Chin. Ann. Math. Ser. B
41(1), 2020, 147–162
DOI: 10.1007/s11401-019-0191-z

Chinese Annals of Mathematics, Series B

Geodesics in the Engel Group with a Sub-Lorentzian Metric — the Space-Like Case*

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Abstract Let E be the Engel group and D be a bracket generating left invariant distribution with a Lorentzian metric, which is a nondegenerate metric of index 1. In this paper, the author constructs a parametrization of a quasi-pendulum equation by Jacobi functions, and then gets the space-like Hamiltonian geodesics in the Engel group with a sub-Lorentzian metric.

Keywords Sub-Lorentzian metric, Engel Group, Geodesics
2000 MR Subject Classification 58E10, 53C50

1 Introduction

Sub-Riemannian geometry is an important branch of geometrical analysis. It has been applied in many fields, such as quantum physics, C-R geometry, control theory, sub-elliptic equations, and so on (see [1, 16–18]). Because of the degeneration of the metric, sub-Riemannian geometry enjoys many differences from the Riemannian one. For example, the Hausdorff dimension is larger than the manifold topological dimention, there exist strictly abnormal minimizers in sub-Riemannian manifolds (see [11, 15]). If we change the positively definite metric to an indefinite degenerate metric with index one, it leads to another geometry, sub-Lorentzian geometry. It is a special semi-sub-Riemannian geometry (see [14]). For more details about sub-Lorentzian geometry, the reader is referred to [4, 8, 12]. Since there are three kinds of horizontal curves (time-like, space-like, null) in sub-Lorentzian manifolds, researches on geodesics and reachable sets in sub-Lorentzian manifolds are more complicated than in sub-Riemannian manifolds. In [7], Chang, Markina, and Vasiliev have systematically studied the geodesics in an anti-de Sitter space with a sub-Lorentzian metric and a sub-Riemannian metric respectively. In [9], Grochowski computed reachable sets starting from a point in the Heisenberg sub-Lorentzian manifold on $\mathbb{R}^3$. In [10], Grochowski studied the normal form for Martinet sub-Lorentzian structure of Hamiltonian type, and calculated reachable sets, future null conjugate and cut loci. It was shown in [13] that the Heisenberg group $\mathbb{H}$ with a Lorentzian metric on $\mathbb{R}^3$ possesses the uniqueness of Hamiltonian geodesics of time-like or space-like type.

Manuscript received April 17, 2015. Revised August 8, 2017.
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*This work was supported by the Science and Technology Development Fund of Nanjing Medical University (No. 2017NJMU005).

The Engel group was first named by Cartan [6] in 1901. It is the simplest sub-Lorentzian manifold with nontrivial abnormal extremal trajectories, and is a Goursat manifold. In [2], Ardentov and Sachkov computed extremal trajectories on the sub-Riemannian Engel group. In [5], we computed nonspace-like geodesics on the Engel group with sub-Lorentzian metric, but did not calculate space-like geodesics. Because if we use the same method in the space-like case, the computation becomes too complicated. In this paper, we introduce a new method, construct a parameter transformation by Jacobi elliptic functions, and get the space-like Hamiltonian geodesics.

The paper is organized in the following way. In Section 2, we introduce some preliminaries as well as definitions of sub-Lorentzian manifolds, the Engel group. In Section 3, we give the Hamiltonian system, and compute the space-like abnormal geodesics. In Section 4, we introduce a new elliptic coordinate to solve the quasi pendulum equation, and obtain a complete description of the space-like Hamiltonian geodesics in the Engel group.

2 Preliminaries

Let M be a smooth n-dimensional manifold, D be a smooth distribution on M and g be a smooth varying Lorentzian metric on D. The triple (M, D, g) is called a sub-Lorentzian manifold.

We introduce a time orientation first. A vector $v \in D_p$ is said to be time-like if $g(v, v) < 0$; space-like if $g(v, v) > 0$ or $v = 0$; null (light-like) if $g(v, v) = 0$ and $v \neq 0$; and non-space-like if $g(v, v) \leq 0$. An absolutely continuous curve $\gamma(t)$ is said to be horizontal if its derivative $\gamma'(t)$ exists almost everywhere and lies in $D(\gamma(t))$. A horizontal curve is said to be time-like if its tangent vector is time-like a.e.. Similarly, space-like, null and non-space-like curves can be defined. By a time orientation of (M, D, g), we mean a continuous time-like vector field on M. If X is a time orientation on (M, D, g), then a non-space-like vector $v \in D_p$ is said to be future directed if $g(v, X(p)) < 0$, and past directed if $g(v, X(p)) > 0$. Throughout this paper, "f.d." stands for "future directed", "t." for "time-like", and "nspc." for "non-space-like". The sub-Lorentzian length of a horizontal curve l is defined by

$$l(\gamma(t)) = \int_0^1 \|\gamma'(t)\| \mathrm{d}t = \int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|} \mathrm{d}t.$$

A distribution $D \subset TM$ is called bracket generating if any local frame $\{X_i\}_{1 \leq i \leq r}$ for D, together with all of its iterated Lie brackets $[X_i, X_j], [X_i, [X_j, X_k]], \cdots$ span the tangent bundle TM. Bracket generating distributions are sometimes also called completely nonholonomic or distributions that satisfying Hörmander's condition.

Theorem 2.1 (Chow Theorem) *Fix a point $q \in M$. If the distribution $D \subset TM$ is bracket generating, then the set of points that can be connected to q by a horizontal curve is the component of M containing q.*

By the Chow theorem, we know that if D is bracket generating and M is connected, then any two points of M can be joined by a horizontal curve.

Now we introduce the Hamiltonian function with respect to the sub-Lorentzian structure

$$H(\lambda, u) = \frac{\nu}{2}\Big(-u_1^2 + \sum_{i=2}^{k} u_i^2\Big) + \Big\langle \lambda, \sum_{i=1}^{k} u_i X_i \Big\rangle, \tag{2.1}$$

where $\lambda = (q, \xi) \in T_q^* M$, $u \in R^k$, $\nu = 0$ or -1, $(\nu, \xi) \neq 0$. $\vec{H} \in T(T^*M)$ is the Hamiltonian vector field induced by H, λ_s is the flow of the vector field $\vec{H}$, i.e.,

$$\dot{\lambda}_s = \vec{H}(\lambda_s),$$

where u satisfies the maximum condition $\frac{\partial H}{\partial u} = 0$. If $\nu = 0$, the integral curve λ is called an abnormal lift, and the projection γ is an abnormal extremal. If $\nu = -1$, λ is called a normal lift, the projection γ is a normal geodesics (or a normal extremal). Immediately from the definition, we have the following lemma (cf. [5]).

