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Stochastic Maximum Principle for Forward-Backward Regime Switching Jump Diffusion Systems and Applications to Finance*

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Abstract The authors prove a sufficient stochastic maximum principle for the optimal control of a forward-backward Markov regime switching jump diffusion system and show its connection to dynamic programming principle. The result is applied to a cash flow valuation problem with terminal wealth constraint in a financial market. An explicit optimal strategy is obtained in this example.

Keywords Stochastic maximum principle, Dynamic programming principle, Forward-backward stochastic differential equation, Regime switching, Jump diffusion

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1 Introduction

The stochastic maximum principle is one of the principal approaches in solving optimal control problems. The key idea of the stochastic maximum principle is to derive a set of necessary conditions that must be satisfied by any optimal control and these necessary conditions become sufficient under certain convexity conditions (see [2, 7, 9, 12, 24]). These works can be regarded as the references on the controls of stochastic differential equations (SDEs for short). On the other hand, since the introduction of nonlinear backward stochastic differential equations (BSDEs for short, see [11]), the stochastic maximum principles for optimal control problems derived by BSDEs or forward-backward SDEs (FBSDEs for short) have been studied by many authors (see [3, 14, 19–22]).

There is a very extensive literature on the stochastic maximum principles for various types of optimal control problems. For jump diffusion processes, see [6], for Markov regime switching diffusion processes, see [4], and for Markov regime switching jump diffusion processes, see [23], in which sufficient maximum principles for SDEs were developed. Stochastic maximum principles for forward-backward controlled systems with Poisson jumps or Markov chains were studied in [10] and [17], respectively. In this paper, we prove a sufficient stochastic maximum principle

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for the optimal control of a forward-backward Markov regime switching jump diffusion system. This work extends the results of [23], which only discussed a forward case.

For another important approach to study forward-backward stochastic optimal control problems, Peng [13] first obtained the generalized dynamic programming principle and introduced the generalized Hamilton-Jacobi-Bellman (HJB for short) equation. Shi [16] generalized the results of [13] by considering the controlled FBSDE with jump. In this paper, we establish the connection between maximum principle and dynamic programming principle of Peng's type in the Markov regime switching jump diffusion context. Relations among the adjoint processes, generalized Hamiltonian function, and value function are given under certain differentiability conditions.

Finally, we use the sufficient maximum principle to discuss the cash flow valuation problem with terminal wealth constraint in a financial model. Using Lagrange multiplier technique, the problem is converted to an unconstrained optimization problem. We prove that the system for this unconstrained problem is governed by a controlled FBSDE, which is naturally reduced to the framework of our paper. And then, the explicit optimal strategy is given with linear state feedback form by virtue of delicate analysis technique.

The paper is organized as follows. The next section presents system dynamics and the optimal control problem. In Section 3, we prove the sufficient stochastic maximum principle. Section 4 establishes the relationship between maximum principle and dynamic programming principle. We illustrate the use of the maximum principle by solving a cash flow valuation problem with terminal wealth constraint in Section 5.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a complete probability space on which defined a standard one-dimensional Brownian motion, a continuous-time Markov chain, and a Poisson random measure. The Markov chain $\alpha(t)$ takes values in a finite space $S = \{\alpha_1, \alpha_2, \cdots, \alpha_D\}$, where $D \in N$, $\alpha_i \in R^D$, and the j-th component of α_i is the Kronecker delta δ_{ij} for each $i, j = 1, 2, \cdots, D$.

We define the generator $\Lambda = \{\lambda_{ij}\}_{i,j=1,2,\cdots,D}$ of the chain, which is also called the rate matrix, or Q-matrix. We also assume that the Markov chain has stationary transition probabilities $P_{\alpha_i\alpha_j}(t) = P(\alpha(t) = \alpha_j|\alpha(0) = \alpha_i),\ i, j = 1, 2, \cdots, D$. From [5], the chain $\alpha(t)$ has a semi-martingale representation $\alpha(t) = \alpha(0) + \int_0^t \Lambda^T\alpha(s)\mathrm{d}s + M(t)$, here $M(t)$ is an R^D-valued, $\mathcal{F}_t$-martingale.

For each $i, j = 1, 2, \cdots, D$ with $i \neq j$, let $J^{ij}(t)$ be the number of jumps from state α_i to state α_j up to time t. By the above semi-martingale representation, we have

$$\begin{aligned}
J^{ij}(t) &= \sum_{0<s\leq t} \langle\alpha(s-), \alpha_i\rangle\langle\alpha(s), \alpha_j\rangle \\
&= \sum_{0<s\leq t} \langle\alpha(s-), \alpha_i\rangle\langle\alpha(s) - \alpha(s-), \alpha_j\rangle \\
&= \int_0^t \langle\alpha(s-), \alpha_i\rangle\langle \mathrm{d}\alpha(s), \alpha_j\rangle \\
&= \lambda_{ij}\int_0^t \langle\alpha(s-), \alpha_i\rangle \mathrm{d}s + m_{ij}(t),
\end{aligned}$$

where $m_{ij}(t)=\int_0^t\langle\alpha(s-),\alpha_i\rangle\langle\mathrm{d}M(s),\alpha_j\rangle$ is an $\mathcal{F}_t$-martingale. Now, for each fixed $j=1,2,\cdots,D$, let $\Phi_j(t)$ be the number of jumps into state α_j up to time t. Then

$$\begin{aligned}\Phi_j(t)&=\sum_{i=1,i\neq j}^{D}J^{ij}(t)\\&=\sum_{i=1,i\neq j}^{D}\lambda_{ij}\int_0^t\langle\alpha(s-),\alpha_i\rangle\mathrm{d}s+\sum_{i=1,i\neq j}^{D}m_{ij}(t)\\&=\lambda_j(t)+\widetilde{\Phi}_j(t),\end{aligned}$$

where we define $\lambda_j(t)=\sum_{i=1,i\neq j}^{D}\lambda_{ij}\int_0^t\langle\alpha(s-),\alpha_i\rangle\mathrm{d}s$ and $\widetilde{\Phi}_j(t)=\sum_{i=1,i\neq j}^{D}m_{ij}(t)$. For each $j=1,2,\cdots,D$, $\widetilde{\Phi}_j(t)$ is again an $\mathcal{F}_t$-martingale. Then, $\lambda(t)=(\lambda_1(t),\lambda_2(t),\cdots,\lambda_D(t))^{\mathrm{T}}$ and $\widetilde{\Phi}(t)=(\widetilde{\Phi}_1(t),\widetilde{\Phi}_2(t),\cdots,\widetilde{\Phi}_D(t))^{\mathrm{T}}$ are the compensator and compensated martingale measure related to the Markov chain, respectively.

We now introduce the Poisson random measure. Denote $R^+=[0,\infty)$ and $\mathcal{B}(R^+)$ the Borel σ-field of R^+. Let $\mathcal{E}\subset R\backslash\{0\}$ be a nonempty Borel set and $\mathcal{B}(\mathcal{E})$ the Borel σ-field generated by open subset O of $\mathcal{E}$, whose closure $\overline{O}$ does not contain the point 0. Suppose that $N(\mathrm{d}t,\mathrm{d}e)$ is the Poisson random measure on $(R^+\times\mathcal{E},\mathcal{B}(R^+)\otimes\mathcal{B}(\mathcal{E}))$ with the compensator $n(\mathrm{d}t,\mathrm{d}e)=\nu(\mathrm{d}e)\mathrm{d}t$, where $\nu(\mathrm{d}e)$ is the Lévy density of jump size of the random measure $N(\mathrm{d}t,\mathrm{d}e)$ on $(\mathcal{E},\mathcal{B}(\mathcal{E}))$. In what follows, we write the compensated Poisson martingale measure as

$$\widetilde{N}(\mathrm{d}t,\mathrm{d}e)=N(\mathrm{d}t,\mathrm{d}e)-n(\mathrm{d}t,\mathrm{d}e).$$

We assume that the Brownian motion, the Markov chain, and the Poisson random measure defined above are independent of each other. This assumption can ensure that the integration by parts formula (see Lemma 3.1) and the Itô's formula (see Lemma 4.1) hold for the regime switching jump diffusions. Assume further that the initial market mode $\alpha(0)$ of the Markov chain is α_{i_0}.

