.

Theorem 1.1 *Let* Meas$(\cdot)$ *denote the Lebesgue measure for sets. Under Assumptions* A, B, C, *for given* $1 \gg \gamma > 0$, *there exists* ε^* *with* $0 < \varepsilon^* = \varepsilon^*(n, \gamma) \ll \gamma$, *and exists a subset* $\Pi \subset [1, 2]$ *with*

$$\operatorname{Meas} \Pi \geq 1 - O(\gamma)$$

such that for any $0 < \varepsilon < \varepsilon^*$ *and for any* $\tau \in \Pi$, *there is a quasi-periodic coordinate transform* $u = \Phi(\theta, x)v|_{\theta = \omega t}$ *with the map* $\theta \mapsto \Phi(\theta, \cdot)$ *being of class* $C^{N-\mu}(\mathbb{T}^n, L(\mathcal{H}_0^p[0, \pi], \mathcal{H}_0^p[0, \pi]))$ *for any* $\mu \in (0, 1)$ *and satisfying*

$$\|\Phi(\theta, \cdot) - \mathrm{id}\|_{L(\mathcal{H}_0^p[0,\pi], \mathcal{H}_0^p[0,\pi])} \leq C_\mu \varepsilon$$

where id *is the identity of* $\mathcal{H}_0^p[0, \pi] \to \mathcal{H}_0^p[0, \pi]$, C_μ *is a constant depending on* μ, *and* $L(\mathcal{H}_0^p[0, \pi], \mathcal{H}_0^p[0, \pi])$ *is the class of all bounded linear operators from* $\mathcal{H}_0^p[0, \pi]$ *to itself, which changes* (1.3) *subject to* (1.4) *into*

$$\mathrm{i} v_t - v_{xx} + M_\xi v = 0, \quad v(t, 0) = v(t, \pi) = 0, \tag{1.6}$$

where M_ξ *is a real Fourier multiplier:*

$$M_\xi \sin(kx) = (M + \xi_k) \sin(kx), \quad k \in \mathbb{N}$$

with constants $\xi_k = \xi(\tau, \varepsilon) \in \mathbb{R}$ *and* $\xi_k = O(\varepsilon)$. *Moreover, the Schrödinger operator* $\mathcal{L}$ *is of pure point spectrum property and of zero Lyapunov exponent.*

Remark 1.1 Actually, (1.3) can be written as a Hamiltonian system. Thus, the coordinate transform $u = \Phi(\omega t, x)v$ can be chosen to be symplectic. Following [3], the coordinate transform $u = \Phi(\omega t, x)v$ can be chosen to be unitary.

Remark 1.2 We will combine the Jackson-Moser-Zehnder approximation technique(see [7] for example) and KAM technique (see [13, 16]), which is also applied to the case in [9–10]. Thus our result extends theirs. We also mention [20] where the reducibility is dealt with for a finite smooth and unbounded perturbation W.

2 Preliminaries

2.1 Analytical approximation lemma

In this subsection, we cite an approximation lemma which can be obtained from [17–18].

We start by recalling some definitions and setting some notations. Assume that X is a Banach space with the norm $\|\cdot\|_X$. First recall that $C^\mu(\mathbb{R}^n; X)$ for $0 < \mu < 1$ denotes the space of bounded Hölder continuous functions $f : \mathbb{R}^n \to X$ with the form

$$\|f\|_{C^\mu, X} = \sup_{0<|x-y|<1} \frac{\|f(x) - f(y)\|_X}{|x - y|^\mu} + \sup_{x \in \mathbb{R}^n} \|f(x)\|_X.$$

If $\mu = 0$, then $\|f\|_{C^\mu, X}$ denotes the sup-norm. For $\ell = k + \mu$ with $k \in \mathbb{N}$ and $0 \leq \mu < 1$, we denote by $C^\ell(\mathbb{R}^n; X)$ the space of functions $f : \mathbb{R}^n \to X$ with Hölder continuous partial derivatives, i.e., $\partial^\alpha f \in C^\mu(\mathbb{R}^n; X_\alpha)$ for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ with the assumption that $|\alpha| := |\alpha_1| + \cdots + |\alpha_n| \leq k$ and X_α is the Banach space of bounded operators $T : \prod^{|\alpha|}(\mathbb{R}^n) \to X$ with the norm

$$\|T\|_{X_\alpha} = \sup\{\|T(u_1, u_2, \cdots, u_{|\alpha|})\|_X : \|u_i\| = 1,\ 1 \leq i \leq |\alpha|\}.$$

We define the norm

$$\|f\|_{C^\ell} = \sup_{|\alpha| \leq \ell} \|\partial^\alpha f\|_{C^\mu, X_\alpha}.$$

Lemma 2.1 [Jackson-Moser-Zehnder] *Let $f \in C^\ell(\mathbb{R}^n; X)$ for some $\ell > 0$ with finite C^ℓ norm over $\mathbb{T}^n$. Let ϕ be a radical-symmetric, C^∞ function, having as support the closure of the unit ball centered at the origin, where ϕ is completely flat and takes value 1, and let $K = \widehat{\phi}$ be its Fourier transform. For all $\sigma > 0$, define*

$$f_\sigma(x) := K_\sigma * f = \frac{1}{\sigma^n} \int_{\mathbb{R}^n} K\Big(\frac{x-y}{\sigma}\Big) f(y) \mathrm{d}y.$$

Then there exists a constant $C \geq 1$ depending only on ℓ and n such that the following holds: For any $\sigma > 0$, the function $f_\sigma(x)$ is a real-analytic function from $\mathbb{C}^n$ to X such that if Δ_σ^n denotes the n-dimensional complex strip of width σ,

$$\Delta_\sigma^n := \{x \in \mathbb{C}^n \mid |\mathrm{Im}\, x_j| \leq \sigma,\ 1 \leq j \leq n\},$$

then $\forall \alpha \in \mathbb{N}^n$ such that $|\alpha| \leq \ell$ one has

$$\sup_{x \in \Delta_\sigma^n} \Big\| \partial^\alpha f_\sigma(x) - \sum_{|\beta| \leq \ell - |\alpha|} \frac{\partial^{\beta+\alpha} f(\mathrm{Re} x)}{\beta!} (\mathrm{i}\, \mathrm{Im}\, x)^\beta \Big\|_{X_\alpha} \leq C \|f\|_{C^\ell} \sigma^{\ell - |\alpha|}, \tag{2.1}$$

and for all $0 \leq s \leq \sigma$,

$$\sup_{x \in \Delta_s^n} \|\partial^\alpha f_\sigma(x) - \partial^\alpha f_s(x)\|_{X_\alpha} \leq C \|f\|_{C^\ell} \sigma^{\ell - |\alpha|}. \tag{2.2}$$

The function f_σ preserves periodicity (i.e., if f is T-periodic in any of its variable x_j, so is f_σ). Finally, if f depends on some parameter $\xi \in \Pi \subset \mathbb{R}^n$ and if the Lipschitz-norm of f and its x-derivatives with respect to $\xi \in \Pi$ are uniformly bounded by $\|f\|_{C^\ell}^{\mathcal{L}}$, then all the above estimates hold with $\|\cdot\|$ replaced by $\|\cdot\|^{\mathcal{L}}$.

For the following result, the reader can refer to [20] for detail. For brevity, we will replace $\|\cdot\|_X$ by $\|\cdot\|$. Fix a sequence of fast decreasing numbers $s_\nu \downarrow 0$, $\upsilon \geq 0$ and $s_0 \leq \frac{1}{2}$. For a X-valued function $P(\phi)$, construct a sequence of real analytic functions $P^{(\upsilon)}(\phi)$ such that the following conclusions holds:

(1) $P^{(\upsilon)}(\phi)$ is real analytic on the complex strip $\mathbb{T}_{s_\upsilon}^n = \{x \in \mathbb{C}^n/\pi\mathbb{Z}^n : |\mathrm{Im}\, x| \leq s_\upsilon\}$ of the width s_υ around $\mathbb{T}^n$.

