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Degeneracy and Finiteness Theorems for Meromorphic Mappings in Several Complex Variables*

Si Duc QUANG[1]

Abstract The author proves that there are at most two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(n \geq 2)$ sharing $2n+2$ hyperplanes in general position regardless of multiplicity, where all zeros with multiplicities more than certain values do not need to be counted. He also shows that if three meromorphic mappings f^1, f^2, f^3 of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(n \geq 5)$ share $2n+1$ hyperplanes in general position with truncated multiplicity, then the map $f^1 \times f^2 \times f^3$ is linearly degenerate.

Keywords Second main theorem, Uniqueness problem, Meromorphic mapping, Multiplicity
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1 Introduction

In 1926, Nevanlinna [4] showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ cannot have the same inverse images for five distinct values, and that g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values. These results are usually called the five values and the four values theorems of Nevanlinna. After that, many authors have extended and improved the results of Nevanlinna to the case of meromorphic mappings into complex projective spaces. These theorems are called uniqueness theorems or finiteness theorems. To state some of them, first of all we recall the following.

For a divisor ν on $\mathbb{C}^m$, which is regarded as a function with values in $\mathbb{Z}$, and for a positive integer k or $k = \infty$, we set

$$\nu_{\leq k}(z) = \begin{cases} 0, & \text{if } \nu(z) > k, \\ \nu(z), & \text{if } \nu(z) \leq k. \end{cases}$$

Similarly, we define $\nu_{>k}(z)$. The zero divisor of a meromorphic φ will be denoted by ν_φ.

Let f be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$, and H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ given by $H = \{a_0\omega_0 + \cdots + a_n\omega_n = 0\}$, where $(a_0, \cdots, a_n) \neq (0, \cdots, 0)$. Set $(f, H) = \sum\limits_{i=0}^{n} a_i f_i$. We see that $\nu_{(f,H)}$ does not depend on the choice of the reduced representation of f and the representation of H.

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[1]Department of Mathematics, Hanoi National University of Education, 136-Xuan Thuy, Cau Giay, Hanoi, Vienam; Thang Long Institute of Mathematics and Applied Sciences, Nghiem Xuan Yem, Hoang Mai, Ha Noi, Vietnam. E-mail: quangsd@hnue.edu.vn
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Let $H_1, \cdots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and let $k_1, \cdots, k_q$ be q positive integers or $+\infty$. Assume that f is linearly nondegenerate and satisfies

$$\dim\{z; \nu_{(f,H_i),\le k_i}(z)\cdot \nu_{(f,H_j),\le k_j}(z) > 0\} \le m-2, \quad 1 \le i < j \le q.$$

Let d be an integer. We consider the set $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^q, d)$ of all meromorphic maps $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

(a) $\min\ (\nu_{(f,H_i),\le k_i}, d) = \min\ (\nu_{(g,H_i),\le k_i}, d), \quad 1 \le j \le q,$

(b) $f(z) = g(z)$ on $\bigcup_{i=1}^{q} \{z; \nu_{(f,H_i),\le k_i}(z) > 0\}$.

If $k_1 = \cdots = k_q = +\infty$, we will write $\mathcal{F}(f, \{H_i\}_{i=1}^q, d)$ for $\mathcal{F}(f, \{H_i, \infty\}_{i=1}^q, d)$. We see that conditions (a) and (b) mean the sets of all intersecting points with multiplicity at most k_i (truncated to level d) of f and g with the hyperplane H_i are the same, and two mappings f and g are identify on these sets.

Denote by $\sharp S$ the cardinality of the set S. There have been many results on uniqueness problem for the case of $k_1 = \cdots = k_q = +\infty$. Firstly, in 1983, Smiley [9] proved that $\sharp\mathcal{F}(f, \{H_i\}_{i=1}^q, 1) = 1$ for $q = 3n+2$. In 2006, Thai and Quang [10] showed that the result of Smiley is still valid for $q = 3n+1$ and $n \ge 2$. In 2009, Dethloff and Tan [2] proved that this result still holds for $q = [2.75n]$ with n big enough, and then Chen and Yan in [1] reduced the number q to $2n+3$. After that, in 2011 Quang [6] improved these results to the case of $q = 2n+3$ and $k_1 = \cdots = k_q > \frac{4n^3+11n^2+n-2}{3n+2}$. As far as we known, there is still no uniqueness theorem for meromorphic mappings sharing less than $2n+3$ hyperplanes regardless of multiplicities.

For the case $q = 2n+2$, in 2011 Yan and Chen [11] proved a degeneracy theorem as follows.

Theorem A *If $q = 2n+2$, then the map $f^1 \times f^2 \times f^3$ of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is linearly degenerate for every three maps $f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$.*

The first finiteness theorem for the case of meromorphic mappings sharing $2n+2$ hyperplanes is given by Quang [7] in 2012 and its correction [8] as follows.

Theorem B *If $n \ge 3$ and $q = 2n+2$, then $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^q, 1) \le 2$.*

We would also like to emphasize here that in the above two results, all intersecting points of the mappings and the hyperplanes are considered, i.e., $k_i = +\infty$ for all i. The techniques used in the proofs of Theorems A and B are based on the estimation of the counting function of the Cartan's auxiliary function. But they do not work for the case where $k_i < +\infty$. Our first purpose in this paper is to improve the above result by considering that case (including the case of $n = 2$). Namely, we will prove the following.

Theorem 1.1 *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(n \ge 2)$. Let $H_1, \cdots, H_{2n+2}$ be $2n+2$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and let $k_1, \cdots, k_{n+2}$ be positive integers or $+\infty$. Assume that*

$$\dim\{z; \nu_{(f,H_i),\le k_i}(z)\cdot \nu_{(f,H_j),\le k_j}(z) > 0\} \le m-2, \quad 1 \le i < j \le 2n+2$$

and

$$\sum_{i=1}^{2n+2} \frac{1}{k_i+1} < \min\Big\{\frac{n+1}{3n^2+n}, \frac{5n-9}{24n+12}, \frac{n^2-1}{10n^2+8n}\Big\}.$$

Then $\sharp\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1) \le 2$.

In order to prove this theorem, we first prove that $f^1 \wedge f^2 \wedge f^3 = 0$ for every three maps f^1, f^2, f^3 in $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$ if $\sum\limits_{i=1}^{2n+2} \frac{1}{k_i+1} < \frac{n+1}{3n^2+n}$ (see Lemma 3.2). And then, we improve the estimate of the counting function of the Cartan's auxiliary function (see Lemma 3.6).

The last purpose of this paper is to prove a degeneracy theorem for three mappings sharing $2n+1$ hyperplanes. Namely, we will proved the following.

Theorem 1.2 *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(n \geq 5)$. Let $H_1, \cdots, H_{2n+1}$ be $2n+1$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and let $k_1, \cdots, k_{2n+1}$ be positive integers or $+\infty$ such that*

$$\dim\{z; \nu_{(f,H_i),\leq k_i}(z) \cdot \nu_{(f,H_j),\leq k_j}(z) > 0\} \leq m-2, \quad 1 \leq i < j \leq 2n+2.$$

If there exists a positive integer p with $p \leq n$ and

$$\sum_{i=1}^{2n+1} \frac{1}{k_i+1} < \frac{np-3n-p}{4n^2+3np-n},$$

then the map $f^1 \times f^2 \times f^3$ of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is linearly degenerate for every three maps $f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+1}, p)$.

2 Basic Notions in Nevanlinna Theory

In this paper, we will use the standard notation from Nevanlinna theory due to [6–8]. As usual, we denote by $N_\varphi^{[M]}(r), N_{\varphi,\leq k}^{[M]}(r)$ and $N_{\varphi,>k}^{[M]}(r)$ the counting functions of the divisors $\nu_\varphi, \nu_{\varphi,\leq k}$ and $\nu_{\varphi,>k}$ respectively, where φ is a meromorphic function on $\mathbb{C}^m$. For brevity we will omit the superscript $^{[M]}$ if $M = \infty$.

For a set $S \subset \mathbb{C}^m$, we define the characteristic function of S by

$$\chi_S(z) = \begin{cases} 1, & \text{if } z \in S, \\ 0, & \text{if } z \notin S. \end{cases}$$

If the closure $\overline{S}$ of S is an analytic subset of $\mathbb{C}^m$, then we denote by $N(r, S)$ the counting function of the reduced divisor whose support is the union of all irreducible components of $\overline{S}$ with codimension one.

For a meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ with $f(\mathbb{C}^m) \not\subset H$, we denote by $T_f(r)$ the characteristic function of f and $m_{f,H}(r)$ the proximity function of f with respect to H (if $f(\mathbb{C}^m) \not\subset H$). The proximity function of a nonzero meromorphic function φ is defined by

$$m(r, \varphi) := \int_{S(r)} \log\max\,(|\varphi|, 1)\sigma_n.$$

As usual, by the notation "$||P$" we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E \mathrm{d}r < \infty$.

The following results play essential roles in Nevanlinna theory (see [5]).

Theorem 2.1 (The First Main Theorem) *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T_f(r), \quad r > 1.$$

Theorem 2.2 (The Second Main Theorem) *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then*

$$||(q-n-1)T_f(r) \le \sum_{i=1}^{q} N^{[n]}_{(f,H_i)}(r) + o(T_f(r)).$$

For meromorphic functions F, G, H on $\mathbb{C}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}^m_+$, we define the Cartan's auxiliary function as follows:

$$\Phi^\alpha(F,G,H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \dfrac{1}{F} & \dfrac{1}{G} & \dfrac{1}{H} \\ \mathcal{D}^\alpha\Big(\dfrac{1}{F}\Big) & \mathcal{D}^\alpha\Big(\dfrac{1}{G}\Big) & \mathcal{D}^\alpha\Big(\dfrac{1}{H}\Big) \end{vmatrix}.$$

Lemma 2.1 (see [3, Proposition 3.4]) *If $\Phi^\alpha(F,G,H) = 0$ and $\Phi^\alpha\big(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}\big) = 0$ for all α with $|\alpha| \le 1$, then one of the following assertions holds*:
(i) $F = G,\ G = H$ *or* $H = F$,
(ii) $\frac{F}{G}, \frac{G}{H}$ *and* $\frac{H}{F}$ *are all constants.*

Lemma 2.2 *Let f^1, f^2, f^3 be three maps in $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^q, 1)$. Assume that f^i has a representation $f^i = (f^i_0 : \cdots : f^i_n)$, $1 \le i \le 3$. Suppose that there exist $s, t, l \in \{1, \cdots, q\}$ such that*

$$P := \det \begin{pmatrix} (f^1, H_s) & (f^1, H_t) & (f^1, H_l) \\ (f^2, H_s) & (f^2, H_t) & (f^2, H_l) \\ (f^3, H_s) & (f^3, H_t) & (f^3, H_l) \end{pmatrix} \not\equiv 0.$$

Then we have

$$T(r) \ge \sum_{i=s,t,l} (N(r, \min_{1 \le u \le 3}\{\nu_{(f^u,H_i),\le k_i}\}) - N^{[1]}_{(f,H_i),\le k_i}(r)) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) + o(T(r)),$$

where $T(r) = \sum\limits_{u=1}^{3} T_{f^u}(r)$.