The state processes $(X(t),Y(t),Z(t),\overline{Z}(t,e),\widehat{Z}(t))\in R^4\times R^D$ corresponding to the control process $u(t)\in U\subset R$ are modeled by the following decoupled FBSDE:

$$\begin{cases}\mathrm{d}X(t)=b(t,X(t),u(t),\alpha(t))\mathrm{d}t+\sigma(t,X(t),u(t),\alpha(t))\mathrm{d}W(t)\\\qquad\qquad+\displaystyle\int_{\mathcal{E}}\overline{\sigma}(t,X(t),u(t),\alpha(t),e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e)+\langle\widehat{\sigma}(t,X(t),u(t),\alpha(t)),\mathrm{d}\widetilde{\Phi}(t)\rangle,\\-\mathrm{d}Y(t)=f(t,\Theta(t),u(t),\alpha(t))\mathrm{d}t-Z(t)\mathrm{d}W(t)-\displaystyle\int_{\mathcal{E}}\overline{Z}(t,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e)-\langle\widehat{Z}(t),\mathrm{d}\widetilde{\Phi}(t)\rangle,\\X(0)=x_0,\quad Y(T)=\mu X(T),\end{cases}\tag{2.1}$$

where $\mu\in R$ is a given constant and b, σ, $\overline{\sigma}$, $\widehat{\sigma}$, f are given functions with appropriate dimensions. Here we denote $\Theta(t)=(X(t),Y(t),Z(t))$ for notational simplicity.

Consider a performance criterion defined as

$$J(u(t))=E\Big[\int_0^T l(t,\Theta(t),u(t),\alpha(t))\mathrm{d}t+g(X(T),\alpha(T))+h(Y(0))\Big],\tag{2.2}$$

where l, g, h are given functions.

We say that the control process $u(t)$ is admissible if (2.1) has a unique solution. Write $\mathcal{U}$ for the set of admissible controls. The stochastic control problem is to find an optimal control $u^*(t)\in\mathcal{U}$ such that $J(u^*(t))=\inf\limits_{u(t)\in\mathcal{U}} J(u(t))$.

Let θ denote (x,y,z) and $\mathcal{R}$ denote the set of all functions $r:\mathcal{E}\mapsto R$. Define the Hamiltonian $H:[0,T]\times R^3\times U\times S\times R^3\times\mathcal{R}\times R^D\mapsto R$ by $H(t,\theta,u,\alpha_i,\phi,\varphi,\psi,\overline{\psi},\widehat{\psi})=b(t,x,u,\alpha_i)\varphi+\sigma(t,x,u,\alpha_i)\psi+\int_{\mathcal{E}}\overline{\sigma}(t,x,u,\alpha_i,e)\overline{\psi}\nu(\mathrm{d}e)+\sum\limits_{j=1}^{D}\widehat{\sigma}_j(t,x,u,\alpha_i)\widehat{\psi}_j\lambda_{ij}+f(t,\theta,u,\alpha_i)\phi+l(t,\theta,u,\alpha_i)$. We also assume that the Hamiltonian H is differentiable with respect to θ.

Now we introduce an FBSDE satisfied by the adjoint processes $(\phi(t),\varphi(t),\psi(t),\overline{\psi}(t,e),\widehat{\psi}(t))\in R^4\times R^D$ (from now on the argument t is suppressed sometimes for simplicity whenever no confusion arises):

$$\begin{cases}\mathrm{d}\phi(t)=\Big[\dfrac{\partial f}{\partial y}(t,\Theta,u,\alpha)\phi(t)+\dfrac{\partial l}{\partial y}(t,\Theta,u,\alpha)\Big]\mathrm{d}t\\ \qquad\qquad+\Big[\dfrac{\partial f}{\partial z}(t,\Theta,u,\alpha)\phi(t)+\dfrac{\partial l}{\partial z}(t,\Theta,u,\alpha)\Big]\mathrm{d}W(t),\\ -\mathrm{d}\varphi(t)=\Big[\dfrac{\partial b}{\partial x}(t,X,u,\alpha)\varphi(t)+\dfrac{\partial \sigma}{\partial x}(t,X,u,\alpha)\psi(t)+\displaystyle\int_{\mathcal{E}}\dfrac{\partial \overline{\sigma}}{\partial x}(t,X,u,\alpha,e)\overline{\psi}(t,e)\nu(\mathrm{d}e)\\ \qquad\qquad+\dfrac{\partial \widehat{\sigma}}{\partial x}(t,X,u,\alpha)^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{\psi}(t)+\dfrac{\partial f}{\partial x}(t,\Theta,u,\alpha)\phi(t)+\dfrac{\partial l}{\partial x}(t,\Theta,u,\alpha)\Big]\mathrm{d}t\\ \qquad\qquad-\psi(t)\mathrm{d}W(t)-\displaystyle\int_{\mathcal{E}}\overline{\psi}(t,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e)-\langle\widehat{\psi}(t),\mathrm{d}\widetilde{\Phi}(t)\rangle,\\ \phi(0)=\dfrac{\partial h}{\partial y}(Y(0)),\quad \varphi(T)=\mu\phi(T)+\dfrac{\partial g}{\partial x}(X(T),\alpha(T)),\end{cases}\tag{2.3}$$

where $\mathrm{Diag}(\lambda(t))$ represents a diagonal matrix with the elements of $\lambda(t)$ on the diagonal.

3 Sufficient Stochastic Maximum Principle

Theorem 3.1 *Let $u^*\in\mathcal{U}$ with a corresponding solution $(X^*,Y^*,Z^*,\overline{Z}^*,\widehat{Z}^*)$ of (2.1) and suppose that there exists a solution $(\phi^*,\varphi^*,\psi^*,\overline{\psi}^*,\widehat{\psi}^*)$ of the corresponding adjoint equation (2.3), such that for all $u\in\mathcal{U}$,*

$$E\int_0^T(X(t)-X^*(t))^2\Big[\psi^*(t)^2+\int_{\mathcal{E}}\overline{\psi}^*(t,e)^2\nu(\mathrm{d}e)+\widehat{\psi}^*(t)^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{\psi}^*(t)\Big]\mathrm{d}t<\infty,$$
$$E\int_0^T(Y(t)-Y^*(t))^2\Big[\frac{\partial f}{\partial z}(t,\Theta^*,u^*,\alpha)^2+\frac{\partial l}{\partial z}(t,\Theta^*,u^*,\alpha)^2\Big]\mathrm{d}t<\infty,$$
$$E\int_0^T\phi^*(t)^2\Big[Z(t)^2+\int_{\mathcal{E}}\overline{Z}(t,e)^2\nu(\mathrm{d}e)+\widehat{Z}(t)^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{Z}(t)\Big]\mathrm{d}t<\infty,$$
$$E\int_0^T\varphi^*(t)^2\Big[\sigma(t,X,u,\alpha)^2+\int_{\mathcal{E}}\overline{\sigma}(t,X,u,\alpha,e)^2\nu(\mathrm{d}e)+\widehat{\sigma}(t,X,u,\alpha)^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{\sigma}(t,X,u,\alpha)\Big]\mathrm{d}t<\infty.$$

Furthermore, we assume that the following conditions hold (to simply the notations, in what follows we write $H(t,\theta,u)=H(t,\theta,u,\alpha(t),\phi^*(t),\varphi^*(t),\psi^*(t),\overline{\psi}^*(t,e),\widehat{\psi}^*(t)))$:

Condition 1 For all $t\in[0,T]$, $H(t,\Theta^*(t),u^*(t))=\inf\limits_{u\in U}H(t,\Theta^*(t),u)$.

Condition 2 *For each fixed* $t \in [0,T]$, $\widehat{H}(t,\theta) = \inf_{u\in U} H(t,\theta,u)$ *exists and is a convex function of* θ.

Condition 3 *The functions* $g(x,\alpha_i)$ *and* $h(y)$ *are convex for each* α_i, $i = 1,2,\cdots,D$.
Then u^* *is an optimal control and* $(X^*, Y^*, Z^*, \overline{Z}^*, \widehat{Z}^*)$ *is the corresponding optimal state processes.*

To prove this theorem we first need the following lemma on the integration by parts formula, whose proof is similar to that of Lemma 3.2 in [23], so we omit it.