(2) The sequence of functions $P^{(\upsilon)}(\phi)$ satisfies the bounds:

$$\sup_{\phi \in \mathbb{T}^n} \|P^{(\upsilon)}(\phi) - P(\phi)\| \leq C \|P\|_{C^\ell} s_\upsilon^\ell, \tag{2.3}$$

$$\sup_{\phi \in \mathbb{T}_{s_{\upsilon+1}}^n} \|P^{(\upsilon+1)}(\phi) - P^{(\upsilon)}(\phi)\| \leq C \|P\|_{C^\ell} s_\upsilon^\ell, \tag{2.4}$$

where C denotes the constant (varying in different places) depending only on n and ℓ.

(3) The first approximate $P^{(0)}$ is "small" with the perturbation P. Precisely speaking, for arbitrary $\phi \in \mathbb{T}^n_{s_0}$, we have

$$\begin{aligned}\|P^{(0)}(\phi)\| &\le \Big\|P^{(0)}(\phi) - \sum_{|\alpha|\le\ell}\frac{\partial^\alpha P(\mathrm{Re}\phi)}{\alpha!}(\mathrm{iIm}\,\phi)^\alpha\Big\| + \Big\|\sum_{|\alpha|\le\ell}\frac{\partial^\alpha P(\mathrm{Re}\phi)}{\alpha!}(\mathrm{i}\,\mathrm{Im}\,\phi)^\alpha\Big\| \\ &\le C\Big(\|P\|_{C^\ell}s_0^\ell + \sum_{0\le m\le\ell}\|P\|_{C^m}s_0^m\Big) \\ &\le C\|P\|_{C^\ell}\sum_{m=0}^{\ell}s_0^m \\ &\le C\|P\|_{C^\ell}\sum_{m=0}^{\infty}s_0^m \\ &\le C\|P\|_{C^\ell}, \end{aligned} \tag{2.5}$$

where constant C is independent of s_0, and the last inequality holds due to the hypothesis that $s_0 \le \frac{1}{2}$.

(4) From (2.3) we have

$$P(\phi) = P^{(0)}(\phi) + \sum_{\upsilon=0}^{+\infty}(P^{(\upsilon+1)}(\phi) - P^{(\upsilon)}(\phi)), \quad \phi \in \mathbb{T}^n. \tag{2.6}$$

2.2 Lemmas

In this subsection, we present some lemmas that will be needed to develop this paper.

Lemma 2.2 (see [5]) *For $0 < \delta < 1$, $\nu > 1$, one has*

$$\sum_{k\in\mathbb{Z}^n}\mathrm{e}^{-2|k|\delta}|k|^\nu < \Big(\frac{\nu}{\mathrm{e}}\Big)^\nu\frac{(1+\mathrm{e})^n}{\delta^{\nu+n}}.$$

Lemma 2.3 (see [16]) *If $A = (A_{ij})$ is a bounded linear operator on ℓ^2, then also $B = (B_{ij})$ with*

$$B_{ij} = \frac{|A_{ij}|}{|i-j|}, \quad i \ne j$$

and $B_{ii} = 0$ is a bounded linear operator on ℓ^2, and $\|B\| \le \big(\frac{\pi}{\sqrt{3}}\big)\|A\|$, where $\|\cdot\|$ is the $\ell^2 \to \ell^2$ operator norm.

3 Main Results

Consider the differential equation:

$$\mathcal{L}u = \mathrm{i}u_t - u_{xx} + Mu + \varepsilon W(\omega t, x)u = 0 \tag{3.1}$$

subject to the boundary condition

$$u(t,0) = u(t,\pi) = 0. \tag{3.2}$$

It is well-known that the Sturm-Liouville problem

$$-y'' + My = \lambda y \tag{3.3}$$

with the boundary condition

$$y(0) = y(\pi) = 0 \tag{3.4}$$

has the eigenvalues and eigenfunctions, respectively,

$$\lambda_k = k^2 + M, \qquad k = 1, 2, \cdots, \tag{3.5}$$

$$\phi_k(x) = \sin kx, \quad k = 1, 2, \cdots. \tag{3.6}$$

Write

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t)\phi_k(x). \tag{3.7}$$

Note that W is an even function of x. Write

$$W(\omega t, x) = \sum_{k=0}^{\infty} v_k(\omega t)\varphi_k(x), \tag{3.8}$$

where

$$\varphi_k(x) = \cos(2kx), \quad k = 1, 2, \cdots.$$

For any $u, v \in L^2[0, \pi]$, define $(u,\, v) = \int_0^\pi u(x)\overline{v}(x)\mathrm{d}x$. Consider

$$W(\omega t, x)u(x) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{jlk} v_j u_l \phi_k(x),$$

where

$$\begin{aligned} c_{jlk} &= \int_0^\pi \varphi_j\phi_l\phi_k \mathrm{d}x = \int_0^\pi \cos(2jx)\cdot\sin(lx)\cdot\sin(kx)\,\mathrm{d}x \\ &= \begin{cases} 0, & k \neq \pm l \pm 2j, \\ \dfrac{\pi}{4}, & k = l \pm 2j \geq 1, \\ -\dfrac{\pi}{4}, & k = -l \pm 2j \geq 1. \end{cases} \end{aligned} \tag{3.9}$$

Then (3.1) can be expressed as

$$\sum_{k=1}^{\infty}\Big(\mathrm{i}\,\dot{u}_k + \lambda_k u_k + \varepsilon\sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{jlk} v_j u_l\Big)\phi_k = 0,$$

which implies that

$$\mathrm{i}\,\dot{u}_k + \lambda_k u_k + \varepsilon\sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{jlk} v_j u_l = 0. \tag{3.10}$$

This is a Hamiltonian system

$$\begin{cases} \mathrm{i}\,\dot{u}_k = -\dfrac{\partial H}{\partial \overline{u}_k}, & k \geq 1, \\ \mathrm{i}\,\dot{\overline{u}}_k = \dfrac{\partial H}{\partial u_k}, & k \geq 1, \end{cases} \tag{3.11}$$

with Hamiltonian

$$H(u, \overline{u}) = \sum_{k=1}^{\infty} \lambda_k u_k \overline{u}_k + \varepsilon \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} c_{jlk} v_j(\theta) u_l \overline{u}_k. \tag{3.12}$$

For two sequences $x = (x_j \in \mathbb{C},\ j = 1, 2, \cdots)$, $y = (y_j \in \mathbb{C},\ j = 1, 2, \cdots)$, define

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j.$$

Then we can write

$$H = \langle \Lambda u, \overline{u} \rangle + \varepsilon \langle R(\theta) u, \overline{u} \rangle, \tag{3.13}$$

where

$$\begin{aligned} &\Lambda = \mathrm{diag}(\lambda_j : j = 1, 2, \cdots), \quad \theta = \omega t, \\ &R(\theta) = (R_{kl}(\theta) : k, l = 1, 2, \cdots), \quad R_{kl}(\theta) = \sum_{j=0}^{\infty} c_{jlk} v_j(\theta). \end{aligned} \tag{3.14}$$

For $p \in \mathbb{N} = \{0, 1, 2, \cdots\}$, h_p denotes the Hilbert space of all complex sequences $z = (z_1, z_2, \cdots)$ with

$$\|z\|_p^2 = \sum_{k=1}^{\infty} k^{2p} |z_k|^2 < \infty. \tag{3.15}$$

Let

$$\langle y, z \rangle_p := \sum_{k=1}^{\infty} k^{2p} y_k \overline{z}_k, \quad \forall y, z \in h_p.$$

For $p \geq 0$, let $\mathcal{H}^p[0, \pi]$ be a Sobolev space. Define $\mathcal{H}_0^p[0, \pi] = \{u \in \mathcal{H}^p[0, \pi] : u(0) = u(\pi) = 0\}$.