Lemma 2.1 (cf. [5]) *The causal character* (*time-like, space-like, null*) *of normal sub-Lorentzian geodesics does not depend on time.*

At last, we describe the Engel group E. We consider the Engel group E with coordinates $q = (x_1, x_2, y, z) \in \mathbb{R}^4$. The group law is denoted by $\odot$ and defined as

$$\begin{aligned} &(x_1, x_2, y, z) \odot (x_1', x_2', y', z') \\ &= \Big(x_1 + x_1', x_2 + x_2', y + y' + \frac{x_1 x_2' - x_1' x_2}{2}, z + z' + \frac{x_2 x_2'}{2}(x_2 + x_2') + x_1 y' + \frac{x_1 x_2'}{2}(x_1 + x_1')\Big). \end{aligned}$$

A vector field X is said to be left-invariant if it satisfies $\mathrm{d}L_q X(e) = X(q)$, where L_q denotes the left translation $p \to L_q(p) = q \odot p$ and e is the identity of E. This definition implies that any left-invariant vector field on E is a linear combination of the following vector fields:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial y} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial z}, \\ X_3 &= \frac{\partial}{\partial y} + x_1 \frac{\partial}{\partial z}, \qquad X_4 = \frac{\partial}{\partial z}. \end{aligned} \tag{2.2}$$

The distribution $D = \mathrm{span}\{X_1, X_2\}$ of E satisfies the bracket generating condition, since $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$. The Engel group is a nilpotent Lie group, since $[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0$. We define a smooth Lorentzian metric on D by

$$g(X_1, X_1) = -1, \quad g(X_2, X_2) = 1, \quad g(X_1, X_2) = 0. \tag{2.3}$$

X_1 is the time orientation.

The horizontal curve on the Engel group has the property (cf. [5]).

Lemma 2.2 (cf. [5]) *For a horizontal curve on the Engel group, the property of time-likeness* (*space-likeness, light-likeness*) *and future-directness* (*past-directness*) *is preserved under left translations.*

3 Abnormal Extremal Trajectories

In this section we write down the Hamiltonian system and compute the abnormal extremals. By Lemma 2.2 horizontal space-like curves are left invariant, so we can assume that the initial point is the origin, i.e., $x_1(0) = x_2(0) = y(0) = z(0) = 0$, and space-like initial velocity is $-u_1^2(0) + u_2^2(0) = 1$. Let us introduce the vector of costate variables $\eta = (\nu, \xi_1, \xi_2, \xi_3, \xi_4)$ and define the Hamiltonian function

$$H(\xi, q(t), u) = \nu\frac{-u_1^2 + u_2^2}{2} + \xi_1 u_1 + \xi_2 u_2 + \xi_3\frac{x_1 u_2 - x_2 u_1}{2} + \xi_4\frac{x_1^2 + x_2^2}{2}u_2. \tag{3.1}$$

From Pontryagin's maximum principle for this Hamiltonian function we obtain a Hamiltonian system for the costate variables

$$\dot{\xi}_1 = -H_{x_1} = -\frac{\xi_3 u_2}{2} - \xi_4 x_1 u_2, \quad \dot{\xi} = -H_{x_2} = \frac{\xi_3 u_1}{2} - \xi_4 x_2 u_2, \quad \dot{\xi}_3 = \dot{\xi}_4 = 0, \tag{3.2}$$

and the maximum condition

$$\max_{u\in\mathbb{R}^2} H(\xi(t), q(t), u(t)) = H(\xi(t), \widehat{q}(t), \widehat{u}(t)), \quad \nu \le 0, \tag{3.3}$$

where $\widehat{u}(t), \widehat{q}(t)$ are the optimal processes, and the condition $\eta(t) \neq 0$ is the non-triviality of the costate variables.

Let $\nu = 0$. From the maximum condition (3.3) we obtain

$$H_{u_1} = \xi_1 - \frac{\xi_3 x_2}{2} = 0, \tag{3.4}$$

$$H_{u_2} = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4(x_1^2 + x_2^2)}{2} = 0. \tag{3.5}$$

Differentiating equations (3.4)–(3.5), we obtain

$$0 = \dot{\xi}_1 - \frac{\xi_3\dot{x}_2}{2} = \dot{\xi}_1 - \frac{\xi_3 u_2}{2} = -u_2(\xi_3 + \xi_4 x_1), \tag{3.6}$$

$$0 = \dot{\xi}_2 + \frac{\xi_3\dot{x}_1}{2} + \xi_4(x_1\dot{x}_1 + x_2\dot{x}_2) = u_1(\xi_3 + \xi_4 x_1). \tag{3.7}$$

For the space-like case, $-u_1^2 + u_2^2 = 1$, so $\xi_3 + \xi_4 x_1 = 0$. If $\xi_4 = 0$, then $\xi_3 = 0$, and therefore $\xi = 0$. It is a contradiction with the non-triviality of the costate variables, hence $\xi_4 \neq 0$. In this case, we can get that $x_1 = \frac{-\xi_3}{\xi_4}$ is a constant, and $u_1 = 0,\ u_2 = \pm 1$, so the space-like abnormal extremal trajectories are the following expression:

$$\gamma(s) = \Big(0, \pm s, 0, \pm\frac{s^3}{6}\Big). \tag{3.8}$$

4 Normal Geodesics

Now we consider the normal case $\nu = -1$. It follows from the maximum condition (3.3) that $H_{u_1} = H_{u_2} = 0$. Hence

$$u_1 = -\Big(\xi_1 - \frac{x_2\xi_3}{2}\Big), \quad u_2 = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4(x_1^2 + x_2^2)}{2}. \tag{4.1}$$

Let $h_i = (\xi, X_i),\ i = 1, 2, 3, 4$ be the Hamiltonians which are linear on fibers. It is easy to get

$$h_1 = \xi_1 - \frac{x_2\xi_3}{2}, \quad h_2 = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4(x_1^2 + x_2^2)}{2}, \quad h_3 = \xi_3 + x_1\xi_4, \quad h_4 = \xi_4, \tag{4.2}$$

so $u_1 = -\zeta_1$ and $u_2 = \zeta_2$.