Lemma 3.1 *Suppose that* $\Gamma^{(j)}(t)$, $j = 1,2$, *are processes defined by the following SDEs:*

$$\begin{cases} \mathrm{d}\Gamma^{(j)}(t) = b^{(j)}(t)\mathrm{d}t + \sigma^{(j)}(t)\mathrm{d}W(t) + \int_{\mathcal{E}} \overline{\sigma}^{(j)}(t,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e) + \langle \widehat{\sigma}^{(j)}(t), \mathrm{d}\widetilde{\Phi}(t)\rangle, \\ \Gamma^{(j)}(0) = \gamma^{(j)}, \quad j = 1,2, \end{cases}$$

where $b^{(j)}(t)$, $\sigma^{(j)}(t)$, $\overline{\sigma}^{(j)}(t,e) \in R$ *and* $\widehat{\sigma}^{(j)}(t) \in R^D$, $j = 1,2$. *Then*

$$\begin{aligned} &\Gamma^{(1)}(T)\Gamma^{(2)}(T) \\ &= \gamma^{(1)}\gamma^{(2)} + \int_0^T \Gamma^{(1)}(t)\mathrm{d}\Gamma^{(2)}(t) + \int_0^T \Gamma^{(2)}(t)\mathrm{d}\Gamma^{(1)}(t) + \int_0^T \sigma^{(1)}(t)\sigma^{(2)}(t)\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathcal{E}} \overline{\sigma}^{(1)}(t,e)\overline{\sigma}^{(2)}(t,e)\nu(\mathrm{d}e)\mathrm{d}t + \int_0^T \widehat{\sigma}^{(1)}(t)^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{\sigma}^{(2)}(t)\mathrm{d}t. \end{aligned}$$

Proof of Theorem 3.1 For any $u \in \mathcal{U}$ and corresponding state processes $(X, Y, Z, \overline{Z}, \widehat{Z})$, by Condition 3,

$$\begin{aligned} &J(u(t)) - J(u^*(t)) \\ &\geq E\Big[\frac{\partial g}{\partial x}(X^*(T),\alpha(T))(X(T) - X^*(T))\Big] + E\Big[\frac{\partial h}{\partial y}(Y^*(0))(Y(0) - Y^*(0))\Big] \\ &\quad + E\int_0^T [l(t,\Theta,u,\alpha) - l(t,\Theta^*,u^*,\alpha)]\mathrm{d}t. \end{aligned} \tag{3.1}$$

Noting the initial value of $\phi^*(t)$ in (2.3), we have

$$\begin{aligned} &(Y(0) - Y^*(0))\frac{\partial h}{\partial y}(Y^*(0)) \\ &= (Y(0) - Y^*(0))\phi^*(0). \end{aligned} \tag{3.2}$$

From (2.1), (2.3) and Lemma 3.1, we obtain that the above is equal to

$$\begin{aligned} &E[(Y(T) - Y^*(T))\phi^*(T)] \\ &- E\int_0^T (Y(t) - Y^*(t))\Big[\frac{\partial f}{\partial y}(t,\Theta^*,u^*,\alpha)\phi^*(t) + \frac{\partial l}{\partial y}(t,\Theta^*,u^*,\alpha)\Big]\mathrm{d}t \\ &+ E\int_0^T [f(t,\Theta,u,\alpha) - f(t,\Theta^*,u^*,\alpha)]\phi^*(t)\mathrm{d}t \\ &- E\int_0^T (Z(t) - Z^*(t))\Big[\frac{\partial f}{\partial z}(t,\Theta^*,u^*,\alpha)\phi^*(t) + \frac{\partial l}{\partial z}(t,\Theta^*,u^*,\alpha)\Big]\mathrm{d}t. \end{aligned} \tag{3.3}$$

By the definitions in (2.1) and (2.3) of $Y(T)$ and $\varphi^*(T)$, we have

$$\begin{aligned} &E[(Y(T) - Y^*(T))\phi^*(T)] \\ &= E\Big[(X(T) - X^*(T))(\varphi^*(T) - \frac{\partial g}{\partial x}(X^*(T), \alpha(T)))\Big]. \end{aligned} \tag{3.4}$$

From (2.1), (2.3) and Lemma 3.1, $E[(X(T) - X^*(T))\varphi^*(T)]$ is equal to

$$\begin{aligned} &E\int_0^T (X(t) - X^*(t))\Big[- \frac{\partial b}{\partial x}(t, X^*, u^*, \alpha)\varphi^*(t) - \frac{\partial \sigma}{\partial x}(t, X^*, u^*, \alpha)\psi^*(t) \\ &- \int_{\mathcal{E}} \frac{\partial \overline{\sigma}}{\partial x}(t, X^*, u^*, \alpha, e)\overline{\psi}^*(t, e)\nu(\mathrm{d}e) - \frac{\partial \widehat{\sigma}}{\partial x}(t, X^*, u^*, \alpha)^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{\psi}^*(t) \\ &- \frac{\partial f}{\partial x}(t, \Theta^*, u^*, \alpha)\phi^*(t) - \frac{\partial l}{\partial x}(t, \Theta^*, u^*, \alpha)\Big]\mathrm{d}t \\ &+ E\int_0^T [b(t, X, u, \alpha) - b(t, X^*, u^*, \alpha)]\varphi^*(t)\mathrm{d}t \\ &+ E\int_0^T [\sigma(t, X, u, \alpha) - \sigma(t, X^*, u^*, \alpha)]\psi^*(t)\mathrm{d}t \\ &+ E\int_0^T \int_{\mathcal{E}}[\overline{\sigma}(t, X, u, \alpha, e) - \overline{\sigma}(t, X^*, u^*, \alpha, e)]\overline{\psi}^*(t, e)\nu(\mathrm{d}e)\mathrm{d}t \\ &+ E\int_0^T [\widehat{\sigma}(t, X, u, \alpha) - \widehat{\sigma}(t, X^*, u^*, \alpha)]^{\mathrm{T}}\mathrm{Diag}(\lambda(t))\widehat{\psi}^*(t)\mathrm{d}t. \end{aligned} \tag{3.5}$$

Combining (3.1)–(3.5), we get

$$\begin{aligned} &J(u(t)) - J(u^*(t)) \\ &\geq E\int_0^T \Big[H(t, \Theta(t), u(t)) - H(t, \Theta^*(t), u^*(t)) - \frac{\partial H}{\partial x}(t, \Theta^*(t), u^*(t))(X(t) - X^*(t)) \\ &- \frac{\partial H}{\partial y}(t, \Theta^*(t), u^*(t))(Y(t) - Y^*(t)) - \frac{\partial H}{\partial z}(t, \Theta^*(t), u^*(t))(Z(t) - Z^*(t))\Big]\mathrm{d}t. \end{aligned}$$

Then we show that the integrand on the right-hand side of the above equation is nonnegative. By Condition 1, $H(t, \Theta^*(t), u^*(t)) = \widehat{H}(t, \Theta^*(t))$. Then by Condition 2, $\widehat{H}(t, \theta)$ is a convex function of θ and for all (θ, u), $H(t, \theta, u) \geq \widehat{H}(t, \theta)$. Therefore, for all (θ, u),

$$H(t, \theta, u) - H(t, \Theta^*(t), u^*(t)) \geq \widehat{H}(t, \theta) - \widehat{H}(t, \Theta^*(t)). \tag{3.6}$$

Since $\theta \mapsto \widehat{H}(t, \theta)$ is convex, it follows that by a standard separating hyperplane argument (see [15]), there exists a sub-gradient $\xi_j(t) \in R^3$, $j = 1, 2, 3$, for $\widehat{H}(t, \theta)$ at $\theta = \Theta^*(t)$, i.e., for all θ,

$$\begin{aligned} 0 \leq \widehat{H}(t, \theta) &- \widehat{H}(t, \Theta^*(t)) - \xi_1(t)(x - X^*(t)) \\ &- \xi_2(t)(y - Y^*(t)) - \xi_3(t)(z - Z^*(t)). \end{aligned} \tag{3.7}$$

Define

$$\begin{aligned} \eta(t, \theta) = H(t, \theta, u^*(t)) &- H(t, \Theta^*(t), u^*(t)) - \xi_1(t)(x - X^*(t)) \\ &- \xi_2(t)(y - Y^*(t)) - \xi_3(t)(z - Z^*(t)). \end{aligned}$$

By (3.6)–(3.7), $\eta(t,\theta)\ge 0$ for all θ. Moreover, $\eta(t,\Theta^*(t))=0$. Therefore, $\frac{\partial\eta}{\partial\theta}(t,\Theta^*(t))=0$. That is

$$\Big(\frac{\partial H}{\partial x},\frac{\partial H}{\partial y},\frac{\partial H}{\partial z}\Big)(t,\Theta^*(t),u^*(t))=(\xi_1(t),\xi_2(t),\xi_3(t)).$$

Substituting these into (3.7) and from (3.6), we get

$$\begin{aligned}0\le{}& H(t,\Theta(t),u(t))-H(t,\Theta^*(t),u^*(t))-\frac{\partial H}{\partial x}(t,\Theta^*(t),u^*(t))(X(t)-X^*(t))\\&-\frac{\partial H}{\partial y}(t,\Theta^*(t),u^*(t))(Y(t)-Y^*(t))-\frac{\partial H}{\partial z}(t,\Theta^*(t),u^*(t))(Z(t)-Z^*(t)),\end{aligned}$$

and conclude $J(u(t))-J(u^*(t))\ge 0$, which proves that $u^*(t)$ is optimal.