Recall that

$$\mathcal{W}(\theta, x) \in C^p([0, \pi], \mathbb{R}) \quad \text{for fixed } \theta \in \mathbb{T}^n, \quad \mathcal{W}(\theta, x) \in C^N(\mathbb{T}^n, \mathbb{R}) \quad \text{for fixed } x \in [0, \pi].$$

Note that the Fourier transformation (3.7) is isometric from $u \in \mathcal{H}_0^p[0, \pi]$ to $(u_k : k = 1, 2, \cdots) \in h_p$. By (3.14), we have

$$\sup_{\theta \in \mathbb{T}^n} \Big\| \sum_{|\alpha| \leq N} \partial_\theta^\alpha R(\theta) \Big\|_{h_p \to h_p} \leq C, \tag{3.16}$$

where $\| \cdot \|_{h_p \to h_p}$ is the operator norm from h_p to h_p, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\alpha_j\ (\geq 0)\ (j = 1, 2, \cdots, n)$ are integers.

Now we apply analytical approximation lemma to the perturbation $R(\phi)$. Take a sequence of real numbers $\{s_v \geq 0\}_{v=0}^{\infty}$ with $s_v > s_{v+1}$ going fast to zero. Take $P(\theta) = R(\theta)$ in (2.6). Then by (2.6) we can write

$$R(\theta) = R_0(\theta) + \sum_{l=1}^{\infty} R_l(\theta), \tag{3.17}$$

where $R_0(\theta)$ is analytic in $\mathbb{T}_{s_0}^n$ with

$$\sup_{\theta \in \mathbb{T}_{s_0}^n} \|R_0(\theta)\|_{h_p \to h_p} \leq C, \tag{3.18}$$

and $R_l(\theta)$ $(l \geq 1)$ is analytic in $\mathbb{T}_{s_l}^n$ with

$$\sup_{\theta \in \mathbb{T}_{s_l}^n} \|R_l(\theta)\|_{h_p \to h_p} \leq C s_{l-1}^N. \tag{3.19}$$

3.1 Iterative parameters of domains

Let

- $\varepsilon_0 = \varepsilon,\ \varepsilon_\nu = \varepsilon^{(\frac{4}{3})^\nu},\ \nu = 0, 1, 2, \cdots$, which measures the size of perturbation at ν-th step.
- $s_\nu = \varepsilon_{\nu+1}^{\frac{1}{N}},\ \nu = 0, 1, 2, \cdots$, which measures the strip-width of the analytic domain $\mathbb{T}_{s_\nu}^n$,

$$\mathbb{T}_{s_\nu}^n = \{\theta \in \mathbb{C}^n/(\pi\mathbb{Z})^n : |\operatorname{Im}\theta| \leq s_\nu\}.$$

- $C(\nu)$ is a constant which may be varying in different places, and it is of the form

$$C(\nu) = C_1 2^{C_2 \nu},$$

where C_1, C_2 are constants.

- $K_\nu = 100 s_\nu^{-1} \left(\frac{4}{3}\right)^\nu |\log \varepsilon|$.
- $\gamma_\nu = \frac{\gamma}{2^\nu}$, $0 < \gamma \ll 1$.
- A family of subsets $\Pi_\nu \subset [1, 2]$ with $[1, 2] \supset \Pi_0 \supset \cdots \supset \Pi_\nu \supset \cdots$, and

$$\operatorname{Meas} \Pi_\nu \geq \operatorname{Meas} \Pi_{\nu-1} - C\gamma_{\nu-1}.$$

- For an operator-value (or a vector value) function $B(\theta, \tau)$, whose domain is $(\theta, \tau) \in \mathbb{T}_{s_\nu}^n \times \Pi_\nu$, set

$$\|B\|_{\mathbb{T}_{s_\nu}^n \times \Pi_\nu} = \sup_{(\theta,\tau) \in \mathbb{T}_{s_\nu}^n \times \Pi_\nu} \|B(\theta, \tau)\|_{h_p \to h_p},$$

where $\|\cdot\|_{h_p \to h_p}$ is the operator norm, and set

$$\|B\|_{\mathbb{T}_{s_\nu}^n \times \Pi_\nu}^{\mathcal{L}} = \sup_{(\theta,\tau) \in \mathbb{T}_{s_\nu}^n \times \Pi_\nu} \|\partial_\tau B(\theta, \tau)\|_{h_p \to h_p}.$$

3.2 Iterative lemma

In the following, for a function $f(\omega)$, denote by ∂_ω the derivative of $f(\omega)$ with respect to ω in Whitney's sense.

Lemma 3.1 *Let $R_{0,0} = R_0$, $R_{l,0} = R_l$, where R_0, R_l are defined by (3.17). Assume that we have a family of Hamiltonian functions H_ν :*

$$H_\nu = \sum_{j=1}^{\infty} \lambda_j^{(\nu)} u_j \overline{u}_j + \sum_{l\geq\nu}^{\infty} \varepsilon_l \langle R_{l,\nu} u, \overline{u}\rangle, \quad \nu = 0, 1, \cdots, m, \tag{3.20}$$

where $R_{l,\nu} = R_{l,\nu}(\theta, \tau)$ is an operator-valued function defined on the domain $\mathbb{T}^n_{s_\nu} \times \Pi_\nu$, and

$$\theta = \omega t. \tag{3.21}$$

(A1)$_\nu$

$$\lambda_j^{(0)} = \lambda_j = j^2 + M, \quad \lambda_j^{(\nu)} = \lambda_j + \sum_{i=0}^{\nu-1} \varepsilon_i \mu_j^{(i)}, \quad \nu \geq 1 \tag{3.22}$$

and $\mu_j^{(i)} = \mu_j^{(i)}(\tau) : \Pi_i \to \mathbb{R}$ with

$$|\mu_j^{(i)}|_{\Pi_i} := \sup_{\tau\in\Pi_i} |\mu_j^{(i)}(\tau)| \leq C(i), \quad 0 \leq i \leq \nu - 1, \tag{3.23}$$

$$|\mu_j^{(i)}|^{\mathcal{L}}_{\Pi_i} := \sup_{\tau\in\Pi_i} |\partial_\tau \mu_j^{(i)}(\tau)| \leq C(i), \quad 0 \leq i \leq \nu - 1. \tag{3.24}$$

(A2)$_\nu$ *$R_{l,\nu} = R_{l,\nu}(\theta, \tau)$ is defined in $\mathbb{T}^n_{s_l} \times \Pi_\nu$ with $l \geq \nu$, and is analytic in θ for fixed $\tau \in \Pi_\nu$, and*

$$\|R_{l,\nu}\|_{\mathbb{T}^n_{s_l}\times\Pi_\nu} \leq C(\nu), \tag{3.25}$$

$$\|R_{l,\nu}\|^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_\nu} \leq C(\nu). \tag{3.26}$$

Then there exists a compact set $\Pi_{m+1} \subset \Pi_m$ with

$$\operatorname{Meas} \Pi_{m+1} \geq \operatorname{Meas} \Pi_m - C\gamma_m, \tag{3.27}$$

and exists a symplectic coordinate transfrom

$$\Psi_m : \mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1} \to \mathbb{T}^n_{s_m} \times \Pi_m, \tag{3.28}$$

$$\|\Psi_m - \mathrm{id}\|_{h_p\to h_p} \leq \varepsilon_m^{\frac{1}{2}}, \quad (\theta, \tau) \in \mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1} \tag{3.29}$$

such that the Hamiltonian function H_m is changed into

$$\begin{aligned} H_{m+1} &\triangleq H_m \circ \Psi_m \\ &= \sum_{j=1}^{\infty} \lambda_j^{(m+1)} u_j \overline{u}_j + \sum_{l\geq m+1}^{\infty} \varepsilon_l \langle R_{l,m+1} u, \overline{u}\rangle, \end{aligned} \tag{3.30}$$

which is defined on the domain $\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}$, and $\lambda_j^{(m+1)} (j = 1, 2, \cdots)$ satisfy the assumptions (A1)$_{m+1}$ *and $R_{l,m+1}$ satisfy the assumptions* (A2)$_{m+1}$.