According to Pontryagin's maximum principle, we get the following Hamilton system:

$$\dot{x}_1 = -h_1, \tag{4.3}$$

$$\dot{x}_2 = h_2, \tag{4.4}$$

$$\dot{y} = \frac{1}{2}(x_2 h_1 + x_1 h_2), \tag{4.5}$$

$$\dot{z} = \frac{1}{2}h_2(x_1^2 + x_2^2), \tag{4.6}$$

$$\dot{h}_1 = -h_2 h_3, \tag{4.7}$$

$$\dot{h}_2 = -h_1 h_3, \tag{4.8}$$

$$\dot{h}_3 = -h_1 h_4, \tag{4.9}$$

$$\dot{h}_4 = 0. \tag{4.10}$$

For the space-like case, the curves are considered on the level surface $H = -h_1^2 + h_2 = 1$, whence we go over to another coordinate system (θ, c, α) :

$$h_1 = \sin h\theta, \quad h_3 = c,$$

$$h_2 = \cos h\theta, \quad h_4 = \alpha.$$

So in the normal case, the Hamiltonian system becomes

$$\dot{\theta} = -c, \tag{4.11}$$

$$\dot{c} = -\alpha \sin h\theta, \tag{4.12}$$

$$\dot{\alpha} = 0, \tag{4.13}$$

$$\dot{x}_1 = -\sin h\theta, \tag{4.14}$$

$$\dot{x}_2 = \cos h\theta, \tag{4.15}$$

$$\dot{y} = \frac{1}{2}(x_2 \sin h\theta + x_1 \cos h\theta), \tag{4.16}$$

$$\dot{z} = \frac{1}{2}\cos h\theta(x_1^2 + x_2^2). \tag{4.17}$$

The vertical part is reduced to an equation which is quite similar to a pendulum one:

$$\ddot{\theta} = \alpha \sin h\theta, \quad \dot{\alpha} = 0. \tag{4.18}$$

For this reason, we call the formula quasi-pendulum equation. Figures 1–2 are phase graphs for $\alpha = 1$ and $\alpha = -1$.

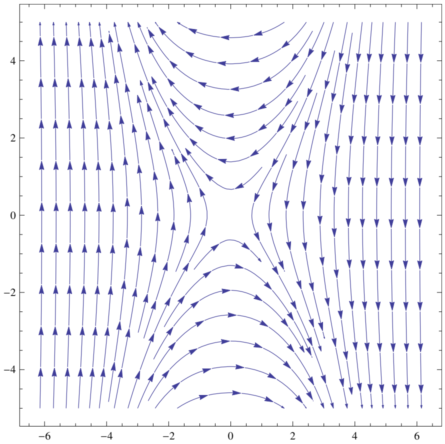

Figure 1 $\ddot{\theta} = \sin h\theta$.

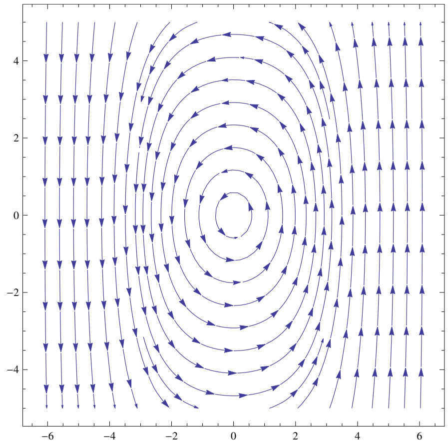

Figure 2 $\ddot{\theta} = -\sin h\theta$.

Let us introduce the energy integral

$$E = \frac{1}{2}c^2 - \alpha \cos h\theta, \quad \dot{E} = 0.$$

The family of all normal geodesics is parameterized by the points of the set

$$C = \{(h_1, h_2, h_3, h_4) \in \mathbb{R}^4 \mid h_2^2 - h_1^2 = 1\} = \{(\theta, c, \alpha) \in \mathbb{R}^3)\}.$$

We divide the set C into 10 parts as follows:

$$\begin{aligned}
C &= C_0^0 \cup C^0 \cup C_1^- \cup C_2^- \cup C_1^+ \cup C_2^+ \cup C_3^+ \cup C_4^+ \cup C_5^+ \cup C_6^+, \quad \lambda = (\theta, c, \alpha),\\
C_0^0 &= \{\lambda \in C \mid \alpha = 0,\ c = 0\},\\
C^0 &= \{\lambda \in C \mid \alpha = 0,\ c \neq 0\},\\
C_1^- &= \{\lambda \in C \mid \alpha < 0,\ E \in (-\alpha, +\infty)\},\\
C_2^- &= \{\lambda \in C \mid \alpha < 0,\ E = -\alpha\},\\
C_1^+ &= \{\lambda \in C \mid \alpha > 0,\ E \in (-\infty, -\alpha)\},\\
C_2^+ &= \{\lambda \in C \mid \alpha > 0,\ E \in (-\alpha, \alpha)\},\\
C_3^+ &= \{\lambda \in C \mid \alpha > 0,\ E \in (\alpha, +\infty)\},\\
C_4^+ &= \{\lambda \in C \mid \alpha > 0,\ E = \alpha\},\\
C_5^+ &= \{\lambda \in C \mid \alpha > 0,\ E = -\alpha,\ c \neq 0\},\\
C_6^+ &= \{\lambda \in C \mid \alpha > 0,\ E = -\alpha,\ c = 0\}.
\end{aligned}$$

The sets $C_i^+,\ i = 1, 2, 3, 4, 5$ are further subdivided into subsets that depending on the sign of $\theta,\ c$,

$$C_{i+}^+ = C_i^+ \cap \{c > 0\}, \quad C_{i-}^+ = C_i^+ \cap \{c < 0\},$$
$$C_i^{++} = C_i^+ \cap \{\theta > 0\}, \quad C_i^{+-} = C_i^+ \cap \{\theta < 0\}.$$

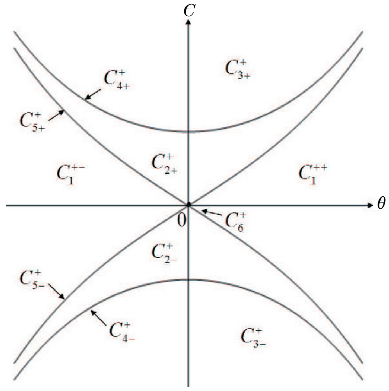

Figure 3 Partition C for $\alpha > 0$.