4 Relationship to Dynamic Programming Principle

As in pure diffusion case, the adjoint processes can be expressed in terms of derivatives of the value function. We first cast our optimal control problem into a Markovian framework and consider the Markovian (feedback) control, that is, the control $u(t)$ of the form $u(t,X(t),\alpha(t))$, and in order to connect the stochastic maximum principle derived in the previous section with the dynamic programming principle of Peng's type (see [13]), we should reduce the cost functional (2.2) to $J(u(t))=Y(0)$, corresponding to $g(x,\alpha_i)=0$, $h(y)=y$ and $l(t,\theta,u,\alpha_i)=0$ for all $i=1,2,\cdots,D$.

Write $J(t,x,\alpha_i;u)=Y(t)$, where (t,x,α_i) represent the initial time and initial states, respectively, i.e., $X(t)=x$, $\alpha(t)=\alpha_i$. Furthermore, we define $V(t,x,\alpha_i)=\inf\limits_{u\in\mathcal{U}}J(t,x,\alpha_i;u)$.

To proceed, we need to use the following Itô's formula for the Markov regime-switching jump-diffusion processes, whose proof can be found in [23].

Lemma 4.1 *Suppose that we are given an real-valued process $X(t)$ satisfying the following SDE:*

$$\begin{aligned}\mathrm{d}X(t)={}&b(t,X,u,\alpha)\mathrm{d}t+\sigma(t,X,u,\alpha)\mathrm{d}W(t)+\int_{\mathcal{E}}\overline{\sigma}(t,X,u,\alpha,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e)\\&+\langle\widehat{\sigma}(t,X,u,\alpha),\mathrm{d}\widetilde{\Phi}(t)\rangle,\end{aligned}$$

and a function $\phi(t,x,\alpha_i)\in C^{1,2}([0,T]\times R)$ for each $\alpha_i\in S$. Then

$$\begin{aligned}&\phi(T,X(T),\alpha(T))-\phi(0,X(0),\alpha(0))\\&=\int_0^T\Big\{\frac{\partial\phi}{\partial t}(t,X,\alpha)+\frac{\partial\phi}{\partial x}(t,X,\alpha)b(t,X,u,\alpha)+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(t,X,\alpha)\sigma(t,X,u,\alpha)^2\\&\quad+\int_{\mathcal{E}}\Big[\phi(t,X+\overline{\sigma}(t,X,u,\alpha,e),\alpha)-\phi(t,X,\alpha)-\frac{\partial\phi}{\partial x}(t,X,\alpha)\overline{\sigma}(t,X,u,\alpha,e)\Big]\nu(\mathrm{d}e)\\&\quad+\sum_{j=1}^D\Big[\phi(t,X+\widehat{\sigma}_j(t,X,u,\alpha),\alpha_j)-\phi(t,X,\alpha)-\frac{\partial\phi}{\partial x}(t,X,\alpha)\widehat{\sigma}_j(t,X,u,\alpha)\Big]\lambda_j(t)\Big\}\mathrm{d}t\\&\quad+\frac{\partial\phi}{\partial x}(t,X,\alpha)\sigma(t,X,u,\alpha)\mathrm{d}W(t)+\int_{\mathcal{E}}[\phi(t,X+\overline{\sigma}(t,X,u,\alpha,e),\alpha)-\phi(t,X,\alpha)]\widetilde{N}(\mathrm{d}t,\mathrm{d}e)\\&\quad+\sum_{j=1}^D[\phi(t,X+\widehat{\sigma}_j(t,X,u,\alpha),\alpha_j)-\phi(t,X,\alpha)]\mathrm{d}\widetilde{\Phi}_j(t).\end{aligned}$$

In the following, for technical reason, we assume $\overline{\sigma}(t,x,u,\alpha_i,e)=\overline{\sigma}(t,u,\alpha_i,e)$ and $\widehat{\sigma}(t,x,u,\alpha_i)=\widehat{\sigma}(t,u,\alpha_i)$.

In a similar way to [13] and by Lemma 4.1, we obtain that the value function $V(t,x,\alpha_i)$ satisfies the following HJB equation:

$$\begin{cases} 0=-\dfrac{\partial v}{\partial t}(t,x,\alpha_i)+\sup\limits_{u\in U} G\Big(t,x,-v(t,x,\alpha_i),-\dfrac{\partial v}{\partial x}(t,x,\alpha_i),-\dfrac{\partial^2 v}{\partial x^2}(t,x,\alpha_i),u,\alpha_i\Big), \\ \mu x=V(T,x,\alpha_i), \end{cases} \tag{4.1}$$

where the generalized Hamiltonian G associated with $v\in C^{1,2}([0,T]\times R)$ for each α_i is defined as

$$\begin{aligned} &G\Big(t,x,-v(t,x,\alpha_i),-\frac{\partial v}{\partial x}(t,x,\alpha_i),-\frac{\partial^2 v}{\partial x^2}(t,x,\alpha_i),u,\alpha_i\Big)\\ =&-\frac{\partial v}{\partial x}(t,x,\alpha_i)b(t,x,u,\alpha_i)-\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,\alpha_i)\sigma(t,x,u,\alpha_i)^2\\ &-\int_{\mathcal{E}}\Big[v(t,x+\overline{\sigma}(t,u,\alpha_i,e),\alpha_i)-v(t,x,\alpha_i)-\frac{\partial v}{\partial x}(t,x,\alpha_i)\overline{\sigma}(t,u,\alpha_i,e)\Big]\nu(\mathrm{d}e)\\ &-\sum_{j=1}^{D}\Big[v(t,x+\widehat{\sigma}_j(t,u,\alpha_i),\alpha_j)-v(t,x,\alpha_i)-\frac{\partial v}{\partial x}(t,x,\alpha_i)\widehat{\sigma}_j(t,u,\alpha_i)\Big]\lambda_{ij}\\ &-f(t,x,v(t,x,\alpha_i),\frac{\partial v}{\partial x}(t,x,\alpha_i)\sigma(t,x,u,\alpha_i),u,\alpha_i). \end{aligned} \tag{4.2}$$

Now we present a theorem which establishes the relationship between our stochastic maximum principle and the dynamic programming principle of Peng's type.

Theorem 4.1 *Assume that $V(t,x,\alpha_i)\in C^{1,2}([0,T]\times R)$ for each $\alpha_i\in S$. Let u^* be the optimal control and $(X^*,Y^*,Z^*,\overline{Z}^*,\widehat{Z}^*)$ be the corresponding optimal state processes. Then for all $s\in[t,T]$, we have*

$$\frac{\partial V}{\partial s}(s,X^*,\alpha)=G\Big(s,X^*,-V(s,X^*,\alpha),-\frac{\partial V}{\partial x}(s,X^*,\alpha),-\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha),u^*,\alpha\Big). \tag{4.3}$$

Furthermore, if $V(t,x,\alpha_i)\in C^{1,3}([0,T]\times R)$, we define the following processes:

$$\begin{aligned} \phi^*(s)&=\exp\Big\{\int_t^s\Big[\frac{\partial f}{\partial y}(r,\Theta^*,u^*,\alpha)-\frac{1}{2}\frac{\partial f}{\partial z}(r,\Theta^*,u^*,\alpha)^2\Big]\mathrm{d}r\\ &\qquad+\int_t^s\frac{\partial f}{\partial z}(r,\Theta^*,u^*,\alpha)\mathrm{d}W(r)\Big\},\\ \varphi^*(s)&=\frac{\partial V}{\partial x}(s,X^*,\alpha)\phi^*(s),\\ \psi^*(s)&=\Big[\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)+\frac{\partial V}{\partial x}(s,X^*,\alpha)\frac{\partial f}{\partial z}(s,\Theta^*,u^*,\alpha)\Big]\phi^*(s),\\ \overline{\psi}^*(s,e)&=\Big[\frac{\partial V}{\partial x}(s,X^*+\overline{\sigma}(s,u^*,\alpha,e),\alpha)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\Big]\phi^*(s),\\ \widehat{\psi}_j^*(s)&=\Big[\frac{\partial V}{\partial x}(s,X^*+\widehat{\sigma}_j(s,u^*,\alpha),\alpha_j)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\Big]\phi^*(s),\quad j=1,2,\cdots,D. \end{aligned} \tag{4.4}$$

Then $(\phi^*(s),\varphi^*(s),\psi^*(s),\overline{\psi}^*(s,e),\widehat{\psi}^*(s))$ are the adjoint processes and satisfy the FBSDE (2.3).