3.3 Derivation of homological equations

Now we want to find a symplectic transformation Ψ_ν such that the terms $R_{l,v}$ (with $l=v$) disappear. Let F be a linear Hamiltonian of the form

$$F = \langle F(\theta,\tau)u, \overline{u}\rangle, \tag{3.31}$$

where $\theta=\omega t$, $(F(\theta,\tau))^{\mathrm{T}} = F(\theta,\tau)$. Moreover, let

$$\Psi = \Psi_m = X^t_{\varepsilon_m F}\big|_{t=1}, \tag{3.32}$$

where $X^t_{\varepsilon_m F}$ is the flow is the Hamiltonian. Vector field $X_{\varepsilon_m F}$ is the Hamiltonian $\varepsilon_m F$ with the symplectic structure $\mathrm{i}\,\mathrm{d}u\wedge\mathrm{d}\overline{u} = \mathrm{i}\sum\limits_{j=1}^{\infty}\mathrm{d}u_j\wedge\mathrm{d}\overline{u}_j$. Let

$$H_{m+1} = H_m\circ\Psi_m. \tag{3.33}$$

By (3.20), we write

$$H_m = N_m + R_m \tag{3.34}$$

with

$$N_m = \sum_{j=1}^{\infty}\lambda_j^{(m)}u_j\overline{u}_j, \tag{3.35}$$

$$R_m = \sum_{l=m}^{\infty}\varepsilon_l R_{lm} = \sum_{l=m}^{\infty}\varepsilon_l\langle R_{l,m}(\theta)u,\overline{u}\rangle, \tag{3.36}$$

where $(R_{l,m}(\theta))^{\mathrm{T}} = \overline{R_{l,m}(\theta)}$ when $\theta\in\mathbb{T}^n$. Since the Hamiltonian $H_m = H_m(\omega t,u,\overline{u})$ depends on time t, we introduce a fictitious action $I=$ constant, and let $\theta=\omega t$ be angle variable. Then the non-autonomous $H_m(\omega t,u,\overline{u})$ can be written as

$$\omega I + H_m(\theta,u,\overline{u})$$

with symplectic structure $\mathrm{d}I\wedge\mathrm{d}\theta+\mathrm{i}\,\mathrm{d}u\wedge\mathrm{d}\overline{u}$. By combining (3.31)–(3.36) and Taylor formula, we have

$$\begin{aligned} H_{m+1} &= H_m\circ X^1_{\varepsilon_m F}\\ &= N_m + \varepsilon_m\{N_m,F\} + \varepsilon_m^2\int_0^1(1-\tau)\{\{N_m,F\},F\}\circ X^\tau_{\varepsilon_m F}\mathrm{d}\tau + \varepsilon_m\omega\cdot\partial_\theta F\\ &\quad + \varepsilon_m R_{mm} + \Big(\sum_{l=m+1}^{\infty}\varepsilon_l R_{lm}\Big)\circ X^1_{\varepsilon_m F} + \varepsilon_m^2\int_0^1\{R_{mm},F\}\circ X^\tau_{\varepsilon_m F}\mathrm{d}\tau, \end{aligned} \tag{3.37}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket with respect to $\mathrm{i}\,\mathrm{d}u\wedge d\overline{u}$, that is

$$\{H(u,\overline{u}),F(u,\overline{u})\} = -\mathrm{i}\Big(\frac{\partial H}{\partial u}\cdot\frac{\partial F}{\partial\overline{u}} - \frac{\partial H}{\partial\overline{u}}\cdot\frac{\partial F}{\partial u}\Big).$$

For any

$$f(\theta) = \sum_{k\in\mathbb{Z}^n}\widehat{f}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle},\quad \theta\in\mathbb{T}^n,$$

define a truncation operator Γ_{K_m} as follows:

$$\Gamma_{K_m} f(\theta) = (\Gamma_{K_m} f)(\theta) \triangleq \sum_{|k| \le K_m} \widehat{f}(k) \mathrm{e}^{\mathrm{i}\,\langle k,\theta\rangle},$$
$$(1-\Gamma_{K_m}) f(\theta) = ((1-\Gamma_{K_m}) f)(\theta) \triangleq \sum_{|k| > K_m} \widehat{f}(k) \mathrm{e}^{\mathrm{i}\,\langle k,\theta\rangle},$$

where K_m is defined in Subsection 3.1. Then

$$f(\theta) = \Gamma_{K_m} f(\theta) + (1-\Gamma_{K_m}) f(\theta).$$

Let

$$\omega \cdot \partial_\theta F + \{N_m, F\} + \Gamma_{K_m} R_{mm} = \langle [R_{mm}]u, \overline{u}\rangle, \tag{3.38}$$

where

$$[R_{mm}] := \mathrm{diag}(\widehat{R}_{mmjj}(0) : j = 1, 2, \cdots), \tag{3.39}$$

and $R_{mmij}(\theta)$ is the matrix element of $R_{m,m}(\theta)$ and $\widehat{R}_{mmij}(k)$ is the k-Fourier coefficient of $R_{mmij}(\theta)$. Then

$$H_{m+1} = N_{m+1} + C_{m+1} R_{m+1}, \tag{3.40}$$

where

$$N_{m+1} = N_m + \varepsilon_m \langle [R_{mm}]u, \overline{u}\rangle = \sum_{j=1}^{\infty} \lambda_j^{(m+1)} u_j \overline{u}_j, \tag{3.41}$$

$$\lambda_j^{(m+1)} = \lambda_j^{(m)} + \varepsilon_m \widehat{R}_{mmjj}(0) = \lambda_j + \sum_{l=1}^{m} \varepsilon_l \mu_j^{(l)}, \quad \mu_j^{(m)} := \widehat{R}_{mmjj}(0). \tag{3.42}$$

$$C_{m+1} R_{m+1} = \varepsilon_m (1-\Gamma_{K_m}) R_{mm} \tag{3.43}$$
$$+ \varepsilon_m^2 \int_0^1 (1-\tau)\{\{N_m, F\}, F\} \circ X_{\varepsilon_m F}^{\tau} \mathrm{d}\tau \tag{3.44}$$
$$+ \varepsilon_m^2 \int_0^1 \{R_{mm}, F\} \circ X_{\varepsilon_m F}^{\tau} \mathrm{d}\tau \tag{3.45}$$
$$+ \Big(\sum_{l=m+1}^{\infty} \varepsilon_l R_{lm}\Big) \circ X_{\varepsilon_m F}^{1}. \tag{3.46}$$

The equation (3.38) is called homological equation. Developing the Poisson bracket $\{N_m, F\}$ and comparing the coefficients of $u_i \overline{u}_j$ $(i, j = 1, 2, \cdots)$, we get

$$\omega \cdot \partial_\theta F(\theta,\tau) + \mathrm{i}\,(F(\theta,\tau)\Lambda^{(m)} - \Lambda^{(m)} F(\theta,\tau)) = \Gamma_{K_m} R_{m,m}(\theta) - [R_{mm}], \tag{3.47}$$

where

$$\Lambda^{(m)} = \mathrm{diag}(\lambda_j^{(m)} : j = 1, 2, \cdots), \tag{3.48}$$

and we assume $\Gamma_{K_m} F(\theta,\tau) = F(\theta,\tau)$. Let $F_{ij}(\theta)$ be the matrix elements of $F(\theta,\tau)$. Then (3.47) can be rewritten as

$$\omega \cdot \partial_\theta F_{ij}(\theta) - \mathrm{i}\,(\lambda_i^{(m)} - \lambda_j^{(m)}) F_{ij}(\theta) = \Gamma_{K_m} R_{mmij}(\theta), \quad i \ne j, \tag{3.49}$$

and

$$\omega \cdot \partial_\theta F_{ii}(\theta) = \Gamma_{K_m} R_{mmii}(\theta) - \widehat{R}_{mmii}(0), \tag{3.50}$$

where $i, j = 1, 2, \cdots$.