Proof Denote by S the closure of the set

$$\bigcup_{1 \le u \le 3} I(f^u) \cup \bigcup_{1 \le i < j \le 2n+2} \{z; \nu_{(f,H_i),\le k_i}(z) \cdot \nu_{(f,H_j),\le k_j}(z) > 0\}.$$

Then S is an analytic subset of codimension at least two of $\mathbb{C}^m$.

For $z \notin S$, we consider the following two cases.

Case 1 z is a zero of (f, H_i) with multiplicity at most k_i, where $i \in \{s, t, l\}$. For instance, we suppose that $i = s$. We set

$$m = \min\{\nu_{(f^1,H_s),\le k_s}(z), \nu_{(f^2,H_s),\le k_s}(z), \nu_{(f^3,H_s),\le k_s}(z)\}.$$

Then there exists a neighborhood U of z and a holomorphic function h defined on U such that $\mathrm{Zero}(h) = U \cap \mathrm{Zero}(f, H_s)$ and dh has no zero on $\mathrm{Zero}(h)$. Then the functions $\varphi_u = \frac{(f^u, H_s)}{h^m}$ $(1 \le u \le 3)$ are holomorphic in a neighborhood of z. On the other hand, since $f^1 = f^2 = f^3$ on $\mathrm{Supp}\,\nu_{(f,H_s),\le k_s}$, we have

$$P_{uv} := (f^u, H_t)(f^v, H_l) - (f^u, H_l)(f^v, H_t) = 0 \quad \text{on } \mathrm{Supp}\,\nu_{(f,H_s),\le k_s}, \quad 1 \le u < v \le 3.$$

Therefore, there exist holomorphic functions ψ_{uv} on a neighborhood of z such that $P_{uv} = h\psi_{uv}$. Then we have

$$P = h^{m+1}(\varphi_1\psi_{23} - \varphi_2\psi_{13} + \varphi_3\psi_{12})$$

on a neighborhood of z. This yields that

$$\nu_P(z) \geq m + 1 = \sum_{i=s,t,l} \big(\min_{1\leq u\leq 3}\{\nu_{(f^u,H_i),\leq k_i}(z)\} - \nu^{[1]}_{(f,H_i),\leq k_i}(z)\big) + 2\sum_{i=1}^{q} \nu^{[1]}_{(f,H_i),\leq k_i}(z).$$

Case 2 z is a zero point of (f, H_i) with multiplicity at most k_i, where $i \notin \{s, t, l\}$. There exists an index v such that $(f^1, H_v)(z) \neq 0$. Since $f^1(z) = f^2(z) = f^3(z)$, we have that $(f^u, H_v)(z) \neq 0$ $(1 \leq u \leq 3)$ and

$$P = \prod_{u=1}^{3}(f^u, H_v) \cdot \det \begin{pmatrix} \frac{(f^1,H_s)}{(f^1,H_v)} & \frac{(f^1,H_t)}{(f^1,H_v)} & \frac{(f^1,H_l)}{(f^1,H_v)} \\ \frac{(f^2,H_s)}{(f^2,H_v)} & \frac{(f^2,H_t)}{(f^2,H_v)} & \frac{(f^2,H_l)}{(f^2,H_v)} \\ \frac{(f^3,H_s)}{(f^3,H_v)} & \frac{(f^3,H_t)}{(f^3,H_v)} & \frac{(f^3,H_l)}{(f_3,H_v)} \end{pmatrix}$$

$$= \prod_{u=1}^{3}(f^u, H_v) \cdot \det \begin{pmatrix} \frac{(f^1,H_s)}{(f^1,H_v)} & \frac{(f^1,H_t)}{(f^1,H_v)} & \frac{(f^1,H_l)}{(f^1,H_v)} \\ \frac{(f^2,H_s)}{(f^2,H_v)} - \frac{(f^1,H_s)}{(f^1,H_v)} & \frac{(f^2,H_t)}{(f^2,H_v)} - \frac{(f^1,H_t)}{(f^1,H_v)} & \frac{(f^2,H_l)}{(f^2,H_v)} - \frac{(f^1,H_l)}{(f^1,H_v)} \\ \frac{(f^3,H_s)}{(f^3,H_v)} - \frac{(f^1,H_s)}{(f^1,H_v)} & \frac{(f^3,H_t)}{(f^3,H_v)} - \frac{(f^1,H_t)}{(f^1,H_v)} & \frac{(f^3,H_l)}{(f^3,H_v)} - \frac{(f^1,H_l)}{(f^1,H_v)} \end{pmatrix}$$

vanishes at z with multiplicity at least two. Therefore, we have

$$\nu_P(z) \geq 2 = \sum_{i=s,t,l} \Big(\min_{1\leq u\leq 3}\{\nu_{(f^u,H_i),\leq k_i}(z)\} - \nu^{[1]}_{(f,H_i),\leq k_i}(z)\Big) + 2\sum_{i=1}^{q} \nu^{[1]}_{(f,H_i),\leq k_i}(z).$$

Thus, from the above two cases we have

$$\nu_P(z) \geq \sum_{i=s,t,l} \Big(\min_{1\leq u\leq 3}\{\nu_{(f^u,H_i),\leq k_i}(z)\} - \nu^{[1]}_{(f,H_i),\leq k_i}(z)\Big) + 2\sum_{i=1}^{q} \nu^{[1]}_{(f,H_i),\leq k_i}(z)$$

for all z outside the analytic set S. Integrating both sides of the above inequality, we get

$$N_P(r) \geq \sum_{i=s,t,l} \Big(N\Big(r, \min_{1\leq u\leq 3}\{\nu_{(f^u,H_i),\leq k_i}\}\Big) - N^{[1]}_{(f,H_i),\leq k_i}(r)\Big) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\leq k_i}(r) + o(T(r)).$$

By Jensen's formula and the definition of the characteristic function we have

$$\begin{aligned} N_P(r) &= \int_{S(r)} \log|P|\sigma_m + O(1) \\ &\leq \sum_{u=1}^{3} \int_{S(r)} \log||f^u||\sigma_m + O(1) = T(r) + o(T(r)). \end{aligned}$$

Thus, we have

$$T(r) \geq \sum_{i=s,t,l} \Big(N\Big(r, \min_{1\leq u\leq 3}\{\nu_{(f^u,H_i),\leq k_i}\}\Big) - N^{[1]}_{(f,H_i),\leq k_i}(r)\Big) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\leq k_i}(r) + o(T(r)).$$

The lemma is proved.

3 Proof of Main Theorems

Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $H_1, \cdots, H_{2n+2}$ be $2n+2$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and let $k_i \geq n$ $(1 \leq i \leq 2n+2)$ be positive integers or $+\infty$ with

$$\dim\{z; \nu_{(f,H_i),\leq k_i}(z) \cdot \nu_{(f,H_j),\leq k_j}(z) > 0\} \leq m-2, \quad 1 \leq i < j \leq 2n+2.$$

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 3.1 *If* $\sum_{i=1}^{2n+2} \frac{1}{k_i+1} < \frac{1}{n}$, *then every mapping* $g \in \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$ *is linearly nondegenerate and*

$$||T_g(r) = O(T_f(r)) \quad \text{and} \quad ||T_f(r) = O(T_g(r)).$$

Proof Suppose that there exists a hyperplane H satisfying $g(\mathbb{C}^m) \subset H$. We assume that f and g have reduce representations $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$, respectively. Assume that $H = \Big\{(\omega_0 : \cdots : \omega_n)\Big| \sum_{i=0}^{n} a_i\omega_i = 0\Big\}$. Since f is linearly nondegenerate, $(f, H) \not\equiv 0$. On the other hand, $(f,H)(z) = (g,H)(z) = 0$ for all $z \in \bigcup_{i=1}^{2n+2}\{\nu_{(f,H_i),\leq k_i}\}$, hence

$$N_{(f,H)}(r) \geq \sum_{i=1}^{2n+2} N^{[1]}_{(f,H_i),\leq k_i}(r).$$

It yields that

$$\begin{aligned}||T_f(r) \geq N_{(f,H)}(r) &\geq \sum_{i=1}^{2n+2} N^{[1]}_{(f,H_i),\leq k_i}(r) = \sum_{i=1}^{2n+2}(N^{[1]}_{(f,H_i)}(r) - N^{[1]}_{(f,H_i),>k_i}(r))\\ &\geq \sum_{i=1}^{2n+2}\frac{1}{n}N^{[n]}_{(f,H_i)}(r) - \sum_{i=1}^{2n+2}\frac{1}{k_i+1}T_f(r) \geq \Big(\frac{n+1}{n} - \sum_{i=1}^{2n+2}\frac{1}{k_i+1}\Big)T_f(r) + o(T_f(r)).\end{aligned}$$

Letting $r \to +\infty$, we get

$$\sum_{i=1}^{2n+2}\frac{1}{k_i+1} \geq \frac{1}{n}.$$

This is a contradiction. Hence $g(\mathbb{C}^m)$ cannot be contained in any hyperplanes of $\mathbb{P}^n(\mathbb{C})$. Therefore g is linearly nondegenerate.