We give the partitioning of a section of the cylinder $\{\lambda \in C,\ \alpha = \mathrm{con}\,st > 0\}$ in Figure 3.

For subsets C_0^0, C^0, C_2^-, C_6^+, we compute directly. For $C_1^-, C_1^+, C_2^+, C_3^+, C_4^+$ and C_5^+, we introduce new coordinates (φ, k) in the phase space of the quasi-pendulum. We call (φ, k) the elliptic coordinates. In this coordinates, the vector field $(-c, -\alpha \sin h\theta)$ is straightened out, and the quasi-pendulum equation takes a very simple form. That is to say, when we transform (θ, c) to (φ, k), the quasi-pendulum equation

$$\begin{cases} \dot\theta = -c, \\ \dot c = -\alpha \sin h\theta \end{cases} \tag{4.19}$$

becomes

$$\begin{cases} \dot\varphi = 1, \\ \dot k = 0. \end{cases} \tag{4.20}$$

So the solution of the quasi-pendulum equation in the elliptic coordinates is

$$k = \mathrm{con}\,st, \quad \varphi = \varphi_0 + t,$$

where φ_0 is the initial value, and expressions of space-like extremals can be obtained.

If $\lambda \in C_0^0$, we have $\theta \equiv \mathrm{con}\,st$, and so is $\sin h\theta$. We assume $\sin h\theta = s_0$,

$$x_1(t) = -s_0 t, \tag{4.21}$$

$$x_2(t) = \sqrt{s_0^2 + 1}\,t, \tag{4.22}$$

$$y(t) \equiv 0, \tag{4.23}$$

$$z(t) = \frac{(2s_0^2+1)\sqrt{s_0^2+1}}{6} t^3. \tag{4.24}$$

If $\lambda \in C^0$, we have $c \equiv \mathrm{con}\,st,\ \theta = \theta_0 - ct$,

$$x_1(t) = \frac{\cos h\theta - \cos h\theta_0}{c}, \tag{4.25}$$

$$x_2(t) = \frac{\sin h\theta_0 - \sin h\theta}{c}, \tag{4.26}$$

$$y(t) = \frac{ct - \sin h(ct)}{2c^2}, \tag{4.27}$$

$$\begin{aligned} z(t) = \frac{1}{c^3}\Big[& -\frac{1}{3}\sin h^3\theta - \frac{1}{6}\sin h^3\theta_0 - \cos h^2\theta_0 \sin h\theta + \cos h^2\theta_0 \sin h\theta_0 \\ & + \frac{1}{2}\sin h\theta_0 \sin h^2\theta + \frac{1}{4}\cos h\theta_0 \sin h2\theta - \frac{1}{4}\cos h\theta_0 \sin h2\theta_0 - \frac{c}{2}(\cos h\theta_0)t\Big]. \end{aligned} \tag{4.28}$$

If $\lambda \in C_1^-$, and the energy integral $E \in (-\alpha, +\infty)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{E+\alpha}{E-\alpha}} \in (0,1), \tag{4.29}$$

$$\sin h\frac{\theta}{2} = -ksd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big), \quad \cos h\frac{\theta}{2} = nd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big), \tag{4.30}$$

$$c = 2\frac{k\sqrt{-\alpha}}{k'}cd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big), \quad \varphi \in \Big[0, \frac{4k'K}{\sqrt{-\alpha}}\Big], \tag{4.31}$$

where sd, nd, cd are elliptic functions, k is the modulus, k' is the complementary modulus. It is easy to check that in the elliptic coordinates, the vertical part of Hamiltonian system takes the form:

$$\dot\varphi = 1, \quad \dot k = 0, \quad \dot\alpha = 0, \tag{4.32}$$

and the solutions are

$$\varphi(t) = \varphi_0 + t, \quad k = \mathrm{con}\,st, \quad \alpha = \mathrm{con}\,st. \tag{4.33}$$

Now we compute expressions of space-like geodesics. By transformation (4.29)–(4.31), we get

$$\sin h\theta = 2\sin h\frac{\theta}{2}\cos h\frac{\theta}{2} = -2ksd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big)nd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big), \tag{4.34}$$

$$\cos h\theta = 2\sin h^2\frac{\theta}{2} + 1 = 2k^2 sd^2\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big) + 1. \tag{4.35}$$

Substituting them into the horizontal part of the Hamiltonian system, we have

$$\dot x_1 = 2ksd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big)nd\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big),$$

$$\dot x_2 = 2k^2 sd^2\Big(\frac{\sqrt{-\alpha}\varphi}{k'}\Big) + 1.$$

By the following integral formulas:

$$\int sd\ u\ nd\ u du = -\frac{cd\ u}{k'^2}, \tag{4.36}$$

$$\int sd^2 u du = \frac{1}{k^2k'^2}(E(u) - k'^2u - k^2 sn\ u\ cd\ u), \tag{4.37}$$

we obtain

$$x_1 = -2\frac{k}{\sqrt{-\alpha}k'}[cd(\chi_t) - cd(\chi_0)], \tag{4.38}$$

$$x_2 = \frac{2}{\sqrt{-\alpha}k'}[E(\chi_t) - E(\chi_0) - k^2(sn(\chi_t)cd(\chi_t) - sn(\chi_0)cd(\chi_0))] - t, \tag{4.39}$$

where $\chi_t = \frac{\sqrt{-\alpha}}{k'}\varphi, \chi_0 = \frac{\sqrt{-\alpha}}{k'}\varphi_0$.