Proof From the generalized dynamic programming principle (see [13, Theorem 3.1]), it is easy to obtain

$$V(s,X^*(s),\alpha(s))=Y^*(s). \tag{4.5}$$

In fact, because

$$\begin{aligned}
&V(t,x,\alpha_i)=J(t,x,\alpha_i;u^*)=Y^*(t)\\
&=E\Big\{\int_t^s f(r,\Theta^*(r),u^*(r),\alpha(r))\mathrm{d}r+E\Big[\int_s^T f(r,\Theta^*(r),u^*(r),\alpha(r))\mathrm{d}r+\mu X^*(T)\Big|\mathcal{F}_s\Big]\Big\}\\
&=E\int_t^s f(r,\Theta^*(r),u^*(r),\alpha(r))\mathrm{d}r+EJ(s,X^*(s),\alpha(s);u^*(s))\\
&\geq E\int_t^s f(r,\Theta^*(r),u^*(r),\alpha(r))\mathrm{d}r+EV(s,X^*(s),\alpha(s))\\
&\geq V(t,x,\alpha_i),
\end{aligned}$$

where the last inequality is due to the property of backward semigroup introduced by [13], therefore all the inequalities in the aforementioned become equalities. In particular

$$EJ(s,X^*(s),\alpha(s);u^*(s))=EV(s,X^*(s),\alpha(s)).$$

However, by definition, $V(s,X^*(s),\alpha(s))\leq J(s,X^*(s),\alpha(s);u^*(s))$, thus

$$V(s,X^*(s),\alpha(s))=J(s,X^*(s),\alpha(s);u^*(s)),$$

which gives (4.5) because of the definition

$$J(s,X^*(s),\alpha(s);u^*(s))=Y^*(s).$$

On the other hand, applying Itô's formula to $V(s,X^*(s),\alpha(s))$ and $Y^*(s)$ respectively and comparing the coefficients, by uniqueness of solutions to (2.1), we obtain the following relations:

$$\begin{aligned}
&-f(s,\Theta^*,u^*,\alpha)\\
&=\frac{\partial V}{\partial s}(s,X^*,\alpha)+\frac{\partial V}{\partial x}(s,X^*,\alpha)b(s,X^*,u^*,\alpha)+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)^2\\
&\quad+\int_{\mathcal{E}}\Big[V(s,X^*+\overline{\sigma}(s,u^*,\alpha,e),\alpha)-V(s,X^*,\alpha)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\overline{\sigma}(s,u^*,\alpha,e)\Big]\nu(\mathrm{d}e)\\
&\quad+\sum_{j=1}^{D}\Big[V(s,X^*+\widehat{\sigma}_j(s,u^*,\alpha),\alpha_j)-V(s,X^*,\alpha)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\widehat{\sigma}_j(s,u^*,\alpha)\Big]\lambda_j(s)
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
Z^*(s)&=\frac{\partial V}{\partial x}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha),\\
\overline{Z}^*(s,e)&=V(s,X^*+\overline{\sigma}(s,u^*,\alpha,e),\alpha)-V(s,X^*,\alpha),\\
\widehat{Z}_j^*(s)&=V(s,X^*+\widehat{\sigma}_j(s,u^*,\alpha),\alpha_j)-V(s,X^*,\alpha),\quad 1,2,\cdots,D.
\end{aligned} \tag{4.7}$$

In view of the definition (4.2) of the generalized Hamiltionian G, substituting (4.5) and (4.7) into (4.6) implies the relation (4.3). Next, by the HJB equation (4.1), we have

$$\begin{aligned}0&=-\frac{\partial V}{\partial s}(s,X^*,\alpha)+G\Big(s,X^*,-V(s,X^*,\alpha),-\frac{\partial V}{\partial x}(s,X^*,\alpha),-\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha),u^*,\alpha\Big)\\&\geq-\frac{\partial V}{\partial s}(s,x,\alpha)+G\Big(s,x,-V(s,x,\alpha),-\frac{\partial V}{\partial x}(s,x,\alpha),-\frac{\partial^2 V}{\partial x^2}(s,x,\alpha),u^*,\alpha\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}0=\frac{\partial}{\partial x}\Big\{&-\frac{\partial V}{\partial s}(s,x,\alpha)+G(s,x,-V(s,x,\alpha),\\&-\frac{\partial V}{\partial x}(s,x,\alpha),-\frac{\partial^2 V}{\partial x^2}(s,x,\alpha),u^*,\alpha)\Big\}\Big|_{x=X^*}.\end{aligned}\tag{4.8}$$

Solving for $\frac{\partial^2 V}{\partial s\partial x}(s,X^*,\alpha)$ from (4.8) and substituting into the Itô's expansion of $\frac{\partial V}{\partial x}(s,X^*,\alpha)$, we obtain

$$\begin{aligned}&\mathrm{d}\frac{\partial V}{\partial x}(s,X^*,\alpha)\\&=\Big\{-\frac{\partial V}{\partial x}(s,X^*,\alpha)\frac{\partial b}{\partial x}(s,X^*,u^*,\alpha)-\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)\frac{\partial \sigma}{\partial x}(s,X^*,u^*,\alpha)\\&\quad-\frac{\partial f}{\partial x}(s,\Theta^*,u^*,\alpha)-\frac{\partial f}{\partial y}(s,\Theta^*,u^*,\alpha)\frac{\partial V}{\partial x}(s,X^*,\alpha)\\&\quad-\frac{\partial f}{\partial z}(s,\Theta^*,u^*,\alpha)\Big[\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)+\frac{\partial V}{\partial x}(s,X^*,\alpha)\frac{\partial \sigma}{\partial x}(s,X^*,u^*,\alpha)\Big]\Big\}\mathrm{d}s\\&\quad+\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)\mathrm{d}W(s)\\&\quad+\int_{\mathcal{E}}\Big[\frac{\partial V}{\partial x}(s,X^*+\overline{\sigma}(s,u^*,\alpha,e),\alpha)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\Big]\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\\&\quad+\sum_{j=1}^{D}\Big[\frac{\partial V}{\partial x}(s,X^*+\widehat{\sigma}_j(s,u^*,\alpha),\alpha_j)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\Big]\mathrm{d}\widetilde{\Phi}_j(s).\end{aligned}$$

Finally, applying Itô's formula to $\frac{\partial V}{\partial x}(s,X^*,\alpha)\phi^*(s)$, we get

$$\begin{aligned}&\mathrm{d}\frac{\partial V}{\partial x}(s,X^*,\alpha)\phi^*(s)\\&=-\Big\{\frac{\partial f}{\partial x}(s,\Theta^*,u^*,\alpha)\phi^*(s)+\frac{\partial b}{\partial x}(s,X^*,u^*,\alpha)\frac{\partial V}{\partial x}(s,X^*,\alpha)\phi^*(s)\\&\quad+\frac{\partial \sigma}{\partial x}(s,X^*,u^*,\alpha)\Big[\frac{\partial V}{\partial x}(s,X^*,\alpha)\frac{\partial f}{\partial z}(s,\Theta^*,u^*,\alpha)+\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)\Big]\phi^*(s)\Big\}\mathrm{d}s\\&\quad+\Big[\frac{\partial V}{\partial x}(s,X^*,\alpha)\frac{\partial f}{\partial z}(s,\Theta,u^*,\alpha)+\frac{\partial^2 V}{\partial x^2}(s,X^*,\alpha)\sigma(s,X^*,u^*,\alpha)\Big]\phi^*(s)\mathrm{d}W(s)\\&\quad+\int_{\mathcal{E}}\Big[\frac{\partial V}{\partial x}(s,X^*+\overline{\sigma}(s,u^*,\alpha,e),\alpha)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\Big]\phi^*(s)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\\&\quad+\sum_{j=1}^{D}\Big[\frac{\partial V}{\partial x}(s,X^*+\widehat{\sigma}_j(s,u^*,\alpha),\alpha_j)-\frac{\partial V}{\partial x}(s,X^*,\alpha)\Big]\phi^*(s)\mathrm{d}\widetilde{\Phi}_j(t).\end{aligned}\tag{4.9}$$

The first relation in (4.4) is obtained by solving the forward SDE in (2.3) directly. Hence, from (4.9), we show that $(\varphi^*(t),\psi^*(t),\overline{\psi}^*(t,e),\widehat{\psi}^*(t))$ given by (4.4) solve the adjoint equation (2.3) (noting that the terminal value $\frac{\partial V}{\partial x}(T,X^*(T),\alpha(T))=\mu$). The proof is complete.