Also by the Second Main Theorem (see Theorem 2.2), we have

$$\begin{aligned}||(n+1)T_g(r) &\leq \sum_{i=1}^{2n+2} N^{[n]}_{(g,H_i)}(r) + o(T_g(r)) \leq \sum_{i=1}^{2n+2} n\ N^{[1]}_{(g,H_i)}(r) + o(T_g(r))\\ &= \sum_{i=1}^{2n+2} n(N^{[1]}_{(g,H_i),\leq k_i}(r) + N^{[1]}_{(g,H_i),>k_i}(r)) + o(T_g(r))\\ &\leq \sum_{i=1}^{2n+2} n\Big(N^{[1]}_{(f,H_i),\leq k_i}(r) + \frac{1}{k_i+1}T_g(r)\Big) + o(T_g(r))\\ &\leq \sum_{i=1}^{2n+2} n\Big(T_f(r) + \frac{1}{k_i+1}T_g(r)\Big) + o(T_f(r)+T_g(r)).\end{aligned}$$

Thus

$$\Big(n+1-\sum_{i=1}^{2n+2}\frac{n}{k_i+1}\Big)T_g(r)\le n(2n+2)T_f(r)+o(T_f(r)+T_g(r)).$$

We note that

$$n+1-\sum_{i=1}^{2n+2}\frac{n}{k_i+1}>n>0.$$

Hence $||T_g(r)=O(T_f(r))$. Similarly, we get $||T_f(r)=O(T_g(r))$.

Lemma 3.2 *Assume that* $n\ge 2$ *and*

$$\sum_{i=1}^{2n+2}\frac{1}{k_i+1}<\frac{n+1}{n(3n+1)}.$$

Then for three maps $f^1,f^2,f^3\in\mathcal{F}(f,\{H_i,k_i\}_{i=1}^{2n+2},1)$ *we have* $f^1\wedge f^2\wedge f^3=0$.

Proof By Lemma 3.1, we have that f^s is linearly nondegenerate and $||T_{f^s}(r)=O(T_f(r))$ and $||T_f(r)=O(T_{f^s}(r))$ for all $s=1,2,3$.

Suppose that $f^1\wedge f^2\wedge f^3\not\equiv 0$. For each $1\le i\le 2n+2$, we set

$$N_i(r)=\sum_{u=1}^{3}N^{[n]}_{(f^u,H_i),\le k_i}(r)-(2n+1)N^{[1]}_{(f,H_i),\le k_i}(r).$$

Here, we note that for positive integers a,b,c we have $(\min\{a,b,c\}-1)\ge\min\{a,n\}+\min\{a,n\}+\min\{a,n\}-2n-1$. Then

$$\min_{1\le u\le 3}\{\nu_{(f^u,H_i),\le k_i}(z)\}-\nu^{[1]}_{(f,H_i),\le k_i}(z)\ge\sum_{u=1}^{3}\nu^{[n]}_{(f^u,H_i),\le k_i}(z)-(2n+1)\nu^{[1]}_{(f,H_i),\le k_i}(z)$$

for all $z\in\operatorname{Supp}\nu_{(f,H_i),\le k_i}$. This yields that

$$\begin{aligned}&N\Big(r,\min_{1\le u\le 3}\{\nu_{(f^u,H_i),\le k_i}(z)\}\Big)-N^{[1]}_{(f,H_i),\le k_i}(r)\\ \ge&\sum_{u=1}^{3}N^{[n]}_{(f^u,H_i),\le k_i}(r)-(2n+1)N^{[1]}_{(f,H_i),\le k_i}(r)=N_i(r).\end{aligned}$$

We denote by $\mathcal{I}$ the set of all permutations of the $(2n+2)$-tuple $(1,\cdots,2n+2)$, i.e.,

$$\mathcal{I}=\{I=(i_1,\cdots,i_{2n+2})\ :\ \{i_1,\cdots,i_{2n+2}\}=\{1,\cdots,2n+2\}\}.$$

For each $I=(i_1,\cdots,i_{2n+2})\in\mathcal{I}$ we define a subset E_I of $[1,+\infty)$ as

$$E_I=\{r\ge 1\ :\ N_{i_1}(r)\ge\cdots\ge N_{i_{2n+2}}(r)\}.$$

It is clear that $\bigcup_{I\in\mathcal{I}}E_I=[1,+\infty)$. Therefore, there exists an element of $\mathcal{I}$, for instance it is $I_0=(1,2,\cdots,2n+2)$, satisfying

$$\int_{E_{I_0}}\mathrm{d}r=+\infty.$$

Then, we have $N_1(r)\ge N_2(r)\ge\cdots\ge N_{2n+2}(r)$ for all $r\in E_{I_0}$.

We consider $\mathcal{M}^3$ as a vector space over the field $\mathcal{M}$. For each $i = 1, \cdots, 2n+2$, we set

$$V_i = ((f^1, H_i), (f^2, H_i), (f^3, H_i)) \in \mathcal{M}^3.$$

We put

$$s = \min\{i \ : \ V_1 \wedge V_i \not\equiv 0\}.$$

Since $f^1 \wedge f^2 \wedge f^3 \not\equiv 0$, we have $1 < s < n+1$. Also again by $f^1 \wedge f^2 \wedge f^3 \not\equiv 0$, there exists an index $t \in \{s+1, \cdots, n+1\}$ such that $V_1 \wedge V_s \wedge V_t \not\equiv 0$. This means that

$$P := \det(V_1, V_s, V_t) \not\equiv 0.$$

Set $T(r) = \sum_{u=1}^{3} T_{f^u}(r)$. By Lemma 2.2, for $r \in E_{I_0}$ we have

$$\begin{aligned}
T(r) &\ge \sum_{i=1,s,t} \Big(N\Big(r, \min_{1\le u\le 3}\{\nu_{(f^u,H_i),\le k_i}\}\Big) - N^{[1]}_{(f,H_i),\le k_i}(r)\Big) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) + o(T(r))\\
&\ge N_1(r) + N_s(r) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) + o(T(r))\\
&\ge \frac{1}{n+1}\sum_{i=1}^{2n+2} N_i(r) + 2\sum_{i=1}^{2n+2} N^{[1]}_{(f,H_i),\le k_i}(r) + o(T(r))\\
&= \frac{1}{n+1}\sum_{i=1}^{2n+2}\Big(\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(z) - (2n+1)N^{[1]}_{(f,H_i)}(z)\Big) + 2\sum_{i=1}^{2n+2} N^{[1]}_{(f,H_i),\le k_i}(r)\\
&= \frac{1}{n+1}\sum_{i=1}^{2n+2}\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(z) + \frac{1}{3(n+1)}\sum_{i=1}^{2n+2}\sum_{u=1}^{3} N^{[1]}_{(f^u,H_i),\le k_i}(r)\\
&\ge \Big(1+\frac{1}{3n}\Big)\frac{1}{n+1}\sum_{i=1}^{2n+2}\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(r)\\
&\ge \Big(1+\frac{1}{3n}\Big)\frac{1}{n+1}\sum_{i=1}^{2n+2}\sum_{u=1}^{3}(N^{[n]}_{(f^u,H_i)}(r) - N^{[n]}_{(f^u,H_i),>k_i}(r))\\
&\ge \Big(1+\frac{1}{3n}\Big)\frac{1}{n+1}\sum_{u=1}^{3}\Big(n+1-\sum_{i=1}^{2n+2}\frac{n}{k_i+1}\Big)T_{f^u}(r) + o(T(r))\\
&= \Big(1+\frac{1}{3n} - \frac{3n+1}{3(n+1)}\sum_{i=1}^{2n+2}\frac{1}{k_i+1}\Big)T(r) + o(T(r)).
\end{aligned}$$

Letting $r \to +\infty$ ($r \in E_{I_0}$), we get

$$1 \ge 1 + \frac{1}{3n} - \frac{3n+1}{3(n+1)}\sum_{i=1}^{2n+2}\frac{1}{k_i+1}, \quad \text{i.e.,} \quad \sum_{i=1}^{2n+2}\frac{1}{k_i+1} \ge \frac{n+1}{n(3n+1)}.$$

This is a contradiction.

Hence, $f^1 \wedge f^2 \wedge f^3 \equiv 0$. The lemma is proved.

Now for three mappings $f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$, we define

- $F_k^{ij} = \frac{(f^k,H_i)}{(f^k,H_j)}$ $(0 \le k \le 2,\ 1 \le i, j \le 2n+2)$,

- $V_i = ((f^1, H_i), (f^2, H_i), (f^3, H_i)) \in \mathcal{M}_m^3$,
- $T_i = \{z; \nu_{(f,H_i),\le k_i}(z) > 0\}$, $S_i = \bigcup_{u=1}^{3} \{z; \nu_{(f_u,H_i),>k_i}(z) > 0\}$,
- $R_i = \bigcap_{u=1}^{3} \{z; \nu_{(f_u,H_i),>k_i}(z) > 0\}$,
- $\nu_i = \{z; k_i \ge \nu_{(f^u,H_i)}(z) \ge \nu_{(f^v,H_i)}(z) = \nu_{(f^t,H_i)}(z)$ for a permutation (u, v, t) of $(1, 2, 3)$.

We write $V_i \cong V_j$ if $V_i \wedge V_j \equiv 0$, otherwise we write $V_i \not\cong V_j$. For $V_i \not\cong V_j$, we wirte $V_i \sim V_j$ if there exist $1 \le u < v \le 3$ such that $F_u^{ij} = F_v^{ij}$, otherwise we write $V_i \not\sim V_j$.

Lemma 3.3 *With the assumption of Theorem* 1.1, *let h and g be two elements of the family $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$. If there exists a constant λ and two indices i, j such that*

$$\frac{(h, H_i)}{(h, H_j)} = \lambda \frac{(g, H_i)}{(g, H_j)},$$

then $\lambda = 1$.

Proof By Lemma 3.1, we see that h and g are linearly nondegenerate and have the characteristic functions of the same order with the characteristic function of f. Set $H = \frac{(h,H_i)}{(h,H_j)}$ and $G = \frac{(g,H_i)}{(g,H_j)}$ and

$$S'_t = \{z; \nu_{(h,H_t),>k_t}(z) > 0\} \cup \{z; \nu_{(g,H_t),>k_t}(z) > 0\}, \quad 1 \le t \le 2n + 2.$$

Then $H = \lambda G$. Supposing that $\lambda \ne 1$, since $H = G$ on the set $\bigcup_{t\ne i,j} T_t \setminus (S'_i \cup S'_j)$, we have $\bigcup_{t\ne i,j} T_t \subset S'_i \cup S'_j$. Thus

$$\begin{aligned}
0 &\ge \sum_{t\ne i,j} N^{[1]}_{(f,H_t),\le k_t}(r) - (N(r, S'_i) + N(r, S'_j)) \\
&\ge \frac{1}{2} \sum_{t\ne i,j} (N^{[1]}_{(h,H_t),\le k_t}(r) + N^{[1]}_{(g,H_t),\le k_t}(r)) - (N(r, S'_i) + N(r, S'_j)) \\
&\ge \frac{1}{2n} \sum_{t\ne i,j} (N^{[n]}_{(h,H_t)}(r) + N^{[n]}_{(g,H_t)}(r)) - \sum_{t=1}^{2n+2} (N^{[1]}_{(h,H_t),>k_t}(r) + N^{[1]}_{(g,H_t),>k_t}(r)) \\
&\ge \frac{n-1}{2n} (T_h(r) + T_g(r)) - \sum_{t=1}^{2n+2} \frac{1}{k_t + 1} (T_h(r) + T_g(r)) + o(T_f(r)).
\end{aligned}$$

Letting $r \to +\infty$, we get

$$\frac{n-1}{2n} \le \sum_{t=1}^{2n+2} \frac{1}{k_t + 1}.$$

This is a contradiction. Therefore $\lambda = 1$. The lemma is proved.