Since

$$\dot{y} = \frac{1}{2}(-x_2\dot{x}_1 + x_1\dot{x}_2), \tag{4.40}$$

$$\dot{z} = \frac{1}{2}(x_1^2 + x_2^2)\dot{x}_2, \tag{4.41}$$

we get

$$\begin{aligned} y &= \int x_1\dot{x}_2 \mathrm{dt} - \frac{x_1x_2}{2} \\ &= -2\frac{k}{\sqrt{-\alpha}k'}\int (cd\chi_t - cd\chi_0)(2k^2sd^2\chi_t + 1)\mathrm{dt} - \frac{x_1x_2}{2}. \end{aligned} \tag{4.42}$$

By (4.37) and formulas

$$\int cd\ u\ sd^2u\mathrm{du} = -\frac{1}{2k^3}\ln\Big(\frac{1+ksn\ u}{dn\ u}\Big), \tag{4.43}$$

$$\int cd\ u\mathrm{du} = \frac{1}{k}\ln\Big(\frac{1+ksn\ u}{dn\ u}\Big), \tag{4.44}$$

we get

$$\begin{aligned} y = -4\frac{k^4}{\alpha}(sn\chi_t nd\chi_t - sn\chi_0 nd\chi_0) - 4\mathrm{sgn}\,c\frac{k}{\alpha k'^2}[E(\chi_t) - E(\chi_0) - \sqrt{-\alpha}k't \\ - k^2(sn\chi_t cd\chi_t - sn\chi_0 cd\chi_0)] + 2\mathrm{sgn}\,c\ cd\chi_0\frac{k}{\sqrt{-\alpha}k'}t - \frac{x_1x_2}{2}. \end{aligned} \tag{4.45}$$

Since

$$\dot{z} = \frac{1}{2}(x_1^2 + x_2^2)\dot{x}_2, \tag{4.46}$$

we get

$$\begin{aligned} z &= \int \frac{x_1^2\dot{x}_2}{2}\mathrm{dt} + \frac{x_2^3}{6} \\ &= -\frac{4k^2}{\alpha k'^2}\int (cd^2\chi_t - 2cd\chi_0 cd\chi_t + cd\chi_0)(2k^2sd^2\chi_t + 1)\mathrm{dt} + \frac{x_2^3}{6}. \end{aligned} \tag{4.47}$$

By (4.37), (4.43)–(4.44) and formulas

$$\begin{aligned} \int cd^2u\ sd^2u\mathrm{du} = \frac{1}{k'^2}\Big[\frac{2k^2-2}{3k^4}u + \frac{2-k^2}{3k^4}E(u) \\ + (2k^2 + 2k^4 - 1)sn\ u\ cd\ u - k^2k'^2sd\ u\ cd\ u\Big], \end{aligned} \tag{4.48}$$

$$\int cd^2 u\mathrm{d}u = \frac{1}{k}\ln\Big(\frac{1+ksn\ u}{dn\ u}\Big), \tag{4.49}$$

we get

$$z = \frac{4k^2}{(-\alpha)^{\frac{3}{2}}k'}\Big\{\Big(\frac{1}{3k^2} - cd^2\chi_0\Big)\frac{\sqrt{-\alpha}}{k'}t + \Big(\frac{2k^2-1}{3k^2k'^2} + \frac{2}{k'^2}cd^2\chi_0\Big)(E(\chi_t) - E(\chi_0))$$
$$+ \Big[\frac{2k^2}{k'^2}(2k^2+2k^4-1) + 1 - \frac{2k^2}{k'^2}cd\chi_0\Big](sn\chi_t cd\chi_t - sn\chi_0 cd\chi_0) + 2k^4(sn\chi_t cd\chi_t nd\chi_t$$
$$- sn\chi_0 cd\chi_0 nd\chi_0) + 4k^3 cd\chi_0(sn\chi_t nd^2\chi_t - sn\chi_0 nd^2\chi_0)\Big\} + \frac{x_2^3}{6}. \tag{4.50}$$

If $\lambda \in C_2^-$, we have $\theta = c \equiv 0$. Then we have

$$x_1(t) = 0, \quad x_2(t) = t, \quad y(t) = 0, \quad z(t) = \frac{1}{6}t^3. \tag{4.51}$$

If $\lambda \in C_1^+$, the energy integral $E \in (-\alpha, +\infty)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{E+\alpha}{E-\alpha}} \in (0,1), \tag{4.52}$$

$$\sin h\frac{\theta}{2} = \frac{1}{k}\operatorname{sgn}\theta ds\Big(\frac{\sqrt{\alpha}}{k}\varphi\Big), \quad \cos h\frac{\theta}{2} = \frac{1}{k}ns\Big(\frac{\sqrt{\alpha}}{k}\varphi\Big), \tag{4.53}$$

$$c = \operatorname{sgn}\theta\frac{2}{k}\sqrt{\alpha}cs\Big(\frac{\sqrt{\alpha}}{k}\varphi\Big), \quad \varphi \in \Big(0, \frac{2kK}{\sqrt{\alpha}}\Big). \tag{4.54}$$

So we obtain

$$\dot{x}_1 = -\sin h\theta = -\frac{2}{k^2}\operatorname{sgn}\theta ds\Big(\frac{\sqrt{\alpha}}{k}\varphi\Big)ns\Big(\frac{\sqrt{\alpha}}{k}\varphi\Big), \tag{4.55}$$

$$\dot{x}_2 = \cos h\theta = \frac{2}{k^2}ds^2\Big(\frac{\sqrt{\alpha}}{k}\varphi\Big) + 1. \tag{4.56}$$

Combining with the formulas

$$\int cs\ u\mathrm{d}u = \ln\Big(\frac{1-dn\ u}{sn\ u}\Big), \tag{4.57}$$

$$\int cs^2 u\mathrm{d}u = -dn\ u\ cs\ u - E(u), \tag{4.58}$$

$$\int cs\ u\ ds^2 u\mathrm{d}u = -\frac{k^2}{2}ds^2 u\mathrm{d}u - \frac{1}{2}dn\ u\ ns^2 u, \tag{4.59}$$

$$\int cs^2 u\ ds^2 u\mathrm{d}u = \frac{2-k'^2}{3}(E(u) + dn\ u\ cs\ u) - \frac{1}{3}ns^2 u\ dn\ u\ cs\ u - \frac{k'^2}{3}u, \tag{4.60}$$

$$\int ds^2 u\mathrm{d}u = k'^2 u - dn\ u\ cs\ u - E(u), \tag{4.61}$$

we get

$$x_1 = \operatorname{sgn}\theta\frac{2}{\sqrt{\alpha}k}(cs\phi_t - cs\phi_0), \tag{4.62}$$

$$x_2 = \frac{2}{\sqrt{\alpha}k}[dn\phi_0 cs\phi_0 - dn\phi_t cs\phi_t + E(\phi_0) - E(\phi_t)] + \frac{2k'^2}{k^2}t, \tag{4.63}$$