5 Applications to Finance

In this section, we use the stochastic maximum principle to solve the cash flow valuation problem with terminal wealth constraint in a Markov regime-switching jump-diffusion financial model (see [1] for a similar problem fomulation in a pure diffusion case).

Consider a simple financial market consisting of one risk-free asset and one risky asset. The risk-free asset's price $S_0(t)$ is given by the following stochastic ordinary differential equation (ODE):

$$\mathrm{d}S_0(t) = r(t,\alpha(t))S_0(t)\mathrm{d}t,$$

where $r(t,\alpha_i) > 0$, $i = 1, 2, \cdots, D$, are bounded deterministic functions and can be regarded as the interest rates in different market modes. The risky asset's price process $S(t)$ is described by the following SDE:

$$\mathrm{d}S(t) = S(t)\Big\{b(t,\alpha(t))\mathrm{d}t + \sigma(t,\alpha(t))\mathrm{d}W(t) + \int_{\mathcal{E}} \overline{\sigma}(t,\alpha(t),e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e)\Big\}.$$

We suppose that the non-degeneracy condition $\Lambda(t,\alpha_i) = \sigma(t,\alpha_i)^2 + \int_{\mathcal{E}} \overline{\sigma}(t,\alpha_i,e)^2\nu(\mathrm{d}e) \geq \varepsilon$ is satisfied for all $t \in [0,T]$ and $i = 1, 2, \cdots, D$. Here ε is some positive constant. We also suppose that all the functions $b(t,\alpha_i)$, $\sigma(t,\alpha_i)$, $\overline{\sigma}(t,\alpha_i,e)$, $i = 1, 2, \cdots, D$, are uniformly bounded.

Assume that a principal has paid an agent amount x_0 at time 0. The money is invested in an asset portfolio with total wealth $X(t)$ managed by the agent under a time interval $[0,T]$. At each instant $t \in [0,T]$, the principal ought to receive an amount $c(t)X(t)$ from the agent. The process $c(t)$ can be seen a part of control performed by the agent in order to achieve some goal on behalf of the principal. It is easy to see that the present value of the cash stream $\{c(s)X(s)\}_{t\leq s\leq T}$, discounted to time t with a discount factor $\exp\big\{-\int_t^s \kappa(r,\alpha(r))\mathrm{d}r\big\}$, where $\kappa(t,\alpha_i)$, $i = 1, 2, \cdots, D$, are assumed nonnegative, bounded and deterministic, is given by

$$Y(t) = E\Big[\int_t^T \mathrm{e}^{-\int_t^s \kappa(r,\alpha(r))\mathrm{d}r} c(s)X(s)\mathrm{d}s\Big|\mathcal{F}_t\Big]. \tag{5.1}$$

In what follows, we denote $u(t)$ the amount of the agent's wealth invested in the risky asset at time t. We call $u(t)$ a portfolio of the agent and then $u(t)$ can be seen the other part of control. One has

$$\begin{cases} \mathrm{d}X(t) = [r(t,\alpha(t))X(t) + B(t,\alpha(t))u(t)]\mathrm{d}t + \sigma(t,\alpha(t))u(t)\mathrm{d}W(t) \\ \qquad\qquad + \displaystyle\int_{\mathcal{E}} \overline{\sigma}(t,\alpha(t),e)u(t)\widetilde{N}(\mathrm{d}t,\mathrm{d}e), \\ X(0) = x_0, \end{cases} \tag{5.2}$$

where we set $B(t,\alpha_i) = b(t,\alpha_i) - r(t,\alpha_i)$, $i = 1, 2, \cdots, D$.

Definition 5.1 *A strategy pair $(c(t), u(t))$ is said to be admissible if $u(t)$ and $c(t)X(t)$ are square integrable. The set of all admissible strategies is denoted by $\mathcal{A}$.*

The agent want to come close to the following target at time T: Find admissible strategy $(c(t), u(t))$ which maximizes the principal's preference represented by the utility function U of the cash flow, discounted by his personal discount rate $\exp\{-\int_0^t \beta(s,\alpha_i)\mathrm{d}s\}$, $i = 1, 2, \cdots, D$, where $\beta(t, \alpha_i)$ is assumed satisfying the assumptions similar to $\kappa(t, \alpha_i)$, while the terminal wealth $X(T)$ cannot deviate too much from a given level $d \in R$, which in this case is measured by $E[(X(T)-d)^2]$, under the condition that the total amount received by the principal discounted to time zero is equal to x_0. In particular, we formulate this problem as follows.

Definition 5.2 *The cash flow valuation problem with terminal wealth constraint is the following stochastic optimization problem:*

$$\begin{cases} \min\limits_{(c,u)\in\mathcal{A}} \quad J(c(t), u(t)) = E\Big[-\displaystyle\int_0^T \mathrm{e}^{-\int_0^t \beta(s,\alpha(s))\mathrm{d}s} U(c(t)X(t))\mathrm{d}t + \frac{\delta}{2}(X(T)-d)^2\Big], \\ \text{subject to} \quad E\displaystyle\int_0^T \mathrm{e}^{-\int_0^t \kappa(s,\alpha(s))\mathrm{d}s} c(t)X(t)\mathrm{d}t = x_0, \end{cases} \tag{5.3}$$

where the positive constant δ represents the weight. From (5.1)*, the constraint in the above is in fact $Y(0) = x_0$.*

Using the Lagrange multiplier method, the problem can be reduced to the following unconstrained control problem:

$$\begin{aligned} \min_{(c,u)\in\mathcal{A}} J(c(t), u(t)) = E\Big[-\int_0^T \mathrm{e}^{-\int_0^t \beta(s,\alpha(s))\mathrm{d}s} U(c(t)X(t))\mathrm{d}t \\ + \frac{\delta}{2}(X(T)-d)^2 + \theta(Y(0)-x_0)\Big], \end{aligned} \tag{5.4}$$

where θ is the Lagrange multiplier.

If we introduce an stochastic ODE as follows

$$\begin{cases} \mathrm{d}\Gamma(s) = -\kappa(s,\alpha(s))\Gamma(s)\mathrm{d}s, \\ \Gamma(t) = 1, \end{cases}$$

or, explicitly, $\Gamma(s) = \mathrm{e}^{-\int_t^s \kappa(r,\alpha(r))\mathrm{d}r}$, then by a dual technique similar to that in [18], we can see that $Y(t)$ defined by (5.1) is exactly the solution of the following BSDE:

$$\begin{cases} -\mathrm{d}Y(t) = -[\kappa(t,\alpha(t))Y(t) - c(t)X(t)]\mathrm{d}t - Z(t)\mathrm{d}W(t) \\ \qquad\qquad - \displaystyle\int_{\mathcal{E}} \overline{Z}(t,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e) - \langle \widehat{Z}(t), \mathrm{d}\widetilde{\Phi}(t)\rangle, \\ Y(T) = 0. \end{cases} \tag{5.5}$$

So we can reformulate problem (5.4) as follows, where FBSDE provides a natural setup,

$$\begin{cases} \min\limits_{(c,u)\in\mathcal{A}} \quad J(c(t), u(t)) = E\Big[-\displaystyle\int_0^T \mathrm{e}^{-\int_0^t \beta(s,\alpha(s))\mathrm{d}s} U(c(t)X(t))\mathrm{d}t \\ \qquad\qquad + \dfrac{\delta}{2}(X(T)-d)^2 + \theta(Y(0)-x_0)\Big], \\ \text{subject to} \quad X(t) \text{ and } Y(t) \text{ are given by (5.2) and (5.5), respectively.} \end{cases} \tag{5.6}$$

Finally, we assume that the principal's utility function is of HARA (hyperbolic absolute risk aversion) type. That is, $F(x) = \frac{x^\gamma}{\gamma}$, $\gamma \in (0,1)$. We shall solve the above forward-backward Markov regime switching jump diffusion optimal control problem (5.6) using the sufficient stochastic maximum principle obtained in Section 3.