Lemma 3.4 *Let f^1, f^2, f^3 be three elements of $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$, where k_i $(1 \le i \le 2n + 2)$ are positive integers or $+\infty$. Suppose that $f^1 \wedge f^2 \wedge f^3 \equiv 0$ and $V_i \sim V_j$ for some distinct indices i and j. Then f^1, f^2, f^2 are not distinct.*

Proof Suppose that f^1, f^2, f^2 are distinct. Since $V_i \sim V_j$, we may suppose that $F_1^{ij} = F_2^{ij} \ne F_3^{ij}$. Since $f^1 \wedge f^2 \wedge f^3 \equiv 0$ and $f^1 \ne f^2$, there exists a meromorphic function α such that

$$F_3^{tj} = \alpha F_1^{tj} + (1 - \alpha) F_2^{tj}, \quad 1 \le t \le 2n + 2.$$

This implies that $F_3^{ij} = F_1^{ij} = F_2^{ij}$. This is a contradiction. Hence f^1, f^2, f^3 are not distinct. The lemma is proved.

Lemma 3.5 *With the assumption of Theorem 1.1, let f^1, f^2, f^3 be three maps in $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$. Suppose that f^1, f^2, f^3 are distinct and there are two indices $i, j \in \{1, 2, \cdots, 2n+2\}$ $(i \neq j)$ such that $V_i \not\cong V_j$ and*

$$\Phi_{ij}^{\alpha} := \Phi^{\alpha}(F_1^{ij}, F_2^{ij}, F_3^{ij}) \equiv 0$$

for every $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}_+^m$ with $|\alpha| = 1$. Then for every $t \in \{1, \cdots, 2n+2\} \setminus \{i\}$, the following assertions hold:

(i) *$\Phi_{it}^{\alpha} \equiv 0$ for all $|\alpha| \leq 1$,*

(ii) *if $V_i \not\cong V_t$, then $F_1^{ti}, F_2^{ti}, F_3^{ti}$ are distinct and*

$$\begin{aligned} N^{[1]}_{(f,H_i),\leq k_i}(r) &\geq \sum_{s\neq i,t} N^{[1]}_{(f,H_s),\leq k_s}(r) - N^{[1]}_{(f,H_t),\leq k_t}(r) - 2(N(r,S_i) + N(r,S_t)) \\ &\geq \sum_{s\neq i,t} N^{[1]}_{(f,H_s),\leq k_s}(r) - N^{[1]}_{(f,H_t),\leq k_t}(r) - 2\sum_{u=1}^{3}\sum_{s=i,t} N_{(f^u,H_s),>k_s}(r). \end{aligned}$$

Proof By $V_i \not\cong V_j$, we may assume that $F_2^{ji} - F_1^{ji} \neq 0$.

(a) For all $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| = 1$, we have $\Phi_{ij}^{\alpha} = 0$, and hence

$$\begin{aligned} \mathcal{D}^{\alpha}\Big(\frac{F_3^{ji} - F_1^{ji}}{F_2^{ji} - F_1^{ji}}\Big) &= \frac{1}{(F_2^{ji} - F_1^{ji})^2} \cdot ((F_2^{ji} - F_1^{ji}) \cdot \mathcal{D}^{\alpha}(F_3^{ji} - F_1^{ji}) - (F_3^{ji} - F_1^{ji}) \cdot \mathcal{D}^{\alpha}(F_2^{ji} - F_1^{ji})) \\ &= \frac{1}{(F_2^{ji} - F_1^{ji})^2} \cdot \begin{vmatrix} 1 & 1 & 1 \\ F_1^{ji} & F_2^{ji} & F_3^{ji} \\ \mathcal{D}^{\alpha}(F_1^{ji}) & \mathcal{D}^{\alpha}(F_2^{ji}) & \mathcal{D}^{\alpha}(F_3^{ji}) \end{vmatrix} = 0. \end{aligned}$$

Since the above equality holds for all $|\alpha| = 1$, then there exists a constant $c \in \mathbb{C}$ such that

$$\frac{F_3^{ji} - F_1^{ji}}{F_2^{ji} - F_1^{ji}} = c.$$

By Lemma 3.2, we have $f^1 \wedge f^2 \wedge f^3 = 0$. Then for each index $t \in \{1, \cdots, 2n+2\} \setminus \{i, j\}$, we have

$$\begin{aligned} 0 &= \det\begin{pmatrix} (f_1, H_i) & (f_1, H_j) & (f_1, H_t) \\ (f_2, H_i) & (f_2, H_j) & (f_2, H_t) \\ (f_3, H_i) & (f_3, H_j) & (f_3, H_t) \end{pmatrix} \\ &= \prod_{u=1}^{3}(f^u, H_i) \cdot \det\begin{pmatrix} 1 & F_1^{ji} & F_1^{ti} \\ 1 & F_2^{ji} & F_2^{ti} \\ 1 & F_3^{ji} & F_3^{ti} \end{pmatrix} \\ &= \prod_{u=1}^{3}(f^u, H_i) \cdot \det\begin{pmatrix} F_2^{ji} - F_1^{ji} & F_2^{ti} - F_1^{ti} \\ F_3^{ji} - F_1^{ji} & F_3^{ti} - F_1^{ti} \end{pmatrix}. \end{aligned}$$

Thus

$$(F_2^{ji} - F_1^{ji}) \cdot (F_3^{ti} - F_1^{ti}) = (F_3^{ji} - F_1^{ji}) \cdot (F_2^{ti} - F_1^{ti}).$$

If $F_2^{ti} - F_1^{ti} = 0$, then $F_3^{ti} - F_1^{ti} = 0$, and hence $\Phi_{it}^{\alpha} = 0$ for all $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| < 1$. Otherwise, we have

$$\frac{F_3^{ti} - F_1^{ti}}{F_2^{ti} - F_1^{ti}} = \frac{F_3^{ji} - F_1^{ji}}{F_2^{ji} - F_1^{ji}} = c.$$

This also implies that

$$\begin{aligned}
\Phi_{it}^{\alpha} &= F_1^{it} \cdot F_2^{it} \cdot F_3^{it} \cdot \begin{vmatrix} 1 & 1 & 1 \\ F_1^{ti} & F_2^{ti} & F_3^{ti} \\ \mathcal{D}^{\alpha}(F_1^{ti}) & \mathcal{D}^{\alpha}(F_2^{ti}) & \mathcal{D}^{\alpha}(F_3^{ti}) \end{vmatrix} \\
&= F_1^{it} \cdot F_2^{it} \cdot F_3^{it} \cdot \begin{vmatrix} F_2^{ti} - F_1^{ti} & F_3^{ti} - F_1^{ti} \\ \mathcal{D}^{\alpha}(F_2^{ti} - F_1^{ti}) & \mathcal{D}^{\alpha}(F_3^{ti} - F_1^{ti}) \end{vmatrix} \\
&= F_1^{it} \cdot F_2^{it} \cdot F_3^{it} \cdot \begin{vmatrix} F_2^{ti} - F_1^{ti} & c(F_2^{ti} - F_1^{ti}) \\ \mathcal{D}^{\alpha}(F_2^{ti} - F_1^{ti}) & c\mathcal{D}^{\alpha}(F_2^{ti} - F_1^{ti}) \end{vmatrix} = 0.
\end{aligned}$$

Then one always has $\Phi_{it}^{\alpha} = 0$ for all $t \in \{1, \cdots, 2n+2\} \setminus \{i\}$. The first assertion is proved.

(b) We suppose that $V_i \not\cong V_t$. From the above part, we have

$$cF_2^{si} + (1-c)F_1^{si} = F_3^{si}, \quad s \neq i.$$

By the supposition that f^1, f^2, f^3 are distinct, we have $c \notin \{0, 1\}$. This implies that $F_1^{ti}, F_2^{ti}, F_3^{ti}$ are distinct.

We see that the second inequality is clear, then we prove the remain first inequality. We consider the meromorphic mapping F^t of $\mathbb{C}^m$ into $\mathbb{P}^1(\mathbb{C})$ with a reduced representation

$$F^t = (F_1^{ti} h_t : F_2^{ti} h_t),$$

where h_t is a meromorphic function on $\mathbb{C}^m$. We see that

$$\begin{aligned}
T_{F^t}(r) &= T\Big(r, \frac{F_1^{ti}}{F_2^{ti}}\Big) \leq T(r, F_1^{ti}) + T\Big(r, \frac{1}{F_2^{ti}}\Big) + O(1) \\
&\leq T(r, F_1^{ti}) + T(r, F_2^{ti}) + O(1) \leq T_{f^1}(r) + T_{f^2}(r) + O(1) = O(T_f(r)).
\end{aligned}$$

For a point $z \notin I(F^t) \cup S_i \cup S_t$ which is a zero of some functions $F_u^{ti} h_t$ $(1 \leq u \leq 3)$, z must be either zero of (f, H_i) with multiplicity at most k_i or zero of (f, H_t) with multiplicity at most k_t, and hence

$$\sum_{u=1}^{3} \nu^{[1]}_{F_u^{ti} h_t}(z) = 1 \leq \nu^{[1]}_{(f,H_i),\leq k_i}(z) + \nu^{[1]}_{(f,H_t),\leq k_t}(z).$$

This implies that

$$\sum_{u=1}^{3} \nu^{[1]}_{F_u^{ti} h_t}(z) \leq \nu^{[1]}_{(f,H_i),\leq k_i}(z) + \nu^{[1]}_{(f,H_t),\leq k_t}(z) + \chi_{S_i}(z) + \chi_{S_t}(z)$$

outside an analytic subset of codimension two. By integrating both sides of this inequality, we get

$$\sum_{u=1}^{3} N^{[1]}_{F_u^{ti} h_t}(r) \leq N^{[1]}_{(f,H_i),\leq k_i}(r) + N^{[1]}_{(f,H_t),\leq k_t}(r) + N(r, S_i) + N(r, S_t). \tag{3.1}$$

By the second main theorem, we also have

$$||T_{F^t}(r) \le \sum_{u=1}^{3} N^{[1]}_{F^{ti}_u h_t}(r) + o(T(r)). \tag{3.2}$$

On the other hand, applying the first main theorem to the map F^t and the hyperplane $\{w_0 - w_1 = 0\}$ in $\mathbb{P}^1(\mathbb{C})$, we have

$$T_{F^t}(r) \ge N_{(F^{ti}_1 - F^{ti}_2)h_t}(r) \ge \sum_{\substack{v=1\\ v\ne i,t}}^{2n+2} N^{[1]}_{(f,H_v),\le k_v}(r) - N(r,S_i) - N(r,S_t). \tag{3.3}$$

Therefore, from (3.1)–(3.3) we have

$$||N^{[1]}_{(f,H_i),\le k_i}(r) \ge \sum_{\substack{v=1\\ v\ne i,t}}^{2n+2} N^{[1]}_{(f,H_v),\le k_v}(r) - N^{[1]}_{(f,H_t),\le k_t}(r) - 2(N(r,S_i) + N(r,S_t)) + o(T(r)).$$

The second assertion of the lemma is proved.