3.4 Solutions of the homological equations

Lemma 3.2 *There exists a compact subset $\Pi_{m+1} \subset \Pi_m$ with*

$$\mathrm{Meas}(\Pi_{m+1}) \geq \mathrm{Meas}\,\Pi_m - C\gamma_m \tag{3.51}$$

such that for any $\tau \in \Pi_{m+1}$ (Recall $\omega = \tau\omega_0$), the equation (3.47) has a unique solution $F(\theta, \tau)$, which is defined on the domain $\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}$, with

$$\|F(\theta, \tau)\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}, \tag{3.52}$$

$$\|F(\theta, \tau)\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{6(2n+3)}{N}}. \tag{3.53}$$

Proof By passing to Fourier coefficients, (3.49) can be rewritten as

$$(-\langle k, \omega\rangle + \lambda_i^{(m)} - \lambda_j^{(m)})\widehat{F}_{ij}(k) = \mathrm{i}\,\widehat{R}_{mmij}(k), \tag{3.54}$$

where $i, j = 1, 2, \cdots, k \in \mathbb{Z}^n$ with $|k| \leq K_m$. Recall $\omega = \tau\omega_0$. Let

$$\begin{cases} A_k = |k|^{n+3}, & k \in \mathbb{Z}^n \setminus \{0\}, \\ A_k = 1, & k = 0 \in \mathbb{Z}^n. \end{cases}$$

And let

$$Q_{kij}^{(m)} \triangleq \Big\{\tau \in \Pi_m \mid |-\langle k, \omega_0\rangle\tau + \lambda_i^{(m)} - \lambda_j^{(m)}| < \frac{(|i-j|+1)\gamma_m}{A_k}\Big\}, \tag{3.55}$$

where $i, j = 1, 2, \cdots$, $k \in \mathbb{Z}^n$ with $|k| \leq K_m$, and $k \neq 0$ when $i = j$. Let

$$\Pi_{m+1} = \Pi_m \setminus \bigcup_{|k| \leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{kij}^{(m)}.$$

Then for any $\tau \in \Pi_{m+1}$, we have

$$\big| -\langle k, \omega\rangle + \lambda_i^{(m)} - \lambda_j^{(m)}\big| \geq \frac{(|i-j|+1)\gamma_m}{A_k}. \tag{3.56}$$

Recall that $R_{m,m}(\theta)$ is analytic in the domain $\mathbb{T}^n_{s_m}$ for any $\tau \in \Pi_m$,

$$|\widehat{R}_{mmij}(k)| \leq C(m)\mathrm{e}^{-s_m|k|}. \tag{3.57}$$

It follows

$$\begin{aligned} |\widehat{F}_{ij}(k)| &= \Big|\frac{\widehat{R}_{mmij}(k)}{-\langle k, \omega\rangle + \lambda_i^{(m)} - \lambda_j^{(m)}}\Big| \leq \frac{A_k|\widehat{R}_{mmij}(k)|}{\gamma_m(|i-j|+1)} \\ &\leq \frac{C(m)|k|^{n+3}\mathrm{e}^{-s_m|k|}}{\gamma_m(|i-j|+1)}. \end{aligned} \tag{3.58}$$

Therefore, by (3.58), we have

$$\begin{aligned}
&\sup_{\theta\in\mathbb{T}^n_{s'_m}\times\Pi_{m+1}}|F_{ij}(\theta,\tau)|\\
&\le \frac{C(m)}{\gamma_m(|i-j|+1)}\sum_{|k|\le K_m}|k|^{n+3}\mathrm{e}^{-(s_m-s'_m)|k|}\\
&\le \Big(\frac{n+3}{\mathrm{e}}\Big)^{n+3}(1+\mathrm{e})^n\Big(\frac{2}{s_m-s'_m}\Big)^{2n+3}\cdot\frac{C(m)}{\gamma_m(|i-j|+1)}\quad\text{(by Lemma 2.2)}\\
&\le \frac{1}{(s_m-s'_m)^{2n+3}}\cdot\frac{C\cdot C(m)}{\gamma_m(|i-j|+1)}\\
&\le \varepsilon_m^{-\frac{2(2n+3)}{N}}\cdot\frac{C\cdot C(m)}{\gamma_m(|i-j|+1)},
\end{aligned}$$

where $s'_m=s_m-\frac{s_m-s_{m+1}}{4}$. It is easy to verify that Lemma 2.3 holds for the weight norm $\|\cdot\|_p$. Then by Lemma 2.3, we have

$$\|F(\theta,\tau)\|_{\mathbb{T}^n_{s'_m}\times\Pi_{m+1}}\le C\cdot C(m)\gamma_m^{-1}\varepsilon_m^{-\frac{2(2n+3)}{N}}\le C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}. \tag{3.59}$$

It follows from $s'_m>s_{m+1}$ that

$$\|F(\theta,\tau)\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}}\le\|F(\theta,\tau)\|_{\mathbb{T}^n_{s'_m}\times\Pi_{m+1}}\le C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}.$$

Applying ∂_τ to both sides of (3.54), we have

$$(-\langle k,\omega\rangle+\lambda_i^{(m)}-\lambda_j^{(m)})\partial_\tau\widehat{F}_{ij}(k)=\mathrm{i}\,\partial_\tau\widehat{R}_{mmij}(k)+(*), \tag{3.60}$$

where

$$(*)=-(-\langle k,\omega_0\rangle+\partial_\tau(\lambda_i^{(m)}-\lambda_j^{(m)}))\widehat{F}_{ij}(k). \tag{3.61}$$

Recalling $|k|\le K_m=100s_m^{-1}\big(\frac{4}{3}\big)^m|\log\varepsilon|$, using (3.23)–(3.24) with $\nu=m$, and using (3.59), we have, for any $\tau\in\Pi_{m+1}$,

$$|(*)|\le C\cdot C(m+1)K_m\gamma_m^{-1}\varepsilon_m^{-\frac{2(2n+3)}{N}}\mathrm{e}^{-s'_m|k|}. \tag{3.62}$$

According to (3.26),

$$|\,\partial_\tau\widehat{R}_{mmij}(k)\,|\le C(m+1)\mathrm{e}^{-s_m|k|}. \tag{3.63}$$

By (3.56), (3.60) and (3.62)–(3.63), we have

$$|\,\partial_\tau\widehat{F}_{ij}(k)\,|\le\frac{A_k}{\gamma_m(|i-j|+1)}\cdot C\cdot C(m+1)K_m\gamma_m^{-1}\varepsilon_m^{-\frac{2(2n+3)}{N}}\mathrm{e}^{-s'_m|k|}\quad\text{for } i\ne j. \tag{3.64}$$

Note that $s_m>s'_m>s_{m+1}$. Again using Lemmas 2.2–2.3, we have

$$\begin{aligned}
&\|F(\theta,\tau)\|^{\mathcal{L}}_{\mathbb{T}_{s_{m+1}}\times\Pi_{m+1}}=\|\partial_\tau F(\theta,\tau)\|_{\mathbb{T}_{s_{m+1}}\times\Pi_{m+1}}\\
&\le C^2\cdot C(m+1)K_m\gamma_m^{-1}\varepsilon_m^{-\frac{4(2n+3)}{N}}\le C(m+1)\varepsilon_m^{-\frac{6\,(2n+3)}{N}}.
\end{aligned} \tag{3.65}$$

The proof of the measure estimate (3.51) will be postponed to Subsection 3.7. This completes the proof of Lemma 3.2.