and further more,

$$\begin{aligned}
y &= \int x_1\dot{x}_2 \mathrm{d}t - \frac{x_1x_2}{2} \\
&= \operatorname{sgn}\theta \frac{2}{\sqrt{\alpha}k}\int (cs\phi_t - cs\phi_0)\Big(\frac{2}{k^2}ds^2\phi_t + 1\Big)\mathrm{d}t - \frac{x_1x_2}{2} \\
&= -\operatorname{sgn}\theta \frac{2}{\alpha k^2}(dn\phi_t ns^2\phi_t - dn\phi_0 ns^2\phi_0) - \operatorname{sgn}\theta\frac{4}{\alpha k^2}cs\phi_0\Big[\frac{\sqrt{\alpha}k'^2}{k}t \\
&\quad + dn\phi_0 cs\phi_0 - dn\phi_t cs\phi_t + E(\phi_0) - E(\phi_t)\Big] - \operatorname{sgn}\theta\frac{\sqrt{\alpha}}{k}cs\phi_0 t - \frac{x_1x_2}{2},
\end{aligned} \tag{4.64}$$

$$\begin{aligned}
z &= \int \frac{x_1^2\dot{x}_2}{2}\mathrm{d}t + \frac{x_2^3}{6} \\
&= \frac{4}{k^2\alpha}\int (cs\phi_t - cs\phi_0)^2\Big(\frac{2}{k^2}ds^2\phi_t + 1\Big)\mathrm{d}t + \frac{x_2^3}{6} \\
&= \frac{4}{\alpha^{\frac{3}{2}}k}\Big\{\Big(\frac{2-k^2-6cs^2\phi_0}{3k^2}\Big)[E(\phi_t) - E(\phi_0) + dn\phi_t cs\phi_t - dn\phi_0 cs\phi_0] \\
&\quad - \frac{2}{3k^2}(ns^2\phi_t dn\phi_t cs\phi_t - ns^2\phi_0 dn\phi_0 cs\phi_0) - \operatorname{sgn}c\frac{\sqrt{\alpha}}{k}\Big(-\frac{2k'^2}{3k^2} + \frac{2-k^2}{k^2}cs^2\phi_0\Big)t \\
&\quad + \frac{2}{k^2}cs\phi_0(dn\phi_t ns^2\phi_t - dn\phi_0 ns^2\phi_0)\Big\} + \frac{x_2^3}{6},
\end{aligned} \tag{4.65}$$

where $\phi_t = \frac{\sqrt{\alpha}}{k}\varphi,\ \phi_0 = \frac{\sqrt{\alpha}}{k}\varphi_0$.

If $\lambda \in C_2^+$, the energy integral $E \in (-\alpha, \alpha)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{\alpha - E}{2\alpha}} \in (0,1), \tag{4.66}$$

$$\sin h\frac{\theta}{2} = -\operatorname{sgn}c\ k' sc(\sqrt{\alpha}\varphi), \quad \cos h\frac{\theta}{2} = dc(\sqrt{\alpha}\varphi), \tag{4.67}$$

$$c = 2\operatorname{sgn}c\ k'\sqrt{\alpha}nc(\sqrt{\alpha}\varphi), \quad \varphi \in \Big(-\frac{K}{\sqrt{\alpha}}, \frac{K}{\sqrt{\alpha}}\Big). \tag{4.68}$$

So we obtain

$$\dot{x}_1 = -\sin h\theta = -2\operatorname{sgn}c\ k' sc(\sqrt{\alpha}\varphi)dc(\sqrt{\alpha}\varphi), \tag{4.69}$$

$$\dot{x}_2 = \cos h\theta = 2k'^2 sc^2(\sqrt{\alpha}\varphi) + 1. \tag{4.70}$$

Combining with the formulas

$$\int nc\ u\mathrm{d}u = \frac{1}{k'}\ln\Big(\frac{k'sn\ u + dn\ u}{cn\ u}\Big), \tag{4.71}$$

$$\int nc^2u\mathrm{d}u = \frac{1}{k'^2}[k'^2u - E(u) + dn\ u\ sc\ u], \tag{4.72}$$

$$\int nc\ u\ sc^2u\mathrm{d}u = \frac{1}{2k'^2}sc\ u\ dc\ u - \frac{1}{2k'^3}\ln\Big(\frac{k'sn\ u + dn\ u}{cn\ u}\Big), \tag{4.73}$$

$$\int nc^2u\ sc^2u\mathrm{d}u = \frac{1}{3k'^4}[(1+k^2)E(u) - k'^2u + dn\ u\ sc\ u(k'^2nc^2u - 1 - k^2)], \tag{4.74}$$

$$\int sc^2u\mathrm{d}u = \frac{1}{k'^2}(dn\ u\ sc\ u - E(u)), \tag{4.75}$$

we get

$$x_1 = 2\operatorname{sgn} c\frac{k'}{\sqrt{\alpha}}(nc\psi_t - nc\psi_0), \tag{4.76}$$

$$x_2 = \frac{2}{\sqrt{\alpha}}[dn\psi_t sc\psi_t - dn\psi_0 sc\psi_0 - E(\psi_t) + E(\psi_0)] + t, \tag{4.77}$$

$$\begin{aligned} y &= \int x_1\dot{x}_2 \mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\operatorname{sgn} c\frac{k'}{\sqrt{\alpha}}\int(nc\psi_t - nc\psi_0)(2k'^2 sc^2\psi_t + 1)\mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\operatorname{sgn} c\frac{k'}{\alpha}\{sc\psi_t dc\psi_t - sc\psi_0 dc\psi_0 - 2nc\psi_0[dn\psi_t sc\psi_t - dn\psi_0 sc\psi_0 - E(\psi_t) \\ &\quad + E(\psi_0)] - \sqrt{\alpha}nc\psi_0 t\} - \frac{x_1x_2}{2}, \end{aligned} \tag{4.78}$$