Lemma 3.6 *With the assumption of Theorem* 1.1*, let f^1, f^2, f^3 be three meromorphic mappings in $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$. Assume that there exist $i, j \in \{1, 2, \cdots, 2n+2\}$ $(i \ne j)$ and $\alpha \in \mathbb{Z}^m_+$ with $|\alpha| = 1$ such that $\Phi^\alpha_{ij} \not\equiv 0$. Then we have*

$$\begin{aligned}
T(r) &\ge \sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(r) + \sum_{k=1}^{3} N^{[n]}_{(f^k,H_j),\le k_j}(r) + 2\sum_{\substack{t=1\\ t\ne i,j}}^{2n+2} N^{[1]}_{(f,H_t),\le k_t}(r)\\
&\quad - (2n+1)N^{[1]}_{(f,H_i),\le k_i}(r) - (n+1)N^{[1]}_{(f,H_j),\le k_j}(r) + N(r,\nu_j)\\
&\quad - N(r,S_i) - N(r,S_j) - (2n-2)N(r,R_i) - (n-1)N(r,R_j) + o(T(r))\\
&\ge \sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(r) + \sum_{k=1}^{3} N^{[n]}_{(f^k,H_j),\le k_j}(r) + 2\sum_{\substack{t=1\\ t\ne i,j}}^{2n+2} N^{[1]}_{(f,H_t),\le k_t}(r)\\
&\quad - (2n+1)N^{[1]}_{(f,H_i),\le k_i}(r) - (n+1)N^{[1]}_{(f,H_j),\le k_j}(r) + N(r,\nu_j)\\
&\quad - \sum_{u=1}^{3}\Big(\Big(1 + \frac{n-1}{3}\Big)N^{[1]}_{(f^u,H_j),>k_j} + \Big(1 + \frac{2n-2}{3}\Big)N^{[1]}_{(f^u,H_i),>k_i}(r)\Big) + o(T(r)).
\end{aligned}$$

Proof The second inequality is clear. We remain to prove the first inequality. We have

$$\begin{aligned}
\Phi^\alpha_{ij} &= F^{ij}_1 \cdot F^{ij}_2 \cdot F^{ij}_3 \cdot \begin{vmatrix} 1 & 1 & 1\\ F^{ji}_1 & F^{ji}_2 & F^{ji}_3\\ \mathcal{D}^\alpha(F^{ji}_1) & \mathcal{D}^\alpha(F^{ji}_2) & \mathcal{D}^\alpha(F^{ji}_3) \end{vmatrix}\\
&= \begin{vmatrix} F^{ij}_1 & F^{ij}_2 & F^{ij}_3\\ 1 & 1 & 1\\ F^{ij}_1\mathcal{D}^\alpha(F^{ji}_2) & F^{ij}_2\mathcal{D}^\alpha(F^{ji}_2) & F^{ij}_3\mathcal{D}^\alpha(F^{ji}_3) \end{vmatrix}\\
&= F^{ij}_1\Big(\frac{\mathcal{D}^\alpha(F^{ji}_3)}{F^{ji}_3} - \frac{\mathcal{D}^\alpha(F^{ji}_2)}{F^{ji}_2}\Big) + F^{ij}_2\Big(\frac{\mathcal{D}^\alpha(F^{ji}_1)}{F^{ji}_1} - \frac{\mathcal{D}^\alpha(F^{ji}_3)}{F^{ji}_3}\Big)\\
&\quad + F^{ij}_3\Big(\frac{\mathcal{D}^\alpha(F^{ji}_2)}{F^{ji}_2} - \frac{\mathcal{D}^\alpha(F^{ji}_1)}{F^{ji}_1}\Big).
\end{aligned} \tag{3.4}$$

By the Logarithmic Derivative Lemma, it follows that

$$m(r,\Phi^{\alpha}_{ij})\le\sum_{u=1}^{3}m(r,F^{ij}_u)+2\sum_{u=1}^{3}m\Big(\frac{\mathcal{D}^{\alpha}(F^{ji}_u)}{F^{ji}_v}\Big)+O(1)\le\sum_{u=1}^{3}m(r,F^{ij}_u)+o(T_f(r)).$$

Therefore, we have

$$\begin{aligned}T(r)&\ge\sum_{u=1}^{3}T(r,F^{ij}_u)=\sum_{u=1}^{3}(m(r,F^{ij}_u)+N_{\frac{1}{F^{ij}_u}}(r))=m(r,\Phi^{\alpha}_{ij})+\sum_{u=1}^{3}N_{\frac{1}{F^{ij}_u}}(r)+o(T(r))\\&\ge T(r,\Phi^{\alpha}_{ij})-N_{\frac{1}{\Phi^{\alpha}_{ij}}}+\sum_{u=1}^{3}N_{\frac{1}{F^{ij}_u}}(r)+o(T(r))\\&\ge N_{\Phi^{\alpha}_{ij}}(r)-N_{\frac{1}{\Phi^{\alpha}_{ij}}}+\sum_{u=1}^{3}N_{\frac{1}{F^{ij}_u}}(r)+o(T(r))=N(r,\nu_{\Phi^{\alpha}_{ij}})+\sum_{u=1}^{3}N_{\frac{1}{F^{ij}_u}}(r)+o(T(r)).\end{aligned}$$

Then, in order to prove the lemma, it is sufficient for us to prove

$$\begin{aligned}N(r,\nu_{\Phi^{\alpha}_{ij}})\ge&\sum_{u=1}^{3}N^{[n]}_{(f^u,H_i),\le k_i}(r)+\sum_{k=1}^{3}N^{[n]}_{(f^k,H_j),\le k_j}(r)+2\sum_{\substack{t=1\\t\ne i,j}}^{2n+2}N^{[1]}_{(f,H_t),\le k_t}(r)\\&-(2n+1)N^{[1]}_{(f,H_i),\le k_i}(r)-(n+1)N^{[1]}_{(f,H_j),\le k_j}(r)-\sum_{u=1}^{3}N_{\frac{1}{F^{ij}_u}}(r)+N(r,\nu_j)\\&-N(r,S_i)-N(r,S_j)-(2n-2)N(r,R_i)-(n-1)N(r,R_j)+o(T(r)).\qquad(3.5)\end{aligned}$$

Denote by S the set of all singularities of $f^{-1}(H_t)$ $(1\le t\le q)$. Then S is an analytic subset of codimension at least two in $\mathbb{C}^m$. We set

$$I=S\cup\bigcup_{s\ne t}\{z;\nu_{(f,H_s),\le k_s}(z)\cdot\nu_{(f,H_t),\le k_t}(z)>0\}.$$

Then I is also an analytic subset of codimension at least two in $\mathbb{C}^m$.

In order to prove the inequality (3.5), it is sufficient for us to show that the inequality

$$\begin{aligned}P:\overset{\text{Def}}{=}&\sum_{u=1}^{3}\nu^{[n]}_{(f^u,H_i),\le k_i}+\sum_{u=1}^{3}\nu^{[n]}_{(f^k,H_j),\le k_j}+2\sum_{\substack{t=1\\t\ne i,j}}^{2n+2}\chi_{T_t}-(2n+1)\chi_{T_i}-(n+1)\chi_{T_j}\\&-\sum_{u=1}^{3}\nu^{\infty}_{F^{ij}_u}+\chi_{\nu_j}-\chi_{S_i}-\chi_{S_j}-2(n-1)\chi_{R_i}-(n-1)\chi_{R_j}\le\nu_{\Phi^{\alpha}_{ij}}\qquad(3.6)\end{aligned}$$

holds outside the set I.

Indeed, for $z\notin I$, we distinguish the following cases.

Case 1 $z\in T_t\setminus S_i\cup S_j$ $(t\ne i,j)$. We see that $P(z)=2$. We write Φ^{α}_{ij} in the form

$$\Phi^{\alpha}_{ij}=F^{ij}_1\cdot F^{ij}_2\cdot F^{ij}_3\times\begin{vmatrix}(F^{ji}_1-F^{ji}_2)&(F^{ji}_1-F^{ji}_3)\\\mathcal{D}^{\alpha}(F^{ji}_1-F^{ji}_2)&\mathcal{D}^{\alpha}(F^{ji}_1-F^{ji}_3)\end{vmatrix}.$$

Then by the assumption that f^1,f^2,f^3 coincide on T_t, we have $F^{ji}_1=F^{ji}_2=F^{ji}_3$ on $T_t\setminus S_i$. The property of the wronskian implies that $\nu_{\Phi^{\alpha}_{ij}}(z)\ge 2=P(z)$.

Case 2 $z \in T_t \cap (S_i \cup S_j)$ $(t \neq i, j)$. We note that $z \notin T_i \cup T_j$. Therefore, since $f^1(z) = f^2(z) = f^3(z)$, if $z \in S_i$ (resp. $z \in S_j$) then z is a common zero of $(f^1, H_i), (f^2, H_i), (f^3, H_i)$ with multiplicity more than k_i, i.e., $z \in R_i$ (resp. $z \in R_j$).