3.5 Coordinate transformation Ψ by $\varepsilon_m F$

Recall $\Psi = \Psi_m = X^t_{\varepsilon_m F}\big|_{t=1}$, where $X^t_{\varepsilon_m F}$ is the flow of the Hamiltonian $\varepsilon_m F$, $X_{\varepsilon_m F}$ is the vector field with symplectic $\mathrm{i}\,\mathrm{d}u \wedge \mathrm{d}\overline{u}$. So

$$\mathrm{i}\,\dot{u} = \varepsilon_m \frac{\partial F}{\partial \overline{u}}, \quad -\mathrm{i}\,\dot{\overline{u}} = \varepsilon_m \frac{\partial F}{\partial u}, \quad \dot{\theta} = \omega.$$

More exactly,

$$\begin{cases} \mathrm{i}\,\dot{u} = \varepsilon_m F(\theta,\tau)u, & \theta = \omega t, \\ -\mathrm{i}\,\dot{\overline{u}} = \varepsilon_m F(\theta,\tau)\overline{u}, & \theta = \omega t, \\ \dot{\theta} = \omega. \end{cases}$$

Let $z = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}$,

$$B_m(\theta) = \begin{pmatrix} -\mathrm{i}\,F(\theta,\tau) & 0 \\ 0 & \mathrm{i}\,F(\theta,\tau) \end{pmatrix}, \quad \theta = \omega t. \tag{3.66}$$

Then

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \varepsilon_m B_m(\theta) z, \quad \dot{\theta} = \omega. \tag{3.67}$$

Let $z(0) = z_0 \in h_p \times h_p$, $\theta(0) = \theta_0 \in \mathbb{T}^n_{s_{m+1}}$ be initial value. Then

$$\begin{cases} z(t) = z_0 + \displaystyle\int_0^t \varepsilon_m B_m(\theta_0 + \omega s) z(s)\mathrm{d}s, \\ \theta(t) = \theta_0 + \omega t. \end{cases} \tag{3.68}$$

By Lemmas 3.2,

$$\|B_m(\theta)\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}, \tag{3.69}$$

$$\|B_m(\theta)\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m+1)\varepsilon_m^{-\frac{6(2n+3)}{N}}. \tag{3.70}$$

It follows from (3.68) that

$$z(t) - z_0 = \int_0^t \varepsilon_m B_m(\theta_0 + \omega s) z_0 \mathrm{d}s + \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)(z(s) - z_0)\mathrm{d}s.$$

Moreover, for $t \in [0,1]$, $\|z_0\|_p \le 1$,

$$\|z(t) - z_0\|_p \le \varepsilon_m C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}} + \int_0^t \varepsilon_m \|B_m(\theta_0 + \omega s)\| \|z(s) - z_0\|_p \mathrm{d}s, \tag{3.71}$$

where $\|\cdot\|$ is the operator norm from $h_p \times h_p \to h_p \times h_p$.

By Gronwall's inequality,

$$\|z(t) - z_0\|_p \le C(m+1)\varepsilon_m^{1-\frac{2(2n+3)}{N}} \exp\Big(\int_0^t \varepsilon_m \|B_m(\theta_0 + \omega s)\| \mathrm{d}s\Big) \le \varepsilon_m^{\frac{1}{2}}. \tag{3.72}$$

Thus,

$$\Psi_m : \mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1} \to \mathbb{T}^n_{s_m} \times \Pi_m, \tag{3.73}$$

and

$$\|\Psi_m - \mathrm{id}\|_{h_p \to h_p} \le \varepsilon_m^{\frac{1}{2}}. \tag{3.74}$$

Since (3.67) is linear, Ψ_m is a linear coordinate transform. According to (3.68), construct Picard sequence:

$$\begin{cases} z_0(t) = z_0, \\ z_{j+1}(t) = z_0 + \displaystyle\int_0^t \varepsilon_m B(\theta_0 + \omega s) z_j(s) \mathrm{d}s, \quad j = 0, 1, 2, \cdots. \end{cases}$$

By (3.74), this sequence with $t = 1$ goes to

$$\Psi_m(z_0) = z(1) = (\mathrm{id} + P_m(\theta_0)) z_0, \tag{3.75}$$

where id is the identity of $h_p \times h_p \to h_p \times h_p$, and $P_m(\theta_0)$ is an operator of $h_p \times h_p \to h_p \times h_p$ for any fixed $\theta_0 \in \mathbb{T}^n_{s_{m+1}}$, $\tau \in \Pi_{m+1}$, and is analytic in $\theta_0 \in \mathbb{T}^n_{s_{m+1}}$ with

$$\|P_m(\theta_0)\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le \varepsilon_m^{\frac{1}{2}}. \tag{3.76}$$

Note that (3.67) is a Hamiltonian system. So $P_m(\theta_0)$ is a symplectic linear operator from $h_p \times h_p$ to $h_p \times h_p$.

3.6 Estimates of remainders

The aim of this section is devoted to estimate the remainders:

$$R_{m+1} = (3.43) + \cdots + (3.46).$$

• Estimate of (3.43).

By (3.36), let

$$\widetilde{R}_{mm} = \widetilde{R}_{mm}(\theta) = \begin{pmatrix} 0 & \frac{1}{2} R_{m,m}(\theta) \\ \frac{1}{2} R_{m,m}(\theta) & 0 \end{pmatrix},$$

then

$$R_{mm} = \Big\langle \widetilde{R}_{mm} \begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \Big\rangle.$$

So

$$(1 - \Gamma_{K_m}) R_{mm} \triangleq \Big\langle (1 - \Gamma_{K_m}) \widetilde{R}_{mm} \begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \Big\rangle.$$

By the definition of truncation operator Γ_{K_m},

$$(1 - \Gamma_{K_m}) \widetilde{R}_{mm} = \sum_{|k| > K_m} \widehat{\widetilde{R}}_{mm}(k) \mathrm{e}^{\mathrm{i} \langle k, \theta \rangle}, \quad \theta \in \mathbb{T}^n_{s_m}, \quad \tau \in \Pi_m.$$

Since $\widetilde{R}_{mm} = \widetilde{R}_{mm}(\theta)$ is analytic in $\theta \in \mathbb{T}^n_{s_m}$,

$$\begin{aligned}
&\sup_{(\theta,\tau)\in\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \|(1-\Gamma_{K_m})\widetilde{R}_{mm}\|^2_{h_p\to h_p} \\
&\le \sum_{|k|>K_m} \|\widehat{\widetilde{R}}_{mm}(k)\|_p^2 \mathrm{e}^{2|k|s_{m+1}} \\
&\le \|\widetilde{R}_{mm}\|^2_{\mathbb{T}^n_{s_m}\times\Pi_m} \sum_{|k|>K_m} \mathrm{e}^{-2(s_m-s_{m+1})|k|} \\
&\le \frac{C^2(m)\mathrm{e}^{-2K_m(s_m-s_{m+1})}}{\varepsilon_m} \quad \text{(by (3.25))} \\
&\le C^2(m)\varepsilon_m.
\end{aligned}$$

That is

$$\|(1-\Gamma_{K_m})\widetilde{R}_{mm}\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m C(m+1).$$

Thus

$$\|\varepsilon_m(1-\Gamma_{K_m})\widetilde{R}_{mm}\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m^2 C(m+1) \le \varepsilon_{m+1}C(m+1). \tag{3.77}$$

• Estimate of (3.45).

Let

$$S_m(\theta) = \begin{pmatrix} 0 & \frac{1}{2}F(\theta,\tau) \\ \frac{1}{2}F(\theta,\tau) & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & -\mathrm{i}\,\mathrm{id} \\ \mathrm{i}\,\mathrm{id} & 0 \end{pmatrix}.$$

Then we can write

$$F = \left\langle S_m(\theta)\begin{pmatrix} u \\ \bar{u}\end{pmatrix}, \begin{pmatrix} u \\ \bar{u}\end{pmatrix}\right\rangle = \langle S_m(\theta)z, z\rangle, \quad z = \begin{pmatrix} u \\ \bar{u}\end{pmatrix}.$$

Then

$$\varepsilon_m^2\{R_{mm}, F\} = 4\varepsilon_m^2\langle \widetilde{R}_{mm}(\theta)\mathcal{J}S_m(\theta)z, z\rangle. \tag{3.78}$$

Noting $\mathbb{T}^n_{s_m}\times\Pi_m \supset \mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}$, by (3.25)–(3.26) with $l=m, v=m$,

$$\|\widetilde{R}_{mm}(\theta)\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \|\widetilde{R}_{mm}(\theta)\|_{\mathbb{T}^n_{s_m}\times\Pi_m} \le C(m), \tag{3.79}$$

$$\|\widetilde{R}_{mm}(\theta)\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m). \tag{3.80}$$