$$\begin{aligned} z &= \int \frac{x_1^2\dot{x}_2}{2}\mathrm{d}t + \frac{x_2^3}{6} \\ &= \frac{4k'^2}{\alpha}\int(nc\psi_t - nc\psi_0)^2(2k'^2 sc^2\psi_t + 1)\mathrm{d}t + \frac{x_2^3}{6} \\ &= \frac{4k'^2}{\alpha^{\frac{3}{2}}}\Big[\Big(\frac{2(1+k^2)}{3k'^2} - \frac{1}{k'^2} - 2nc^2\psi_0\Big)(E(\psi_t) - E(\psi_0) - dn\psi_t sc\psi_t + dn\psi_0 sc\psi_0) \\ &\quad + \Big(\frac{1}{3} + nc^2\psi_0\Big)\sqrt{\alpha}t - 2nc\psi_0(sc\psi_t dc\psi_t - sc\psi_0 dc\psi_0)\Big] + \frac{x_2^3}{6}, \end{aligned} \tag{4.79}$$

where

$$\psi_t = \sqrt{\alpha}\varphi, \quad \psi_0 = \sqrt{\alpha}\varphi_0.$$

If $\lambda \in C_3^+$, the energy integral $E \in (\alpha, +\infty)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{E-\alpha}{E+\alpha}} \in (0,1), \tag{4.80}$$

$$\sin h\frac{\theta}{2} = -\operatorname{sgn} c\ sc\Big(\frac{\sqrt{\alpha}}{k'}\varphi\Big), \quad \cos h\frac{\theta}{2} = nc\Big(\frac{\sqrt{\alpha}}{k'}\varphi\Big), \tag{4.81}$$

$$c = \operatorname{sgn} c\ \frac{2}{k'}\sqrt{\alpha}dc\Big(\frac{\sqrt{\alpha}}{k'}\varphi\Big), \quad \varphi = \Big(-\frac{k'K}{\sqrt{\alpha}}, \frac{k'K}{\sqrt{\alpha}}\Big). \tag{4.82}$$

So we obtain

$$\dot{x}_1 = -\sin h\theta = 2\operatorname{sgn} c\ sc\Big(\frac{\sqrt{\alpha}}{k'}\varphi\Big)nc\Big(\frac{\sqrt{\alpha}}{k'}\varphi\Big), \tag{4.83}$$

$$\dot{x}_2 = \cos h\theta = 2sc^2\Big(\frac{\sqrt{\alpha}}{k'}\varphi\Big) + 1. \tag{4.84}$$

Using (4.75) and the following formulas:

$$\int dc\ u\mathrm{d}u = \ln\Big(\frac{1 + sn\ u}{cn\ u}\Big), \tag{4.85}$$

$$\int dc^2 u\mathrm{d}u = u - E(u) + dn\ u\ sc\ u, \tag{4.86}$$

$$\int dc\ u\ sc^2 u\mathrm{d}u = -\frac{1}{2}\ln\Big(\frac{1 + sn\ u}{cn\ u}\Big) + \frac{1}{2}sc\ u\ nc\ u, \tag{4.87}$$

$$\int dc^2u\ sc^2u\mathrm{d}u = \frac{1-2k^2}{3k'^2}E(u) - \frac{1}{3}u + \frac{1}{3}dc\ u\ sc\ u\ nc\ u + \frac{1}{3k'^2}dc\ u\ sn\ u - \frac{2}{3}dc\ u\ sn\ u, \tag{4.88}$$

we get

$$x_1 = 2\mathrm{sgn}\, c\frac{1}{\sqrt{\alpha}k'}(dc\delta_t - dc\delta_0), \tag{4.89}$$

$$x_2 = 2\frac{1}{\sqrt{\alpha}k'}[dn\delta_t sc\delta_t - dn\delta_0 sc\delta_0 - E(\delta_t) + E(\delta_0)] + t, \tag{4.90}$$

$$\begin{aligned} y &= \int x_1\dot{x}_2\mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\mathrm{sgn}\, c\frac{1}{k'\sqrt{\alpha}}\int (dc\delta_t - dc\delta_0)(2sc^2\delta_t + 1)\mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\mathrm{sgn}\, c\frac{1}{\alpha}\Big\{sn\delta_t nc^2\delta_t - sn\delta_0 nc^2\delta_0 - 2\frac{dc\delta_0}{k'^2}\Big[dn\delta_t sc\delta_t - dn\delta_0 sc\delta_0 \\ &\quad - E(\delta_t) + E(\delta_0) - \frac{\sqrt{\alpha}}{k'}dc\delta_0 t\Big]\Big\} - \frac{x_1x_2}{2}, \end{aligned} \tag{4.91}$$

$$\begin{aligned} z &= \int \frac{x_1^2\dot{x}_2}{2}\mathrm{d}t + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha k'^2}\int (dc\delta_t - dc\delta_0)^2(2sc^2\delta_t + 1)\mathrm{d}t + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha^{\frac{3}{2}}k'}\Big[\Big(\frac{-2+k'^2}{3k'^2} + \frac{2dc^2\delta_0}{k'^2}\Big)(E(\delta_t) - E(\delta_0)) + \Big(\frac{1}{3} + dc^2\delta_0\Big)\frac{\sqrt{\alpha}}{k'}t \\ &\quad + \Big(\frac{2-k'^2}{3k'^2} + \frac{2dc^2\delta_0}{k'^2}\Big)(dn\delta_t sc\delta_t - dn\delta_0 sc\delta_0) + \frac{2}{3}dc\delta_t sc\delta_t nc\delta_t \\ &\quad - \frac{2}{3}dc\delta_0 sc\delta_0 nc\delta_0\Big] + \frac{x_2^3}{6}, \end{aligned} \tag{4.92}$$

where

$$\delta_t = \frac{\sqrt{\alpha}}{k'}\varphi, \quad \delta_0 = \frac{\sqrt{\alpha}}{k'}\varphi_0.$$

If $\lambda \in C_4^+$, the energy integral $E = \alpha$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = 0, \tag{4.93}$$

$$\sin h\frac{\theta}{2} = -\mathrm{sgn}\, c\ \tan(\sqrt{\alpha}\varphi), \quad \cos h\frac{\theta}{2} = \sec(\sqrt{\alpha}\varphi), \tag{4.94}$$

$$c = 2\mathrm{sgn}\, c\ \sqrt{\alpha}\sec(\sqrt{\alpha}\varphi), \quad \varphi \in \Big(-\frac{\pi}{2\sqrt{\alpha}}, \frac{\pi}{2\sqrt{\alpha}}\Big). \tag{4.95}$$