Firstly, we suppose that $z \in S_i$, and hence $z \in R_i$. Then we have

$$P(z) \leq -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + 2 - 1 - 2(n-1) \leq -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 1.$$

From (3.4) we see that

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq \min\{\nu_{F_1^{ij}}(z) - 1, \nu_{F_2^{ij}}(z) - 1, \nu_{F_3^{ij}}(z) - 1\} \geq P(z).$$

Now, if $z \notin S_i$ then $z \in S_j$ and $z \in R_j$. In this case we note that z will be zero of all $(f^u, H_j), 1 \leq u \leq 3$, with multiplicity more than k_j, but not be zero of any $(f^u, H_i), 1 \leq u \leq 3$. Therefore

$$P(z) \leq -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) \leq \min\{\nu_{F_1^{ij}}(z), \nu_{F_2^{ij}}(z), \nu_{F_3^{ij}}(z)\} - 2(k_j + 1).$$

Similarly as above, we have

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq \min\{\nu_{F_1^{ij}}(z) - 1, \nu_{F_2^{ij}}(z) - 1, \nu_{F_3^{ij}}(z) - 1\} \geq P(z).$$

Case 3 $z \in T_i \setminus S_j$. We have

$$P(z) = \sum_{u=1}^{3} \nu^{[n]}_{(f^u, H_i), \leq k_i}(z) - (2n+1) \leq \min_{1 \leq u \leq 3}\{\nu^{[n]}_{(f^u, H_i), \leq k_i}(z)\} - 1.$$

We may assume that $\nu_{(f^1,H_i)}(z) \leq \nu_{(f^2,H_i)}(z) \leq \nu_{(f^3,H_i)}(z)$. We write

$$\Phi^{\alpha}_{ij} = F_1^{ij}[F_2^{ij}(F_1^{ji} - F_2^{ji})F_3^{ij}\mathcal{D}^{\alpha}(F_1^{ji} - F_3^{ji}) - F_3^{ij}(F_1^{ji} - F_3^{ji})F_2^{ij}\mathcal{D}^{\alpha}(F_1^{ji} - F_2^{ji})].$$

It is easy to see that $F_2^{ij}(F_1^{ji} - F_2^{ji})$, $F_3^{ij}(F_1^{ji} - F_3^{ji})$ are holomorphic on a neighborhood of z, and

$$\nu^{\infty}_{F_3^{ij}\mathcal{D}^{\alpha}(F_1^{ji} - F_3^{ji})}(z) \leq 1,$$
$$\nu^{\infty}_{F_2^{ij}\mathcal{D}^{\alpha}(F_1^{ji} - F_2^{ji})}(z) \leq 1.$$

Therefore, it implies that

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq \nu^{[n]}_{(f^1, H_i), \leq k_i}(z) - 1 \geq P(z).$$

Case 4 $z \in T_i \cap S_j$. The assumption that f^1, f^2, f^3 coincide on T_i yields that $z \in R_j$. We have

$$P(z) \leq \sum_{u=1}^{3} \nu^{[n]}_{(f^u, H_i), \leq k_i}(z) - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - (2n+1) - n \leq -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 1.$$

Thus

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq \min\{\nu_{F_1^{ij}}(z) - 1, \nu_{F_2^{ij}}(z) - 1, \nu_{F_3^{ij}}(z) - 1\} \geq -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 1 \geq P(z).$$

Case 5 $z \in T_j$. We may assume that

$$\nu_{F_1^{ji}}(z) = d_1 \geq \nu_{F_2^{ji}}(z) = d_2 \geq \nu_{F_3^{ji}}(z) = d_3.$$

Choose a holomorphic function h on $\mathbb{C}^m$ with the multiplicity of zero at z equal to 1 such that $F_u^{ji} = h^{d_u}\varphi_u$ $(1 \leq u \leq 3)$, where φ_u are meromorphic on $\mathbb{C}^m$ and holomorphic on a neighborhood of z. Then

$$\begin{aligned}\Phi_{ij}^{\alpha} &= F_1^{ij}\cdot F_2^{ij}\cdot F_3^{ij}\cdot \begin{vmatrix} F_2^{ji}-F_1^{ji} & F_3^{ji}-F_1^{ji} \\ \mathcal{D}^{\alpha}(F_2^{ji}-F_1^{ji}) & \mathcal{D}^{\alpha}(F_3^{ji}-F_1^{ji}) \end{vmatrix} \\ &= F_1^{ij}\cdot F_2^{ij}\cdot F_3^{ij}\cdot h^{d_2+d_3}\cdot \begin{vmatrix} \varphi_2 - h^{d_1-d_2}\varphi_1 & \varphi_3 - h^{d_1-d_3}\varphi_1 \\ \dfrac{\mathcal{D}^{\alpha}(h^{d_2-d_3}\varphi_2 - h^{d_1-d_3}\varphi_1)}{h^{d_2-d_3}} & \mathcal{D}^{\alpha}(\varphi_3 - h^{d_1-d_3}\varphi_1) \end{vmatrix}.\end{aligned}$$

This yields that

$$\nu_{\Phi_{ij}^{\alpha}}(z) \geq \sum_{u=1}^{3}\nu_{F_u^{ij}}(z) + d_2 + d_3 - \max\{0, \min\{1, d_2-d_3\}\}.$$

If $z \notin S_i$, then $P(z) = -\sum\limits_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + \sum\limits_{u=1}^{3}\min\{n, d_u\} - (n+1) + \chi_{\nu_j}$, and hence

$$\begin{aligned}\nu_{\Phi_{ij}^{\alpha}}(z) &\geq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + \sum_{u=1}^{3}\nu_{F_u^{ij}}^{0}(z) + d_2 + d_3 - 1 + \chi_{\nu_j} \\ &\geq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + d_2 + d_3 - 1 + \chi_{\nu_j} \geq P(z).\end{aligned}$$

Otherwise, if $z \in S_i$ then $z \in R_i$, and hence

$$P(z) \leq \sum_{u=1}^{3}\nu_{(f^u,H_j),\leq k_j}^{[n]} - \sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) - 3n + \chi_{\nu_j} \leq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + \chi_{\nu_j}$$

and

$$\begin{aligned}\nu_{\Phi_{ij}^{\alpha}}(z) &\geq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + \sum_{u=1}^{3}\nu_{F_u^{ij}}^{0}(z) + d_2 + d_3 - 1 + \chi_{\nu_j} \\ &= -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + \max\{0, -d_1\} + \max\{d_2, 0\} + \max\{d_3, 0\} - 1 + \chi_{\nu_j} \geq P(z).\end{aligned}$$

Case 6 $z \in (S_i \cup S_j) \setminus \Big(\bigcup\limits_{t=1}^{2n+2} T_t\Big)$. Similarly as Case 5, we have

$$\begin{aligned}\nu_{\Phi_{ij}^{\alpha}}(z) &\geq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) + \max\{0, -d_1\} + \max\{d_2, 0\} + \max\{d_3, 0\} - 1 \\ &\geq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) - 1 \geq -\sum_{u=1}^{3}\nu_{F_u^{ij}}^{\infty}(z) - \chi_{S_i} - \chi_{S_j} \geq P(z).\end{aligned}$$

From the above six cases, the inequality (3.6) holds. The lemma is proved.

Proof of Theorem 1.1 Suppose that there exist three distinct meromorphic mappings f^1, f^2, f^3 in $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$. By Lemma 3.2, we have $f^1 \wedge f^2 \wedge f^3 \equiv 0$. Without loss of generality, we may assume that

$$\underbrace{V_1 \cong \cdots \cong V_{l_1}}_{\text{group 1}} \ncong \underbrace{V_{l_1+1} \cong \cdots \cong V_{l_2}}_{\text{group 2}} \ncong \underbrace{V_{l_2+1} \cong \cdots \cong V_{l_3}}_{\text{group 3}} \ncong \cdots \ncong \underbrace{V_{l_{s-1}} \cong \cdots \cong V_{l_s}}_{\text{group } s},$$

where $l_s = 2n + 2$.

Denote by P the set of all $i \in \{1, \cdots, 2n+2\}$ satisfying that there exist $j \in \{1, \cdots, 2n+2\} \setminus \{i\}$ such that $V_i \ncong V_j$ and $\Phi^\alpha_{ij} \equiv 0$ for all $\alpha \in \mathbb{Z}^m_+$ with $|\alpha| \le 1$. We consider the following three cases.

Case 1 $\sharp P \ge 2$. Then P contains two elements i, j. Then we have $\Phi^\alpha_{ij} = \Phi^\alpha_{ji} = 0$ for all $\alpha \in \mathbb{Z}^m_+$ with $|\alpha| \le 1$. By Lemma 2.1, there exist two functions, for instance they are F_1^{ij} and F^2_{ij}, and a constant λ such that $F_1^{ij} = \lambda F_2^{ij}$. This yields that $F_1^{ij} = F_2^{ij}$ (by Lemma 3.3). Then by Lemma 3.5(ii), we easily see that $V_i \cong V_j$, i.e., V_i and V_j belong to the same group in the above partition.