Let $\widetilde{S}_m(\theta) = \mathcal{J}S_m(\theta)$. Then by Lemma 3.2, we have

$$\|\widetilde{S}_m(\theta)\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}, \tag{3.81}$$

$$\|\widetilde{S}_m(\theta)\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_m^{-\frac{6(2n+3)}{N}} \tag{3.82}$$

and

$$\|\widetilde{R}_{mm}\mathcal{J}S_m\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \|\widetilde{R}_{mm}\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}}\|\widetilde{S}_m\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}.$$

Set

$$[\widetilde{R}_{mm}, \widetilde{S}_m] = \widetilde{R}_{mm}\widetilde{S}_m + (\widetilde{R}_{mm}\widetilde{S}_m)^{\mathrm{T}}.$$

Note that the vector field is linear. So, by Taylor formula, one has

$$(3.45) = \varepsilon_m^2 \langle \widetilde{R}_m^*(\theta)u, u\rangle,$$

where

$$\widetilde{R}_m^*(\theta) = \sum_{j=1}^{\infty} \frac{2^{j+1}\varepsilon_m^{j-1}}{j!} \underbrace{[\cdots[\widetilde{R}_{mm}, \widetilde{S}_m], \cdots, \widetilde{S}_m]}_{(j-1)\text{-fold}} \widetilde{S}_m.$$

By (3.79) and (3.81),

$$\begin{aligned}\|\widetilde{R}_m^*(\theta)\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} &\le \sum_{j=1}^{\infty} \frac{C(m)C(m+1)\varepsilon_m^{j-1}\big(\varepsilon_m^{-\frac{2(2n+3)}{N}}\big)^j}{j!} \\ &\le C(m)C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}}.\end{aligned}$$

By (3.80) and (3.82),

$$\|\widetilde{R}_m^*(\theta)\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{-\frac{6(2n+3)}{N}}.$$

Thus

$$\|\varepsilon_m^2 \widetilde{R}_m^*\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{2-\frac{2(2n+3)}{N}} \le C(m+1)\varepsilon_{m+1} \tag{3.83}$$

and

$$\|\varepsilon_m^2 \widetilde{R}_m^*\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{2-\frac{6(2n+3)}{N}} \le C(m+1)\varepsilon_{m+1}. \tag{3.84}$$

- Estimate of (3.44).

By (3.38),

$$\{N_m, F\} = \langle [R_{mm}]u, \overline{u}\rangle - \Gamma_{K_m} R_{mm} - \omega \cdot \partial_\theta F \triangleq R^*_{mm}. \tag{3.85}$$

Thus

$$(3.44) = \varepsilon_m^2 \int_0^1 (1-\tau)\{R^*_{mm}, F\} \circ X^\tau_{\varepsilon_m F} \mathrm{d}\tau. \tag{3.86}$$

Note that R^*_{mm} is a quadratic polynomial in u and $\overline{u}$. So we write

$$R^*_{mm} = \langle \mathcal{R}_m(\theta, \tau)z,\ z\rangle, \quad z = \begin{pmatrix} u \\ \overline{u}\end{pmatrix}. \tag{3.87}$$

By (3.23)–(3.24) with $l = \nu = m$, and using (3.81)–(3.82), we have

$$\|\mathcal{R}_m\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)\varepsilon_m^{-\frac{2(2n+3)}{N}}, \quad \|\mathcal{R}_m\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)\varepsilon_m^{-\frac{6(2n+3)}{N}}, \tag{3.88}$$

where $\|\cdot\|$ is the operator norm in $h_p \times h_p \to h_p \times h_p$. Recall $F = \langle S_m(\theta)z, z\rangle$. Set

$$[\mathcal{R}_m, \widetilde{S}_m] = \mathcal{R}_m\widetilde{S}_m + (\mathcal{R}_m\widetilde{S}_m)^{\mathrm{T}}. \tag{3.89}$$

Using Taylor formula to (3.86), we get

$$\begin{aligned}(3.44) &= \frac{\varepsilon_m^2}{2!}\{\{R^*_{mm}, F\}, F\} + \cdots + \frac{\varepsilon_m^j}{j!}\underbrace{\{\cdots\{R^*_{mm}, F\}, \cdots, F\}}_{j\text{-fold}} + \cdots \\ &= \Big\langle\Big(\sum_{j=2}^{\infty}\frac{2^{j+1}\varepsilon_m^j}{j!}\underbrace{[\cdots[\mathcal{R}_m, \widetilde{S}_m], \cdots, \widetilde{S}_m]\widetilde{S}_m}_{(j-1)\text{-fold}}\Big)z, z\Big\rangle \\ &\triangleq \langle\mathcal{R}^{**}(\theta, \tau)z, z\rangle.\end{aligned}$$

By (3.81) and (3.88)–(3.89), we have

$$\begin{aligned}\|\mathcal{R}^{**}(\theta,\tau)\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\le \sum_{j=2}^{\infty}\frac{2^{j+1}}{j!}\|\mathcal{R}_m(\theta,\tau)\|_{\mathbb{T}^n_{s_m}\times\Pi_m}\big(\|\widetilde{S}_m\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}}\varepsilon_m\big)^j \\ &\le \sum_{j=2}^{\infty}\frac{C(m)}{j!}(\varepsilon_m C(m+1)\varepsilon_m^{-\frac{2(2n+3)}{N}})^j \\ &\le C(m+1)\varepsilon_m^{\frac{4}{3}} = C(m+1)\varepsilon_{m+1}.\end{aligned} \tag{3.90}$$

Similarly

$$\|\mathcal{R}^{**}\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_{m+1}. \tag{3.91}$$

- Estimate of (3.46).

$$(3.46) = \sum_{l=m+1}^{\infty}\varepsilon_l(R_{lm}\circ X^1_{\varepsilon_m F}). \tag{3.92}$$

Write

$$R_{lm} = \langle\widetilde{R}_{lm}(\theta)z, z\rangle.$$

Then, by Taylor formula,

$$R_{lm}\circ X^1_{\varepsilon_m F} = R_{lm} + \sum_{j=1}^{\infty}\frac{1}{j!}\langle\widetilde{R}_{lmj}z, z\rangle,$$

where

$$\widetilde{R}_{lmj} = 2^{j+1}\underbrace{[\cdots[\widetilde{R}_{lm}, \widetilde{S}_m], \cdots]\widetilde{S}_m}_{(j-1)\text{-fold}}\varepsilon_m^j.$$

By (3.25), (3.26),

$$\|\widetilde{R}_{lm}\|_{\mathbb{T}^n_{s_l}\times\Pi_m} \le C(l), \quad \|\widetilde{R}_{lm}\|^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_m} \le C(l).$$

Combining the last two inequalities with (3.81)–(3.82), one has

$$\begin{aligned}\|\widetilde{R}_{lmj}\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} &\le \|\widetilde{R}_{lm}\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\cdot(\|\widetilde{S}_m\|_{\mathbb{T}^n_{m+1}\times\Pi_{m+1}}4\varepsilon_m)^j \\ &\le C^2(m)(\varepsilon_m\varepsilon_m^{-\frac{2(2n+3)}{N}})^j\end{aligned}$$

and

$$\begin{aligned}\|\widetilde{R}_{lmj}\|^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} &\leq \|\widetilde{R}_{lm}\|^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}(\|\widetilde{S}_m\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}4\varepsilon_m)^j \\ &\quad + \|\widetilde{R}_{lm}\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}(\|\widetilde{S}_m\|^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\varepsilon_m)^j \\ &\leq C^2(m)(\varepsilon_m\varepsilon_m^{-\frac{6(2n+3)}{N}})^j.\end{aligned}$$

Thus, let

$$\overline{R}_{l,m+1} := R_{lm} + \sum_{j=1}^{\infty}\frac{1}{j!}\widetilde{R}_{lmj},$$

then

$$(3.46) = \sum_{l=m+1}^{\infty}\varepsilon_l\langle\overline{R}_{l,m+1}z, z\rangle \tag{3.93}$$

and

$$\begin{aligned}\|\overline{R}_{l,m+1}\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} &\leq C^2(m) \leq C(m+1), \\ \|\overline{R}_{l,m+1}\|^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} &\leq C^2(m) \leq C(m+1).\end{aligned} \tag{3.94}$$

As a whole, the remainder R_{m+1} can be written as

$$C_{m+1}R_{m+1} = \sum_{l=m+1}^{\infty}\varepsilon_l\langle R_{l,\nu}(\theta)u, \overline{u}\rangle, \quad \nu = m+1,$$

where $R_{l,\nu}(\theta)$ satisfies (3.25) and (3.26) with $\nu = m+1,\ l \geq m+1$. This shows that Assumption $(\mathrm{A2})_\nu$ with $\nu = m+1$ holds.