In this case, the Hamiltonian defined in Section 2 has the following form:

$$\begin{aligned} H = {} & [-\kappa(t,\alpha_i)y + cx]\phi + [r(t,\alpha_i)x + B(t,\alpha_i)u]\varphi + \sigma(t,\alpha_i)u\psi \\ & + \int_{\mathcal{E}} \overline{\sigma}(t,\alpha_i,e)u\overline{\psi}\nu(\mathrm{d}e) - \mathrm{e}^{-\int_0^t \beta(s,\alpha_i)\mathrm{d}s}\frac{(cx)^\gamma}{\gamma}. \end{aligned}$$

The adjoint equation (2.3) becomes

$$\begin{cases} \mathrm{d}\phi(t) = -\kappa(t,\alpha(t))\phi(t)\mathrm{d}t, \\ \phi(0) = \theta \end{cases} \tag{5.7}$$

and

$$\begin{cases} -\mathrm{d}\varphi(t) = [r(t,\alpha(t))\varphi(t) + c(t)\phi(t) - \mathrm{e}^{-\int_0^t \beta(s,\alpha(s))\mathrm{d}s}c(t)^\gamma X(t)^{\gamma-1}]\mathrm{d}t \\ \qquad\qquad - \psi(t)\mathrm{d}W(t) - \int_{\mathcal{E}} \overline{\psi}(t,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e) - \langle \widehat{\psi}(t), \mathrm{d}\widetilde{\Phi}(t)\rangle, \\ \varphi(T) = \delta(X(T) - d). \end{cases} \tag{5.8}$$

We immediately have

$$\phi(t) = \theta\mathrm{e}^{-\int_0^t \kappa(s,\alpha(s))\mathrm{d}s}.$$

Now, let (c^*, u^*) be a candidate for an optimal strategy, and let $(X^*, Y^*, Z^*, \overline{Z}^*, \widehat{Z}^*)$ be the corresponding solution of FBSDE (5.2) and (5.5), $(\phi^*, \varphi^*, \psi^*, \overline{\psi}^*, \widehat{\psi}^*)$ be the corresponding solution of FBSDE (5.7) and (5.8). Thus the value of c^* which maximizes H is

$$\begin{aligned} c^*(t) &= [\mathrm{e}^{-\int_0^t \beta(s,\alpha(s))\mathrm{d}s} X^*(t)^{1-\gamma}\phi^*(t)]^{\frac{1}{\gamma-1}} \\ &= \theta^{\frac{1}{\gamma-1}}\mathrm{e}^{\frac{1}{1-\gamma}\int_0^t [\kappa(s,\alpha(s)) - \beta(s,\alpha(s))]\mathrm{d}s} X^*(t)^{-1}. \end{aligned} \tag{5.9}$$

Substituting $c^*(t)$ in (5.9) into (5.8), then the backward adjoint equation becomes

$$\begin{cases} -\mathrm{d}\varphi^*(t) = r(t,\alpha(t))\varphi^*(t)\mathrm{d}t - \psi^*(t)\mathrm{d}W(t) - \int_{\mathcal{E}} \overline{\psi}^*(t,e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e) - \langle \widehat{\psi}^*(t), \mathrm{d}\widetilde{\Phi}(t)\rangle, \\ \varphi^*(T) = \delta(X(T) - d). \end{cases} \tag{5.10}$$

To find a solution $(\varphi^*(t), \psi^*(t), \overline{\psi}^*(t,e), \widehat{\psi}^*(t))$ to (5.10), we try a process $\varphi^*(t)$ of the following form:

$$\varphi^*(t) = p(t,\alpha(t))X^*(t) + q(t,\alpha(t)), \tag{5.11}$$

where $p(t,\alpha_i)$ and $q(t,\alpha_i)$, $i = 1, 2, \cdots, D$, are deterministic, differential functions which are to be determined. From (5.10), $(p(t,\alpha(t)), q(t,\alpha(t)))$ must satisfy terminal boundary condition: $p(T,\alpha_i) = \delta$, $q(T,\alpha_i) = -\delta d$, for each $i = 1, 2, \cdots, D$.

Applying Itô's formula (see Lemma 4.1) to the right-hand side of (5.11) leads to

$$\begin{aligned}
\mathrm{d}\varphi^*(t) = \Big\{ &\Big[p'(t,\alpha(t)) + r(t,\alpha(t))p(t,\alpha(t)) + \sum_{j=1}^{D}(p(t,\alpha_j) - p(t,\alpha(t)))\lambda_j(t)\Big]X^*(t) \\
&+ q'(t,\alpha(t)) + \sum_{j=1}^{D}[q(t,\alpha_j) - q(t,\alpha(t))]\lambda_j(t) + p(t,\alpha(t))B(t,\alpha(t))u^*(t)\Big\}\mathrm{d}t \\
&+ p(t,\alpha(t))\sigma(t,\alpha(t))u^*(t)\mathrm{d}W(t) + \int_{\mathcal{E}} p(t,\alpha(t))\overline{\sigma}(t,\alpha(t),e)u^*(t)\widetilde{N}(\mathrm{d}t,\mathrm{d}e) \\
&+ \sum_{j=1}^{D}[(p(t,\alpha_j) - p(t,\alpha(t)))X^*(t) + q(t,\alpha_j) - q(t,\alpha(t))]\mathrm{d}\widetilde{\Phi}_j(t).
\end{aligned}$$

Comparing the coefficients with (5.10), we get

$$\begin{aligned}
&- r(t,\alpha(t))p(t,\alpha(t))X^*(t) - r(t,\alpha(t))q(t,\alpha(t)) \\
&= \Big[p'(t,\alpha(t)) + r(t,\alpha(t))p(t,\alpha(t)) + \sum_{j=1}^{D}(p(t,\alpha_j) - p(t,\alpha(t)))\lambda_j(t)\Big]X^*(t) \\
&+ q'(t,\alpha(t)) + \sum_{j=1}^{D}(q(t,\alpha_j) - q(t,\alpha(t)))\lambda_j(t) + p(t,\alpha(t))B(t,\alpha(t))u^*(t), \qquad (5.12)\\
&\psi^*(t) = p(t,\alpha(t))\sigma(t,\alpha(t))u^*(t), \\
&\overline{\psi}^*(t,e) = p(t,\alpha(t))\overline{\sigma}(t,\alpha(t),e)u^*(t), \\
&\widehat{\psi}_j^*(t) = (p(t,\alpha_j) - p(t,\alpha(t)))X^*(t) + q(t,\alpha_j) - q(t,\alpha(t)).
\end{aligned}$$

On the other hand, since the Hamiltonian H is a linear expression in u, the coefficients of u should vanish at optimality, i.e.,

$$B(t,\alpha(t))\varphi^*(t) + \sigma(t,\alpha(t))\psi^*(t) + \int_{\mathcal{E}} \overline{\sigma}(t,\alpha(t),e)\overline{\psi}^*(t,e)\nu(\mathrm{d}e) = 0. \qquad (5.13)$$

Substituting for $\psi^*(t)$ and $\overline{\psi}^*(t,e)$ from (5.12) into (5.13) and noting (5.11), we obtain

$$u^*(t) = -\Lambda(t,\alpha(t))^{-1}B(t,\alpha(t))\Big[X^*(t) + \frac{q(t,\alpha(t))}{p(t,\alpha(t))}\Big]. \qquad (5.14)$$