Without loss of generality, we may assume that $i = 1$ and $j = 2$. Since f^1, f^2, f^3 are supposed to be distinct, the number of each group in the above partition is less than $n+1$. Hence we have $V_1 \cong V_2 \ncong V_t$ for all $t \in \{n+1, \cdots, 2n+2\}$. By Lemma 3.5(ii), we have

$$N^{[1]}_{(f,H_1),\le k_1}(r) + N^{[1]}_{(f,H_t),\le k_t}(r) \ge \sum_{s\ne 1,t} N^{[1]}_{(f,H_s),\le k_s}(r) - 2\sum_{u=1}^{3}\sum_{s=1,t} N^{[1]}_{(f^u,H_s),>k_s}(r),$$

$$N^{[1]}_{(f,H_2),\le k_2}(r) + N^{[1]}_{(f,H_t),\le k_t}(r) \ge \sum_{s\ne 2,t} N^{[1]}_{(f,H_s),\le k_s}(r) - 2\sum_{u=1}^{3}\sum_{s=2,t} N^{[1]}_{(f^u,H_s),>k_s}(r).$$

Summing up both sides of the above two inequalities, we get

$$2N^{[1]}_{(f,H_t),\le k_t}(r) \ge 2\sum_{s\ne 1,2,t} N^{[1]}_{(f,H_s),\le k_s}(r) - 2\sum_{u=1}^{3}(N^{[1]}_{(f^u,H_1),>k_1}(r)$$
$$+ N^{[1]}_{(f^u,H_2),>k_2}(r) + 2N^{[1]}_{(f^u,H_t),>k_t}(r)).$$

After summing-up both sides of the above inequalities over all $t \in \{n+1, \cdots, 2n+2\}$, we easily obtain

$$\sum_{u=1}^{3}\Big((n+2)(N^{[1]}_{(f^u,H_1),>k_1}(r) + N^{[1]}_{(f^u,H_2),>k_2}(r)) + 2\sum_{t=n+1}^{2n+2} N^{[1]}_{(f^u,H_t),>k_t}(r)\Big)$$
$$\ge (n+2)\sum_{t=3}^{n} N^{[1]}_{(f,H_t),\le k_t}(r) + n\sum_{t=n+1}^{2n+2} N^{[1]}_{(f,H_t),\le k_t}(r)$$
$$\ge n\sum_{t=3}^{2n+2} N^{[1]}_{(f,H_t),\le k_t}(r) \ge \frac{n}{3}\sum_{u=1}^{3}\sum_{t=3}^{2n+2} N^{[1]}_{(f_u,H_t),\le k_t}(r)$$
$$\ge \frac{n}{3}\sum_{u=1}^{3}\sum_{t=3}^{2n+2} N^{[1]}_{(f_u,H_t)}(r) - \frac{n}{3}\sum_{u=1}^{3}\sum_{t=3}^{2n+2} N^{[1]}_{(f^u,H_t),>k_t}(r)$$
$$\ge \frac{1}{3}\sum_{u=1}^{3}\sum_{t=3}^{2n+2} N^{[n]}_{(f_u,H_t)}(r) - \frac{n}{3}\sum_{u=1}^{3}\sum_{t=3}^{2n+2} N^{[1]}_{(f^u,H_t),>k_t}(r)$$

$$\geq \frac{n-1}{3}T(r) - \frac{n}{3}\sum_{u=1}^{3}\sum_{t=3}^{2n+2} N^{[1]}_{(f^u,H_t),>k_t}(r) + o(T(r)).$$

Therefore, we have

$$\begin{aligned}\frac{n-1}{3}T(r) &\leq (n+2)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[1]}_{(f^u,H_t),>k_t}(r) \leq (n+2)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} \frac{1}{k_t+1}N_{(f^u,H_t),>k_t}(r)\\ &\leq (n+2)\sum_{t=1}^{2n+2}\frac{1}{k_t+1}T(r).\end{aligned}$$

Letting $r \to +\infty$, we get

$$\frac{n-1}{3(n+2)} \leq \sum_{t=1}^{2n+2}\frac{1}{k_t+1}.$$

This is a contradiction.

Case 2 $\sharp P = 1$. We assume that $P = \{1\}$. We easily see that $V_1 \not\cong V_i$ for all $i = 2, \cdots, 2n+2$ (otherwise $i \in P$, this contradicts $\sharp P = 1$). Then by Lemma 3.5(ii), we have

$$N^{[1]}_{(f,H_1),\leq k_1}(r) \geq \sum_{s\neq 1,i} N^{[1]}_{(f,H_s),\leq k_s}(r) - N^{[1]}_{(f,H_i),\leq k_i}(r) - 2\sum_{u=1}^{3}\sum_{s=1,i} N^{[1]}_{(f^u,H_s),>k_s}(r) + o(T(r)).$$

Summing up both sides of the above inequality over all $i = 2, \cdots, 2n+2$, we get

$$\begin{aligned}(2n+1)N^{[1]}_{(f,H_1),\leq k_1}(r) \geq (2n-1)\sum_{i=2}^{2n+2} N^{[1]}_{(f,H_i),\leq k_i}(r) - 2\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i}(r)\\ - 2(2n+1)\sum_{u=1}^{3} N^{[1]}_{(f^u,H_1),>k_1}(r) + o(T(r)).\end{aligned} \tag{3.7}$$

We also see that $i \notin P$ for all $2 \leq i \leq 2n+2$. Set

$$\sigma(i) = \begin{cases} i+n, & \text{if } i \leq n+2,\\ i-n, & \text{if } n+2 < i \leq 2n+2.\end{cases}$$

Then i and $\sigma(i)$ belong to two distinct groups, i.e., $V_i \not\cong V_{\sigma(i)}$ for all $i \in \{2, \cdots, 2n+2\}$, and hence $\Phi^{\alpha}_{i\sigma(i)} \not\equiv 0$ for some $\alpha \in \mathbb{Z}^m_+$ with $|\alpha| \leq 1$. By Lemma 3.6 we have

$$\begin{aligned}T(r) \geq &\sum_{u=1}^{3}\sum_{t=i,\sigma(i)} N^{[n]}_{(f^u,H_t),\leq k_t}(r) - (2n+1)N^{[1]}_{(f,H_i),\leq k_i}(r) - (n+1)N^{[1]}_{(f,H_{\sigma(i)}),\leq k_{\sigma(i)}}(r)\\ &+ 2\sum_{\substack{t=1\\ t\neq i,\sigma(i)}}^{2n+2} N^{[1]}_{(f,H_t),\leq k_t}(r) - \sum_{u=1}^{3}\Big(\frac{2n+1}{3}N^{[1]}_{(f^u,H_i),>k_i}(r) + \frac{n+2}{3}N^{[1]}_{(f^u,H_{\sigma(i)}),>k_{\sigma(i)}}\Big)\\ &+ o(T(r)).\end{aligned}$$

Summing up both sides of the above inequalities over all $i \in \{2, \cdots, 2n+2\}$, we get

$$(2n+1)T(r) \geq 2\sum_{i=2}^{2n+2}\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\leq k_i}(r) + (n-4)\sum_{i=2}^{2n+2} N^{[1]}_{(f,H_i),\leq k_i}(r)$$

$$+ 2(2n+1)N^{[1]}_{(f,H_1),\le k_1}(r) - (n+1)\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i} + o(T(r))$$

$$\ge 2\sum_{i=2}^{2n+2}\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(r) + \frac{5n-6}{3}\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[1]}_{(f^u,H_i),\le k_i}(r)$$

$$- (8n+4)\sum_{u=1}^{3} N^{[1]}_{(f^u,H_1),>k_1}(r) - (n+5)\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i} + o(T(r))$$

$$\ge \frac{11n-6}{3n}\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[n]}_{(f^u,H_i),\le k_i}(r)$$

$$- (8n+4)\sum_{u=1}^{3} N^{[1]}_{(f^u,H_1),>k_1}(r) - (n+5)\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i} + o(T(r))$$

$$\ge \frac{11n-6}{3n}\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[n]}_{(f^u,H_i)}(r) - (8n+4)\sum_{u=1}^{3} N^{[1]}_{(f^u,H_1),>k_1}(r)$$

$$- \frac{14n+9}{3}\sum_{u=1}^{3}\sum_{i=2}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i} + o(T(r))$$

$$\ge \frac{11n-6}{3}T(r) - (8n+4)\sum_{i=1}^{2n+2}\frac{1}{k_i+1}T(r) + o(T(r)).$$

Letting $r \to +\infty$, we get $\frac{5n-9}{24n+12} \le \sum_{i=1}^{2n+2}\frac{1}{k_i+1}$. This is a contradiction.

Case 3 $P = \emptyset$. Then for all $i \ne j$, by Lemma 3.6 we have

$$T(r) \ge \sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\le k_i}(r) + \sum_{k=1}^{3} N^{[n]}_{(f^k,H_j),\le k_j}(r) + 2\sum_{\substack{t=1\\ t\ne i,j}}^{2n+2} N^{[1]}_{(f,H_t),\le k_t}(r)$$
$$- (2n+1)N^{[1]}_{(f,H_i),\le k_i}(r) - (n+1)N^{[1]}_{(f,H_j),\le k_j}(r) + N(r,\nu_j)$$
$$- \sum_{u=1}^{3}\Big(\Big(1+\frac{n-1}{3}\Big)N^{[1]}_{(f^u,H_j),>k_j}(r) + \Big(1+\frac{2n-2}{3}\Big)N^{[1]}_{(f^u,H_i),>k_i}(r)\Big) + o(T(r)).$$

Summing up both sides of the above inequalities over all pairs (i,j), we get

$$(2n+2)T(r) \ge 2\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[n]}_{(f^u,H_t),\le k_t}(r) + (n-2)\sum_{t=1}^{2n+2} N^{[1]}_{(f,H_t),\le k_t}(r) + \sum_{t=1}^{2n+2} N(r,\nu_t)$$
$$- (n+1)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i}(r) + o(T(r)). \tag{3.8}$$

On the other hand, by Lemma 3.4, we see that $V_j \not\sim V_l$ for all $j \ne l$. Hence, we have

$$P^{jl}_{st} \overset{\text{Def}}{:=} (f^s,H_j)(f^t,H_l) - (f^t,H_j)(f^s,H_l) \not\equiv 0, \quad s \ne t,\ j \ne l.$$

Claim 3.1 With $i \ne j \ne l \ne i$ for every $z \in T_i$, we have

$$\sum_{1\le s<t\le 3} \nu_{P^{jl}_{st}}(z) \ge 4\chi_{T_i}(z) - \chi_{\nu_i}(z).$$

Indeed, for $z \in T_i \setminus \nu_i$, we may assume that $\nu_{(f^1,H_i)}(z) < \nu_{(f^2,H_i)}(z) \leq \nu_{(f^3,H_i)}(z)$. Since $f^1 \wedge f^2 \wedge f^3 \equiv 0$, we have $\det(V_i, V_j, V_l) \equiv 0$, and hence

$$(f^1, H_i)P_{23}^{jl} = (f^2, H_i)P_{13}^{jl} - (f^3, H_i)P_{12}^{jl}.$$

This yields that

$$\nu_{P_{23}^{jl}}(z) \geq 2$$

and hence $\sum_{1\leq s<t\leq 3} \nu_{P_{st}^{jl}}(z) \geq 4 = 4\chi_{T_i}(z) - \chi_{\nu_i}(z)$.

Now, for $z \in \nu_i$, we have $\sum_{1\leq s<t\leq 3} \nu_{P_{st}^{jl}}(z) \geq 3 = 4\chi_{T_i}(z) - \chi_{\nu_i}(z)$. Hence, the claim is proved.