By (3.42),

$$\mu_j^{(m)} = \widehat{R}^{z\overline{z}}_{mmjj}(0).$$

In (3.25)–(3.26), we have

$$\begin{aligned}|\mu_j^{(m)}|_{\Pi_m} &\leq |R_{mmjj}(\theta,\tau)| \leq C(m), \\ |\mu_j^{(m)}|^{\mathcal{L}}_{\Pi_m} &\leq |\partial_\tau R_{mmjj}(\theta,\tau)| \leq C(m).\end{aligned}$$

This shows that Assumption $(\mathrm{A1})_\nu$ with $\nu = m+1$ holds.

3.7 Estimate of measure

Now let us return to (3.55)

$$Q^{(m)}_{kij} \triangleq \Big\{\tau \in \Pi_m \mid |-\langle k,\omega_0\rangle\tau + \lambda_i^{(m)} - \lambda_j^{(m)}| < \frac{(|i-j|+1)\gamma_m}{A_k}\Big\}. \tag{3.95}$$

Case 1 If $i = j$, one has $k \neq 0$.

In this case,

$$Q^{(m)}_{kii} = \Big\{\tau \in \Pi_m \mid |\langle k,\omega_0\rangle\tau| < \frac{\gamma_m}{A_k}\Big\}. \tag{3.96}$$

It follows

$$|\langle k,\omega_0\rangle| < \frac{\gamma_m}{|k|^{n+3}\tau} < \frac{\gamma}{2^m|k|^{n+3}}.$$

Recall $|\langle k,\omega_0\rangle| > \frac{\gamma}{|k|^{n+1}}$. Then $Q_{kii}^{(m)} = \emptyset$. So,

$$\operatorname{Meas} Q_{kii}^{(m)} = 0. \tag{3.97}$$

Case 2 $i \neq j$.

If $Q_{kij}^{(m)} = \emptyset$, then $\operatorname{Meas} Q_{kij}^{(m)} = 0$. So we assume $Q_{kij}^{(m)} \neq \emptyset$ in the sequel. Then $\exists \tau \in \Pi_m$ such that

$$|-\langle k,\omega_0\rangle\tau + \lambda_i^{(m)} - \lambda_j^{(m)}| < \frac{|i-j|+1}{A_k}\gamma_m. \tag{3.98}$$

It follows from (3.22)–(3.23) that

$$\lambda_i^{(m)} - \lambda_j^{(m)} = i^2 - j^2 + O(\varepsilon_0) \geq \frac{2}{3}|i^2 - j^2|. \tag{3.99}$$

When $|i| \geq C|k| \gg |\langle k,\omega\rangle|$ or $|j| \geq C|k| \gg |\langle k,\omega\rangle|$, by (3.99), one has

$$\big|-\langle k,\omega\rangle + \lambda_i^{(m)} - \lambda_j^{(m)}\big| \geq \frac{2}{3}|i+j||i-j| - |\langle k,\omega\rangle| \geq \frac{1}{2}|i+j||i-j| > \frac{(|i-j|+1)\gamma_m}{A_k},$$

which implies $Q_{kij}^{(m)} = \emptyset$. Then

$$\operatorname{Meas} Q_{kij}^{(m)} = 0. \tag{3.100}$$

Now assume

$$|i| < C|k|, \quad |j| < C|k|.$$

Note that

$$-\langle k,\omega\rangle + \lambda_i - \lambda_j = -\langle k,\omega_0\rangle\tau + \lambda_i - \lambda_j = \tau\Big(-\langle k,\omega_0\rangle + \frac{\lambda_i - \lambda_j}{\tau}\Big)$$

and

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(-\langle k,\omega_0\rangle + \frac{\lambda_i - \lambda_j}{\tau}\Big)\Big| = \frac{|i^2-j^2|}{\tau^2} + O(\varepsilon) \geq \frac{1}{8}|i^2 - j^2|. \tag{3.101}$$

It follows that

$$\operatorname{Meas} Q_{kij}^{(m)} \leq \frac{16}{|i^2-j^2|}\Big(\frac{|i^2-j^2|+1}{A_k}\gamma_m\Big). \tag{3.102}$$

Then

$$\operatorname{Meas} \bigcup_{|k|\leq K_m} \bigcup_{\substack{i\leq C|k| \\ j\leq C|k|}} Q_{kij}^{(m)} \leq \sum_{|k|\leq K_m} \frac{C\gamma_m}{A_k} \sum_{i,j=1}^{C|k|} \frac{1}{i+j} \leq \sum_{|k|\leq K_m} \frac{C|k|^2\gamma_m}{A_k} \leq C\gamma_m. \tag{3.103}$$

Combining (3.97), (3.100) and (3.103), we have

$$\operatorname{Meas} \bigcup_{|k|\leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{kij}^{(m)} \leq C\gamma_m. \tag{3.104}$$

Let

$$\Pi_{m+1} = \Pi_m \backslash \bigcup_{|k|\le K_m} \bigcup_{i,j=1}^{\infty} Q_{kij}^{(m)}.$$

Then we have proved the following Lemma 3.3.

Lemma 3.3

$$\operatorname{Meas}\Pi_{m+1} \ge \operatorname{Meas}\Pi_m - C\gamma_m.$$

4 Proof of Theorem 1.1

Let

$$\Pi_\infty = \bigcap_{m=1}^{\infty} \Pi_m,$$

and

$$\Psi_\infty = \lim_{m\to\infty} \Psi_0 \circ \Psi_1 \circ \cdots \circ \Psi_m.$$

By (3.28)–(3.29), one has

$$\Psi_\infty : \mathbb{T}^n \times \Pi_\infty \to \mathbb{T}^n \times \Pi_\infty,$$
$$\|\Psi_\infty - \mathrm{id}\|_{h_p\to h_p} \le \varepsilon_0^{\frac{1}{2}}, \quad \varepsilon_0 = \varepsilon,$$

and by (3.30),

$$H_\infty = H \circ \Psi_\infty = \sum_{j=1}^{\infty} \lambda_j^\infty Z_j \overline{Z}_j,$$

where

$$\lambda_j^\infty = \lim_{m\to\infty} \lambda_j^{(m)}.$$

By (3.22)–(3.23), the limit λ_j^∞ does exist and

$$\lambda_j^\infty = j^2 + M + \xi_j, \quad |\xi_j| \le C\varepsilon.$$

Putting $v = \sum\limits_{k=1}^{\infty} Z_k(t)\sin(kx)$ into (1.6), we find that

$$(Z(t), \overline{Z}(t)) = (Z_k(t), \overline{Z}_k(t) : k = 1, 2, \cdots)$$

satisfies the Hamiltonian equations

$$\mathrm{i}\,\dot{Z}_k = -\frac{\partial H_\infty}{\partial \overline{Z}_k}, \quad \mathrm{i}\,\dot{\overline{Z}}_k = \frac{\partial H_\infty}{\partial Z_k}, \quad k = 1, 2, \cdots.$$

Let

$$\mathcal{F} : h_p \to \mathcal{H}_0^p[0, \pi], \quad Z \mapsto \mathcal{F}Z = \sum_{k=1}^{\infty} Z_k \sin(kx)$$

be the inverse discrete Fourier transform, which defines an isometry between the two spaces. Let $\Phi = \mathcal{F} \circ \Psi_\infty \circ \mathcal{F}^{-1}$. This completes the proof of Theorem 1.1.

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