To obtain the expression of the functions $p(t,\alpha(t))$ and $q(t,\alpha(t))$, we substitute for $\varphi^*(t)$ from (5.11) and for $u^*(t)$ from (5.14) into the first relation in (5.12). This leads to a linear equation in $X^*(t)$. Setting the coefficients of $X^*(t)$ equal to zero, we get the following two systems of ODEs:

$$p'(t,\alpha_i) + [2r(t,\alpha_i) - \rho(t,i)]p(t,\alpha_i) + \sum_{j=1}^{D}[p(t,\alpha_j) - p(t,\alpha_i)]\lambda_{ij} = 0 \qquad (5.15)$$

and

$$q'(t,\alpha_i) + [r(t,\alpha_i) - \rho(t,i)]q(t,\alpha_i) + \sum_{j=1}^{D}[q(t,\alpha_j) - q(t,\alpha_i)]\lambda_{ij} = 0 \qquad (5.16)$$

with the terminal boundary conditions

$$p(T,\alpha_i)=\delta,\quad q(T,\alpha_i)=-\delta d,\quad i=1,2,\cdots,D,$$

where

$$\rho(t,i)=\Lambda(t,\alpha_i)^{-1}B(t,\alpha_i),\quad i=1,2,\cdots,D.$$

The existence and uniqueness of solutions to the above two systems of ODEs are evident as both are linear with uniformly bounded coefficients. In order to get explicit solutions of them, we consider the following processes

$$\widetilde{p}(t,\alpha(t))=\delta E\{\mathrm{e}^{\int_t^T[2r(s,\alpha(s))-\rho(s,\alpha(s))]\mathrm{d}s}|\mathcal{F}_t^\alpha\}\tag{5.17}$$

and

$$\widetilde{q}(t,\alpha(t))=-\delta\mathrm{d}E\{\mathrm{e}^{\int_t^T[r(s,\alpha(s))-\rho(s,\alpha(s))]\mathrm{d}s}|\mathcal{F}_t^\alpha\},\tag{5.18}$$

where $\mathcal{F}_t^\alpha=\sigma\{\alpha(s),\ s\in[0,t]\}$ is the augmented natural filtration generated by the Markov chain.

We want to show that $\widetilde{p}(t,\alpha(t))$ and $\widetilde{q}(t,\alpha(t))$ defined by (5.17) and (5.18) are exactly the solutions of (5.15) and (5.16), respectively. We treat $\widetilde{p}(t,\alpha(t))$ firstly. It is helpful to define the following martingale:

$$R(t)=E\{\mathrm{e}^{\int_0^T[2r(s,\alpha(s))-\rho(s,\alpha(s))]\mathrm{d}s}|\mathcal{F}_t^\alpha\}.\tag{5.19}$$

By the $\mathcal{F}_t^\alpha$-martingale representation theorem, there exists an $\mathcal{F}_t^\alpha$-adapted square integrable process $v(t)$ such that

$$R(t)=R(0)+\sum_{j=1}^D v_j(t)\mathrm{d}\widetilde{\Phi}_j(t).\tag{5.20}$$

From the definitions of $\widetilde{p}(t,\alpha(t))$ and $R(t)$ in (5.17) and (5.19), we have the following relationship:

$$R(t)=\frac{1}{\delta}\widetilde{p}(t,\alpha(t))\mathrm{e}^{\int_0^t[2r(s,\alpha(s))-\rho(s,\alpha(s))]\mathrm{d}s}.\tag{5.21}$$

Applying the Itô's formula to $\widetilde{p}(t,\alpha(t))$, we get

$$\begin{aligned}\mathrm{d}\widetilde{p}(t,\alpha(t))=&\Big\{\widetilde{p}'(t,\alpha(t))+\sum_{j=1}^D[\widetilde{p}(t,\alpha_j)-\widetilde{p}(t,\alpha(t))]\lambda_j(t)\Big\}\mathrm{d}t\\&+\sum_{j=1}^D[\widetilde{p}(t,\alpha_j)-\widetilde{p}(t,\alpha(t))]\mathrm{d}\widetilde{\Phi}_j(t).\end{aligned}$$

We then use Lemma 3.1 to expand the right-hand side of (5.21),

$$\mathrm{d}R(t)=\frac{1}{\delta}\mathrm{e}^{\int_0^t[2r(s,\alpha(s))-\rho(s,\alpha(s))]\mathrm{d}s}\Big\{\widetilde{p}'(t,\alpha(t))+[2r(t,\alpha(t))-\rho(t,\alpha(t))]\widetilde{p}(t,\alpha(t))$$

$$+\sum_{j=1}^{D}[\widetilde{p}(t,\alpha_j)-\widetilde{p}(t,\alpha(t))]\lambda_j(t)\mathrm{d}t+\sum_{j=1}^{D}[\widetilde{p}(t,\alpha_j)-\widetilde{p}(t,\alpha(t))]\mathrm{d}\widetilde{\Phi}_j(t)\Big\}. \tag{5.22}$$

Comparing the dt-part in the above equation with the same part of d$R(t)$ given by (5.20), we find that $\widetilde{p}(t,\alpha(t))$ defined by (5.17) satisfies (5.15). By uniqueness, we conclude that $p(t,\alpha(t))=\widetilde{p}(t,\alpha(t))$. Similarly, we can verify that $\widetilde{q}(t,\alpha(t))$ defined by (5.18) is the unique solution of (5.16).

To find the optimal Lagrange multiplier θ^*, using the technique in [8], we insert (5.9) into the initial constraint $Y(0)=x_0$, then easily derive

$$E\int_0^T \theta^{*\frac{1}{\gamma-1}}\mathrm{e}^{\frac{1}{1-\gamma}\int_0^t[\lambda(s,\alpha(s))-\beta(s,\alpha(s))]\mathrm{d}s}\mathrm{d}t=x_0.$$

Thus we get the optimal θ^* (recalling the definition of transition probabilities of the Markov chain)

$$\begin{aligned}\theta^*&=\Big\{\frac{x_0}{E\int_0^T \mathrm{e}^{\frac{1}{1-\gamma}\int_0^t[\kappa(s,\alpha(s))-\beta(s,\alpha(s))]\mathrm{d}s}\mathrm{d}t}\Big\}^{\gamma-1}\\&=\Big\{\frac{x_0}{E\int_0^T \mathrm{e}^{\frac{1}{1-\gamma}\sum\limits_{j=1}^{D}\int_0^t P_{\alpha_{i_0}\alpha_j}(s)[\kappa(s,\alpha_j)-\beta(s,\alpha_j)]\mathrm{d}s}\mathrm{d}t}\Big\}^{\gamma-1}.\end{aligned} \tag{5.23}$$

By the definition of $u^*(t)$ in (5.14), we can see that $u^*(t)$ is linear in $X^*(t)$. It leads to a linear SDE with bounded coefficients for $X^*(t)$. So $(c^*(t),u^*(t))$ defined by (5.9) and (5.14) is indeed an admissible strategy.

Theorem 5.1 *The optimal strategy for the cash flow valuation with terminal wealth constraint problem* (5.3) *is given by* (5.9) *and* (5.14) *with linear state feedback form:*

$$\begin{aligned}c^*(t)&=\theta^{*\frac{1}{\gamma-1}}\mathrm{e}^{\frac{1}{1-\gamma}\int_0^t[\kappa(s,\alpha(s))-\beta(s,\alpha(s))]\mathrm{d}s}X^*(t)^{-1},\\u^*(t)&=-\Lambda(t,\alpha(t))^{-1}B(t,\alpha(t))\Big[X^*(t)+\frac{q(t,\alpha(t))}{p(t,\alpha(t))}\Big],\end{aligned}$$

where θ^* *is given by* (5.23)*,* $p(t,\alpha(t))$ *and* $q(t,\alpha(t))$ *are given by* (5.17) *and* (5.18)*, respectively.*

6 Concluding Remarks

There are several interesting problems that deserve further investigation. One is to consider the necessary part of the stochastic maximum principle. This needs the derivation of the corresponding variational equations, which can be obtained similarly as that in Tao and Wu [17]. Then the necessary stochastic maximum principle can be achieved by virtue of the duality analysis. On the other hand, the forward-backward regime switching jump diffusion system is assumed to be completely observable in this paper. A more realistic and interesting model is only partially observable. To study partially observable optimal control problem will encounter further difficulty including complex filtering technique. Finally, we have established the connection between maximum principle and dynamic programming principle under the assumption that the value function is smooth enough, which is obviously a very strong restriction. Without involving any derivatives of the value function, we should explore the relations among the adjoint processes, the Hamiltonian function and the value function in the language of viscosity solutions. We shall study these problems in our forthcoming papers.

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