On the other hand, with $i = j$ or $i = l$, for every $z \in \{\nu_{(f,H_i),\leq k_i}(z) > 0\}$ we see that

$$\begin{aligned}\nu_{P_{st}^{jl}}(z) &\geq \min\{\nu_{(f^s,H_i),\leq k_i}(z), \nu_{(f^t,H_i),\leq k_i}(z)\}\\ &\geq \nu^{[n]}_{(f^s,H_i),\leq k_i}(z) + \nu^{[n]}_{(f^t,H_i),\leq k_i}(z) - n\nu^{[1]}_{(f,H_i),\leq k_i}(z),\end{aligned}$$

and hence

$$\sum_{1\leq s<t\leq 3} \nu_{P_{st}^{jl}}(z) \geq 2\sum_{u=1}^{3} \nu^{[n]}_{(f^u,H_i),\leq k_i}(z) - 3n\nu^{[1]}_{(f,H_i),\leq k_i}(z).$$

Combining this inequality and the above claim, we have

$$\sum_{1\leq s<t\leq 3} \nu_{P_{st}^{jl}}(z) \geq \sum_{i=j,l} \Big(2\sum_{u=1}^{3} \nu^{[n]}_{(f^u,H_i),\leq k_i}(z) - 3n\nu^{[1]}_{(f,H_i),\leq k_i}(z)\Big) + \sum_{i\neq j,l} (4\nu^{[1]}_{(f,H_i),\leq k_i}(z) - \chi_{\nu_i}(z)).$$

This yields that

$$\begin{aligned}\sum_{1\leq s<t\leq 3} N_{P_{st}^{jl}}(z) \geq &\sum_{i=j,l} \Big(2\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\leq k_i}(r) - 3nN^{[1]}_{(f,H_i),\leq k_i}(r)\Big)\\ &+ \sum_{i\neq j,l} (4N^{[1]}_{(f,H_i),\leq k_i}(r) - N(r,\nu_i)).\end{aligned} \tag{3.9}$$

On the other hand, by Jensen formula, we easily see that

$$N_{P_{st}^{jl}}(z) \leq T_{f^s}(r) + T_{f^t}(r) + o(T(r)), \quad 1 \leq s < t \leq 3.$$

Then the inequality (3.9) implies that

$$2T(r) \geq \sum_{i=j,l} \Big(2\sum_{u=1}^{3} N^{[n]}_{(f^u,H_i),\leq k_i}(r) - 3nN^{[1]}_{(f,H_i),\leq k_i}(r)\Big) + \sum_{i\neq j,l} (4N^{[1]}_{(f,H_i),\leq k_i}(r) - N(r,\nu_i)).$$

Summing up both sides of the above inequalities over all pair (j,l), we obtain

$$\begin{aligned}2T(r) \geq &\frac{2}{n+1}\sum_{u=1}^{3}\sum_{i=1}^{2n+2} N^{[n]}_{(f^u,H_i),\leq k_i}(r) + \frac{n}{3\times(n+1)}\sum_{u=1}^{3}\sum_{i=1}^{2n+2} N^{[1]}_{(f^u,H_i),\leq k_i}(r)\\ &- \frac{n}{n+1}\sum_{i=1}^{2n+2} N(r,\nu_i) + o(T(r)).\end{aligned}$$

Thus

$$\sum_{i=1}^{2n+2} N(r,\nu_i) \geq \frac{2}{n}\sum_{u=1}^{3}\sum_{i=1}^{2n+2} N^{[n]}_{(f^u,H_i),\leq k_i}(r) + \frac{1}{3}\sum_{u=1}^{3}\sum_{i=1}^{2n+2} N^{[1]}_{(f^u,H_i),\leq k_i}(r) - \frac{2(n+1)}{n}T(r) + o(T(r)).$$

Using this estimate, from (3.8) we have

$$\begin{aligned}(2n+2)T(r) &\geq \Big(2+\frac{2}{n}\Big)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[n]}_{(f^u,H_t),\leq k_t}(r) + \frac{n-1}{3}\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[1]}_{(f_u,H_t),\leq k_t}(r)\\
&\quad - \frac{2(n+1)}{n}T(r) - (n+1)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i}(r) + o(T(r))\\
&\geq \Big(2+\frac{2}{n}+\frac{n-1}{3n}\Big)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[n]}_{(f^u,H_t),\leq k_t}(r) - \frac{2(n+1)}{n}T(r)\\
&\quad - (n+1)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i}(r) + o(T(r))\\
&\geq \Big(2+\frac{2}{n}+\frac{n-1}{3n}\Big)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[n]}_{(f^u,H_t)}(r) - \frac{2(n+1)}{n}T(r)\\
&\quad - \Big(3n+3+\frac{n-1}{3}\Big)\sum_{u=1}^{3}\sum_{t=1}^{2n+2} N^{[1]}_{(f^u,H_i),>k_i}(r) + o(T(r))\\
&\geq \Big(2+\frac{2}{n}+\frac{n-1}{3n}\Big)(n+1)T(r) - \frac{2(n+1)}{n}T(r)\\
&\quad - \Big(3n+3+\frac{n-1}{3}\Big)\sum_{i=1}^{2n+2}\frac{1}{k_i+1}T(r) + o(T(r)).\end{aligned}$$

Letting $r\to+\infty$, we get

$$2n+2 \geq \Big(2+\frac{2}{n}+\frac{n-1}{3n}\Big)(n+1) - \frac{2(n+1)}{n} - \Big(3n+3+\frac{n-1}{3}\Big)\sum_{i=1}^{2n+2}\frac{1}{k_i+1}.$$

Thus $\sum_{i=1}^{2n+2}\frac{1}{k_i+1} \geq \frac{n^2-1}{10n^2+8n}$. This is a contradiction.

Hence the supposition is impossible. Therefore, $\sharp\mathcal{F}(f,\{H_i,k_i\}_{i=1}^{2n+2},1) \leq 2$. We complete the proof of the theorem.

Proof of Theorem 1.2 Let $f^1,f^2,f^3 \in \mathcal{F}(f,\{H_i,k_i\}_{i=1}^{2n+1},p)$. Suppose that $f^1\times f^2\times f^3 : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})$ is linearly nondegenerate, where $\mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})$ is embedded into $\mathbb{P}^{(n+1)^3-1}(\mathbb{C})$ by Segre embedding. Then for every s,t,l we have

$$P := \det((f^i,H_s),(f^i,H_t),(f^i,H_l); 1\leq i\leq 3) \not\equiv 0.$$

By Lemma 2.2 we have

$$T(r) \geq \sum_{i=s,t,l}\big(N(r,\min\{\nu_{(f^u,H_i),\leq k_i}; 1\leq u\leq 3\}) - N^{[1]}_{(f,H_i),\leq k_i}(r)\big)$$

$$+ 2 \sum_{i=1}^{2n+1} N^{[1]}_{(f,H_i),\le k_i}(r) + o(T(r)),$$

where $T(r) = \sum_{u=1}^{3} T_{f^u}(r)$. Summing up both sides of the above inequality over all (s,t,l), we obtain

$$T(r) \ge \frac{1}{2n+1} \sum_{i=1}^{2n+1} (3N(r, \min\{\nu_{(f^u,H_i),\le k_i}; 1 \le u \le 3\}) + (4n-1)N^{[1]}_{(f,H_i),\le k_i}(r)) + o(T(r)). \tag{3.10}$$

Now, for positive integers a, b, c with $\min\{a,p\} = \min\{b,p\} = \min\{c,p\}$, we will show that

$$3\min\{a,b,c\} + (4n-1) \ge \frac{4n-1+3p}{2n+p}(\min\{a,n\} + \min\{b,n\} + \min\{c,n\}). \tag{3.11}$$

Indeed, by replacing a, b, c by $\min\{a,n\}, \min\{b,n\}, \min\{c,n\}$ respectively, without loss of generality we may suppose that $n \ge a \ge b \ge c$. If $c \ge p$, we have

$$\begin{aligned} &3\min\{a,b,c\} + (4n-1) - \frac{4n-1+3p}{2n+p}(\min\{a,n\} + \min\{b,n\} + \min\{c,n\}) \\ &\ge 3c + (4n-1) - \frac{4n-1+3p}{2n+p}(2n+c) = \frac{(2n+1)(c-p)}{2n+p} \ge 0. \end{aligned}$$

Otherwise, if $c < p$ then $a = b = c$, and hence

$$\begin{aligned} &3\min\{a,b,c\} + (4n-1) - \frac{4n-1+3p}{2n+p}(\min\{a,n\} + \min\{b,n\} + \min\{c,n\}) \\ &= 3c + (4n-1) - 3c\frac{4n-1+3p}{2n+p} = \frac{(4n-1)(2n+p-3c) + 6c(n-p)}{2n+p} \ge 0. \end{aligned}$$

Hence the inequality (3.11) holds.

From (3.11), we have

$$\begin{aligned} &3N(r, \min\{\nu_{(f^u,H_i),\le k_i}; 1 \le u \le 3\}) + (4n-1)N^{[1]}_{(f,H_i),\le k_i}(r) \\ &\ge \frac{4n-1+3p}{2n+p} \sum_{u=1}^{3} N^{[n]}_{(f,H_i),\le k_i}(r), \quad 1 \le i \le 2n+1. \end{aligned}$$

Therefore, the inequality (3.10) implies that

$$\begin{aligned} T(r) &\ge \frac{1}{2n+1} \sum_{i=1}^{2n+1} \frac{4n-1+3p}{2n+p} \sum_{u=1}^{3} N^{[n]}_{(f,H_i),\le k_i}(r) + o(T(r)) \\ &\ge \frac{4n-1+3p}{(2n+1)(2n+p)} \sum_{i=1}^{2n+1} \sum_{u=1}^{3} (N^{[n]}_{(f,H_i)}(r) - N^{[n]}_{(f,H_i),>k_i}(r)) + o(T(r)) \\ &\ge \frac{4n-1+3p}{(2n+1)(2n+p)} \Big(n - \sum_{i=1}^{2n+1} \frac{n}{k_i+1}\Big) T(r) + o(T(r)). \end{aligned}$$

Letting $r \to +\infty$, we get

$$1 \ge \frac{4n-1+3p}{(2n+1)(2n+p)} \Big(n - \sum_{i=1}^{2n+1} \frac{n}{k_i+1}\Big),$$

i.e.,

$$\sum_{i=1}^{2n+1} \frac{1}{k_i + 1} \geq \frac{np - 3n - p}{4n^2 + 3np - n}.$$

This is a contradiction.

Hence, the map $f^1 \times f^2 \times f^3$ is linearly degenerate. The theorem is proved.

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