So we obtain

$$\dot{x}_1 = -\sin\theta = 2\mathrm{sgn}\, c\ \tan(\sqrt{\alpha}\varphi)\sec(\sqrt{\alpha}\varphi), \tag{4.96}$$

$$\dot{x}_2 = \cos\theta = 2\tan^2(\sqrt{\alpha}\varphi) + 1, \tag{4.97}$$

and then

$$x_1 = 2\mathrm{sgn}\, c\frac{1}{\sqrt{\alpha}}(\sec\psi_t - \sec\psi_0), \tag{4.98}$$

$$x_2 = \frac{2}{\sqrt{\alpha}}(\tan\psi_t - \tan\psi_0) - t, \tag{4.99}$$

$$\begin{aligned} y &= \int x_1\dot{x}_2 \mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\operatorname{sgn} c\frac{1}{\sqrt{\alpha}}\int(\sec\psi_t - \sec\psi_0)(2\tan^2\psi_t + 1)\mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\operatorname{sgn} c\frac{1}{\alpha}[\tan\psi_t\sec\psi_t - \tan\psi_0\sec\psi_0 - 2\sec\psi_0(\tan\psi_t - \psi_0) + \sec\psi_0\sqrt{\alpha}t] \\ &\quad - \frac{x_1x_2}{2}, \end{aligned} \tag{4.100}$$

$$\begin{aligned} z &= \int \frac{x_1^2\dot{x}_2}{2}\mathrm{d}t + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha}\int(\sec\psi_t - \sec\psi_0)^2(2\tan^2\psi_t + 1)\mathrm{d}t + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha^{\frac{3}{2}}}\Big\{\frac{2}{3}(\tan^3\psi_t - \tan^3\psi_0) + (1 + 2\sec^2\psi_0)(\tan\psi_t - \tan\psi_0) - \sec^2\psi_0\sqrt{\alpha}t \\ &\quad - 2\sec\psi_0(\tan\psi_t\sec\psi_t - \tan\psi_0\sec\psi_0)\Big\} + \frac{x_2^3}{6}, \end{aligned} \tag{4.101}$$

where

$$\psi_t = \sqrt{\alpha}\varphi, \quad \psi_0 = \sqrt{\alpha}\varphi_0.$$

If $\lambda \in C_5^+$, the energy integral $E = -\alpha$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = 1, \tag{4.102}$$

$$\sin h\frac{\theta}{2} = \operatorname{sgn}\theta\operatorname{csc} h(\sqrt{\alpha}\varphi), \quad \cos h\frac{\theta}{2} = \cot h(\sqrt{\alpha}\varphi), \tag{4.103}$$

$$c = 2\operatorname{sgn}\theta\sqrt{\alpha}\operatorname{csch}(\sqrt{\alpha}\varphi), \quad \varphi \in (0, +\infty). \tag{4.104}$$

So we obtain

$$\dot{x}_1 = -\sin\theta = -2\operatorname{sgn}\theta\operatorname{csc} h(\sqrt{\alpha}\varphi)\cot h(\sqrt{\alpha}\varphi), \tag{4.105}$$

$$\dot{x}_2 = \cos\theta = 2\operatorname{csc} h^2(\sqrt{\alpha}\varphi) + 1, \tag{4.106}$$

and then

$$x_1 = 2\operatorname{sgn}\theta\frac{1}{\sqrt{\alpha}}(\operatorname{csc} h\psi_t - \operatorname{csc} h\psi_0), \tag{4.107}$$

$$x_2 = -\frac{2}{\sqrt{\alpha}}(\cot h\psi_t - \cot h\psi_0) + t, \tag{4.108}$$

$$\begin{aligned} y &= \int x_1\dot{x}_2\mathrm{d}t - \frac{x_1x_2}{2} \\ &= 2\operatorname{sgn}\theta\frac{1}{\sqrt{\alpha}}\int(\operatorname{csc} h\psi_t - \operatorname{csc} h\psi_0)(2\operatorname{csc} h^2\psi_t + 1)\mathrm{d}t - \frac{x_1x_2}{2} \\ &= \operatorname{sgn}\theta\frac{2}{\alpha}[-\cot h\psi_t\operatorname{csc} h\psi_t + \cot h\psi_0\operatorname{csc} h\psi_0 + 2\operatorname{csc} h\psi_0(\cot h\psi_t - \cot h\psi_0) \\ &\quad + \sqrt{\alpha}\operatorname{csc}\psi_0 t] - \frac{x_1x_2}{2}, \end{aligned} \tag{4.109}$$

$$
\begin{aligned}
z &= \int \frac{x_1^2 \dot{x}_2}{2} \mathrm{d}t + \frac{x_2^3}{6} \\
&= \frac{4}{\alpha} \int (\csc h\psi_t - \csc h\psi_0)^2 (2\csc h^2 \psi_t + 1) \mathrm{d}t + \frac{x_2^3}{6} \\
&= 4\alpha^{-\frac{3}{2}} \Big\{ (1 - 2\csc h^2 \psi_0)(\cot h\psi_t - \cot h\psi_0) - \frac{2}{3}(\cot h^3 \psi_t - \cot h^3 \psi_0) \\
&\quad - \operatorname{sgn} c \sqrt{\alpha} \csc h^2 \psi_0 t + 2\csc h\psi_0 (\cot h\psi_t \csc h\psi_t - \cot h\psi_0 \csc h\psi_0) \Big\} + \frac{x_2^3}{6},
\end{aligned} \tag{4.110}
$$

where

$$
\psi_t = \sqrt{\alpha}\varphi, \quad \psi_0 = \sqrt{\alpha}\varphi_0.
$$

If $\lambda \in C_6^+$, we have

$$
\theta = c \equiv 0.
$$

The expression is the same as the case of $\lambda \in C_2^-$.

Remark 4.1 System (4.11)–(4.17) has the symmetry

$$
(\theta, c, \alpha, x_1, x_2, y, z, t) \mapsto \left(\theta, \frac{c}{\sqrt{|\alpha|}}, \pm 1, \sqrt{|\alpha|} x_1, \sqrt{|\alpha|} x_2, |\alpha| y, |\alpha|^{\frac{3}{2}} z, \sqrt{|\alpha|} t \right), \tag{4.111}
$$

which transforms the variable φ and k as follows:

$$
(\varphi, k, \alpha) \mapsto (\sqrt{|\alpha|}\varphi, k, \pm 1).
$$

So we can also get space-like geodesics of the general case $\alpha \neq 0$ from the formulas for the special case $\alpha = \pm 1$.

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